Some Localization of the Zeros of the Derivatives of a Complex Polynomial in the Disks or Cardioid Interiorities

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Abstract

In this paper, we localize the zeros of the derivatives of a complex polynomial in some sets. These sets are relevant to the first and second derivative of the polynomial, and they are respectively disks and cardioid interiors.

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1. Introduction

The localization of the zeros of the complex polynomials is very important area of the mathematics. The impossibility to find the zeros of any polynomials using the coefficients makes every statement here very significant. There exist many conjectures which are not proved, like Sendov’s conjecture, Obreshkoff’s conjecture. Their assertion is localized the zeros of the derivative of the any complex polynomial in some areas. Here we present some new results about the zeros of the derivative of the complex polynomials. These are Theorem3, Theorem4 and Theorem5. Their results could be applied for the localization of the zeros of the derivative of the polynomials- these are Theorem3 and Theorem4. In Theorem5 we localize the zeros of the second derivative of the complex polynomial.
We see that they must belong to the cardioid interiorities, created by the zeros of the given polynomial. Many of these results could be applied for the solving of the unproved conjectures.

2. Preliminaries

We note:
\[ D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \] is the open disk.
\[ \overline{D}(a, r) = \{ z \in \mathbb{C} : |z - a| \leq r \} \] is the closed disk.
\[ C(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \] – the open cardioid interior – how to define it: after translation \( t \), \( t(a) = r \in \mathbb{R} \) and then rotation with angle \( \varphi = -\arg a; a \in \mathbb{C} \). Then coordinates must satisfy
\[
(x^2 + y^2 - 2rx)^2 < 4r^2(x^2 + y^2). 
\]
\[ \overline{C}(a, r) \] – the closed cardioid interior.

Sendov’s conjecture: Let us put for \( n \geq 2, p(z) = \prod_{k=1}^{n}(z - z_k), \) where \( z_k \in \overline{D}(0,1), k = 1, 2, \ldots, n \). Then \( p'(z) \) has at least one zero in each of the disks \( \overline{D}(z_k, 1), k = 1, 2, \ldots, n \).

3. Related results

**Theorem 1.** Let the zeros \( z_k, k = 1, 2, \ldots, n \) of a polynomial \( p(z) \in \mathbb{C}[z] \) satisfy \( z_k \in \overline{D}(0,1) \). Then the zeros \( z \) of the polynomial \( q(z) = \gamma p(z) + zp'(z) \), where \( \text{Re} \ \gamma \geq -\frac{n}{2} \) satisfy \( z \in \overline{D}(0,1) \).

**Proof:** Let \( z \) be such that \( q(z) = 0 \) and \( p(z) \neq 0 \). Then
\[
\frac{q(z)}{p(z)} = \gamma + \frac{z}{z - z_1} + \cdots + \frac{z}{z - z_n} = 0. 
\]

Hence
Some localization of the zeros of the derivatives

\( q(z) = \gamma + \frac{z_z_1 + z_z_1}{z - z_1} + \ldots + \frac{z_z_n + z_z_n}{z - z_k} = 0. \)

\[ \frac{q(z)}{p(z)} = \gamma + n \frac{1}{2} \left( \frac{z + z_z_1 (\bar{z} - \bar{z}_1)}{|z - z_1|^2} + \ldots + \frac{z + z_z_n (\bar{z} - \bar{z}_n)}{|z - z_n|^2} \right) = 0. \]

Therefore

\[ \text{Re} \frac{q(z)}{p(z)} = \text{Re} \gamma + \frac{n}{2} + \frac{1}{2} \left( \frac{|z|^2 - |z_z_1|^2}{|z - z_1|^2} + \ldots + \frac{|z|^2 - |z_z_n|^2}{|z - z_n|^2} \right) = 0. \]

If we assume \( z \notin D(0,1) \), then we obtain \( \text{Re} \gamma > 0 \), which is impossible.

**Theorem 2.** If all the zeros \( z_k, k = 1,2,\ldots,n \); of a polynomial \( p(z) \in \mathbb{C}[z] \) satisfy \( z_k \in D(0,1) \) and \( a \) is a zero of \( p(z) \) of modulus 1, then the derivative \( p'(z) \) has at least one zero in \( D \left( \frac{a}{2}, \frac{1}{2} \right) \).

**Proof:** Let \( p(z) = (z-a)q(z) \). If we denote by \( z_1, z_2, \ldots, z_{n-1} \) the zeros of \( q(z) \) and by \( w_1, w_2, \ldots, w_{n-1} \) those of \( p'(z) \), then in the non-trivial case \( q(a) \neq 0 \) we obtain

\[ \sum_{k=1}^{n-1} \text{Re} \frac{a}{a - w_k} = \text{Re} \frac{ap''(a)}{p'(a)} = 2 \text{Re} \frac{q'(a)}{q(a)} = \]

\[ = 2 \sum_{k=1}^{n-1} \text{Re} \frac{a}{a - z_k} \geq 2 \frac{n-1}{2} = n - 1. \]

Here we essentially, that

\[ p'(z) = (z-a)q'(z) + q(z), \]

\[ p''(z) = (z-a)q''(z) + 2q(z), \]

and \( |a| = 1 \).

