Integration over Radius-Decreasing Circles

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Abstract

In this study, the concept of integration over a radius-decreasing circles is introduced. Results on the $n^{th}$ partial lower sum, $n^{th}$ partial upper sum, lower $C$-integral, and upper $C$-integral are presented. Moreover, the area of a closed and bounded region $R$ in the plane $\mathbb{R}^2$ is obtained using integration over a radius-decreasing circles.

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1 Introduction

In calculus, the integral of a function is a generalization of area, mass, and volume. Unlike the closely-related process of differentiation, there are several possible definitions of integration, with different technical underpinnings. They are, however, compatible; any two different ways of integrating a function will give the same result when they are both defined.
The most important integrals are the Riemann integral and the Lebesgue integral. The Riemann integral was created by Bernhard Riemann in 1854 and was the first rigorous definition of the integral. The Lebesgue integral was created by Henry Lebesgue to integrate a wider class of function and prove very strong theorems about interchanging limits and integrals.

Definite integrals may be approximated using several methods of numerical integration. One popular method, called the rectangle method, relies on dividing the region under the function into a series of rectangles and finding the sum.

One difficulty in computing definite integral is that, it is not always possible to find “explicit formula” for antiderivatives. For instance, there is a (nontrivial) proof that there is no nice function (e.g., sin, cos, exp, polynomials, roots and so on) whose derivative is $x^2$. Integrals of positive functions always give the area under the graph and can fail to exist either because the function is not behaving well in terms of its area or because the area is not finite. As the concept of function became wider this task became more difficult, and led to the need for more and more general integrals.

In this paper, we introduce a new way of integrating a function and in general, finding the area of any bounded region through integration over a radius-decreasing circles.

2 Preliminary Results

The following definition introduces the concept of diameter and radius of a closed and bounded region $R$ in $\mathbb{R}^2$.

**Definition 2.1** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. The diameter of $R$, denoted by $\text{diam}(R)$, is given by

$$\text{diam}(R) = \sup_{x,y \in R} d_{\mathbb{R}^2}(x,y).$$

The radius of $R$, denoted by $\text{rad}(R)$, is given by

$$\text{rad}(R) = \frac{\text{diam}(R)}{2}.$$

Definition 3.1.1 implies that both $\text{diam}(R)$ and $\text{rad}(R)$ are greater than or equal to zero.

The following theorem establishes the relationship between the diameter and the radius of a closed and bounded region $R$ in $\mathbb{R}^2$.

**Theorem 2.2** For any closed and bounded region $R$ in $\mathbb{R}^2$, $\text{rad}(R) \leq \text{diam}(R)$.
Proof: Let $R$ be a closed and bounded region in $\mathbb{R}^2$. Since $\text{diam}(R) \geq 0$, it follows that $\text{rad}(R) \geq 0$. Thus,
\[
\text{rad}(R) \leq 2\text{rad}(R) = \text{diam}(R)
\]
Accordingly, $\text{rad}(R) \leq \text{diam}(R)$.

The next theorem characterizes the region $R$ in $\mathbb{R}^2$ whose radius and diameter coincide.

**Theorem 2.3** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. Then $\text{rad}(R) = \text{diam}(R)$ if and only if $R$ is the singleton set.

Proof: Let $R$ be a closed and bounded region in $\mathbb{R}^2$. Assume that $\text{rad}(R) = \text{diam}(R)$. Then $\text{rad}(R) = \text{diam}(R) = 2\text{rad}(R)$. Consequently, $\text{rad}(R) = 2\text{rad}(R)$. Thus, $\text{rad}(R) = 0$ and $\text{diam}(R) = 0$. Then $\sup_{x,y \in R} d_{\mathbb{R}^2}(x,y) = \text{diam}(R) = 0$. Thus $d_{\mathbb{R}^2}(x,y) \leq 0$, $\forall$ $x, y \in R$. But $d_{\mathbb{R}^2}(x,y) \geq 0$, $\forall$ $x, y \in R$. Hence, $d_{\mathbb{R}^2}(x,y) = 0$. Also $x = y$ for all $x, y \in R$. Since $x$ and $y$ are arbitrary, $R = \{x\}$. Therefore, $|R| = 1$. Accordingly, $R$ is a singleton set. Conversely, assume that $R$ is a singleton set. Then for all $x$ and $y$ in $R$, $x = y$. Thus, $d_{\mathbb{R}^2}(x,y) = 0$, $\forall$ $x, y \in R$. Hence, $\text{diam}(R) = \sup_{x,y \in R} d_{\mathbb{R}^2}(x,y) = \sup_{x,y \in R} 0 = 0$. Therefore, $\text{rad}(R) = \frac{1}{2}\text{diam}(R) = \frac{1}{2}(0) = 0 = \text{diam}(R)$.

The notion of proximity is very much useful in almost all fields of mathematics. The proximity point of $R$ is defined as follows:

**Definition 2.4** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. A point $x \in \mathbb{R}^2$ is called the **proximity point** of $R$ if for every $y \in R$, $d_R(x,y) \leq \text{rad}(R)$.

The following theorem which establishes the relationship between the proximity point and the center of a given circle of diameter $\text{diam}(R)$ follows directly from the definition of proximity point and the center of a given region $R$.

**Theorem 2.5** The proximity point of $R$ is exactly the center of the smallest circle that circumscribe the region $R$.

The next notion defines the smallest circle that circumscribe a closed and bounded region $R$ in $\mathbb{R}^2$ and the largest circle that can be inscribed in $R$.

**Definition 2.6** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. The largest circle that can be inscribed in $R$ is called the **circle inferior**. The smallest circle that circumscribe the region $R$ is called the **circle superior**.
3 Integration of a Closed and Bounded Region

Given a closed and bounded region $R$ in $\mathbb{R}^2$, its $n^{th}$ partial lower and upper sums are defined in the following definition.