Hence \( \text{Re} \frac{a}{a - w_k} \geq 1 \) for some \( k, (1 \leq k \leq n - 1) \). That means
\[
\Re \left( \frac{a - \frac{w_k}{2} + \frac{w_k}{2}}{a - w_k} \right) = \Re \left( \frac{1}{2} + \frac{w_k}{2} \right) = \Re \left[ \frac{1}{2} + \frac{1}{2} \frac{(a + w_k)(\bar{a} - \bar{w_k})}{|a - w_k|^2} \right] = \frac{1}{2} + \frac{1}{2} \frac{|a|^2 - |w_k|^2}{|a - w_k|^2} \geq 1,
\]

i.e. \(|a - w_k|^2 + |w_k|^2 \leq |a|^2\) which confirms that \(w_k \in D \left( \frac{a}{2}, \frac{1}{2} \right)\).

4. Main results

**Theorem 3.** If all the zeros of a polynomial \(p(z) \in \mathbb{C}[z]\) are \(z_k, k = 1, 2, ..., n\). Then for each zero \(w\) of the derivative \(p'(z)\) exists some \(k_0 \in \mathbb{N}, 1 \leq k_0 \leq n\), such that \(w_k \in D \left( \frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2} \right)\).

**Proof:** Let \(w \in \mathbb{C}\) be such that \(p'(w) = 0\). We except the trivial case \(p(w) = 0\), which confirms the assertion. Then

\[
w \frac{p'(w)}{p(w)} = \frac{w}{w - z_1} + \frac{w}{w - z_2} + \cdots + \frac{w}{w - z_n} =
\]

\[
\frac{w - \frac{z_1}{2} + \frac{z_1}{2}}{w - z_1} + \frac{w - \frac{z_2}{2} + \frac{z_2}{2}}{w - z_2} + \cdots + \frac{w - \frac{z_n}{2} + \frac{z_n}{2}}{w - z_n} =
\]

\[
= \frac{n}{2} + \frac{1}{2} \left[ \frac{(w + z_1)(\bar{w} - \bar{z_1})}{|w - z_1|^2} + \cdots + \frac{(w + z_n)(\bar{w} - \bar{z_n})}{|w - z_n|^2} \right] = 0.
\]

Therefore

\[
\Re \left( \frac{p'(w)}{p(w)} \right) = \frac{n}{2} + \frac{1}{2} \left[ \frac{|w|^2 - |z_1|^2}{|w - z_1|^2} + \cdots + \frac{|w|^2 - |z_n|^2}{|w - z_n|^2} \right] = 0
\]

We put

\[
\alpha_k = \frac{|w|^2 - |z_k|^2}{|w - z_k|^2}, k = 1, 2, ... n.
\]
Then $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -n$. Consequently there exists $k_0 \in \mathbb{N}$, $1 \leq k_0 \leq n$ such that $\alpha_{k_0} \leq -1$. It means that $|w|^2 + |w - z_{k_0}|^2 \leq |z_{k_0}|^2$, i.e. $w \in D\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$.

**Theorem 4.** If all the zeros of a polynomial $p(z) \in \mathbb{C}[z]$ are $z_k, k = 1, 2, \ldots, n$. Then for each zero $w$ of the derivative $p'(z)$ exists some $k_0 \in \mathbb{N}$, $1 \leq k_0 \leq n$, such that $w \notin D\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$.

**Proof:** Let $w \in \mathbb{C}$ be such that $p'(w) = 0$. We except the trivial case $p(w) = 0$, which confirms the assertion.

Further we repeat the proof of Theorem 3 and we get $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -1$. Consequently, there exists $k_0 \in \mathbb{N}$, $1 \leq k_0 \leq n$ such that $\alpha_{k_0} \geq -1$. It means that $|w|^2 + |w - z_{k_0}|^2 \geq |z_{k_0}|^2$, i.e. $w \notin D\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$.

**Theorem 5.** If all the zeros of the polynomial $p(z) \in \mathbb{C}[z]$ are $z_k, k = 1, 2, \ldots, n$. Then for each zero of the second derivative $p''(z)$ exists some, $k_0 \in \mathbb{N}$, $1 \leq k_0 \leq n$, such that $t \in \mathbb{C}\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$.

**Proof:** We denote by $t$, the zero of the second derivative, i.e. $p''(t) = 0$.

According to Theorem 3 there exists such a zero $w$ of the derivative $p'(z)$, that $t \in D\left(\frac{w}{2}, \frac{|w|}{2}\right)$. For this zero $w$ of the derivative $p'(z)$, according to Theorem 3, there exists a zero of the polynomial $p(z) - z_{k_0}$, such that $w \in D\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$. In order to find the geometric places of the points $t$, let us take $w$ on the boundary of the disk $D\left(\frac{z_{k_0}}{2}, \frac{|z_{k_0}|}{2}\right)$ and $t$ on the boundary of the disk $D\left(\frac{w}{2}, \frac{|w|}{2}\right)$.

For better understanding, let us take $z_{k_0} \in X$, and $\arg w = \alpha$, $\arg t = \alpha + \beta$.

Here we have $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$. Let us put $|z_{k_0}| = \alpha$. Then
\[ x = a \cos \alpha \cos \beta \cos (\alpha + \beta), \]
\[ y = a \cos \alpha \cos \beta \sin (\alpha + \beta), \]
\[(x^2 + y^2 - ax)^2 = (a^2 \cos^2 \alpha \cos^2 \beta - a^2 \cos \alpha \cos \beta \cos (\alpha + \beta))^2 = \]
\[= a^4 \cos^2 \alpha \cos^2 \beta \sin^2 \alpha \sin^2 \beta \leq a^2 a^2 \cos^2 \alpha \cos^2 \beta = a^2 (x^2 + y^2). \]

This means that our geometric place belongs to \( C \left( \frac{z_{k0}}{2}, \frac{z_{k1}}{2} \right) \).

**References**


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