**Definition 3.1** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. The $n^{th}$ partial lower sum of $R$ over a radius-decreasing circles is given by

$$L(R, \langle r_i \rangle) = \sum_{i=1}^{n} c_i \pi r_i^2,$$

where $c_i$ is the number of circles of radius $r_i$.

The $n^{th}$ partial upper sum of $R$ is given by

$$U(R, \langle r_i^* \rangle) = \pi r^2 - \sum_{i=1}^{n} d_i \pi r_i^*^2,$$

where $d_i$ is the number of circles of radius $r_i^*$ (circles inside the circle superior and outside the region $R$) and $r$ is the radius of the smallest circle that circumscribes the region $R$.

The next theorem establishes the relationship between the $n^{th}$ partial lower sum and the $n^{th}$ partial upper sum of $R$ in $\mathbb{R}^2$.

**Theorem 3.2** For any closed and bounded region $R$ in $\mathbb{R}^2$, $L(R, \langle r_i \rangle) \leq U(R, \langle r_i^* \rangle)$.

**Proof**: Let $R$ be a closed and bounded region in $\mathbb{R}^2$. Now, for every pair of strictly decreasing sequences $\langle r_i \rangle$ and $\langle r_i^* \rangle$ of radius of circles inside $R$ and outside the region $R$ contained in the circle, respectively,

$$L(R, \langle r_i \rangle) = \sum_{i=1}^{n} c_i \pi r_i^2$$

$$\leq \pi r^2 - \sum_{i=1}^{n} c_i \pi r_i^*^2$$

$$= U(R, \langle r_i^* \rangle). \quad \square$$

The succeeding definition introduces the notion of the lower and upper $C$-integral of a closed and bounded region $R$ in $\mathbb{R}^2$. 
Definition 3.3 Let \( R \) be a closed and bounded region in \( \mathbb{R}^2 \). The lower \( C \)-integral of \( R \) over a radius-decreasing circles is given by

\[
I_L(R, \langle r_i \rangle) = \lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2,
\]

where \( c_i \) is the number of circles of radius \( r_i \). The upper \( C \)-integral of \( R \) over a radius-decreasing circles is given by

\[
I_U(R, \langle r_i^* \rangle) = \pi r^2 - \lim_{n \to +\infty} \sum_{i=1}^{n} d_i \pi r_i^{*2},
\]

where \( d_i \) is the number of circles of radius \( r_i^* \) (circles inside the circle superior and outside the region \( R \)), and \( r \)=radius of circle superior.

The next theorem establishes the relationship between the lower \( C \)-integral and upper \( C \)-integral of a closed and bounded region \( R \) in \( \mathbb{R}^2 \).

Theorem 3.4 Let \( R \) be a closed and bounded region in \( \mathbb{R}^2 \). Then

\[
I_L(R, \langle r_i \rangle) \leq I_U(R, \langle r_i^* \rangle).
\]

Proof: Let \( R \) be a closed and bounded region in \( \mathbb{R}^2 \). Let \( c_i \) be the number of circles of radius \( r_i \), \( d_i \) is the number of circles of radius \( r_i^* \) (circles inside the circle superior and outside the region \( R \)), and \( r \)=radius of circle superior. Then by Theorem 3.2,

\[
L(R, \langle r_i \rangle) \leq U(R, \langle r_i^* \rangle), \text{ i.e.,}
\]

\[
\sum_{i=1}^{n} c_i \pi r_i^2 \leq \pi r^2 - \sum_{i=1}^{n} d_i \pi r_i^{*2}.
\]

Now, if

\[
\lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2 \text{ and } \lim_{n \to +\infty} \left[ \pi r^2 - \sum_{i=1}^{n} d_i \pi r_i^{*2} \right] \text{ exist.}
\]

Moreover,

\[
\lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2 \leq \lim_{n \to +\infty} \left[ \pi r^2 - \sum_{i=1}^{n} d_i \pi r_i^{*2} \right]
\]

\[
\Rightarrow \lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2 \leq \pi r^2 - \lim_{n \to +\infty} \sum_{i=1}^{n} d_i \pi r_i^{*2}
\]

\[
\Rightarrow \lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2 \leq \pi r^2 - \lim_{n \to +\infty} \sum_{i=1}^{n} d_i \pi r_i^{*2}
\]
Therefore by Definition 3.3, $I_L(R, \langle r_i \rangle) \leq I_U(R, \langle r^*_i \rangle)$. □

The following notion defines the $C$-integrability of a given closed and bounded region $R$ in $\mathbb{R}^2$.

**Definition 3.5** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. We say that $R$ is said to be $C$-integrable if

$$I_L(R, \langle r_i \rangle) = I_U(R, \langle r^*_i \rangle).$$

In view of the above definition, the following theorem is immediate.

**Theorem 3.6** Let $R$ be a closed and bounded region in $\mathbb{R}^2$. Then $R$ is $C$-integrable. Moreover,

$$A(R) = \lim_{n \to +\infty} \sum_{i=1}^{n} c_i \pi r_i^2 = \pi r^2 - \lim_{n \to +\infty} \sum_{i=1}^{n} d_i \pi r^*_i,$$

where $\langle r_i \rangle$ is a decreasing sequence of radii of circles inside the region $R$, $\langle r_i \rangle$ is a decreasing sequence of radii of circles inside the circle superior and outside the region $R$, $c_i$ is the number of circles of radius $r_i$, $d_i$ is the number of circles of radius $r^*_i$, and $r$ is the radius of the circle superior.

**References**


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