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Best Proximity Point Theorems in Partially Ordered Metric Spaces

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Abstract

The notion of generalized Berinde type contraction non-self maps in partially ordered metric spaces is introduced, and some best proximity point theorems for this class are established.

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1 Introduction and preliminaries

Fixed point theory is one of the most powerful and fruitful tools in the study of many branches of mathematics, mathematical sciences and economics ([6, 21]). Many authors (for example, [1, 2, 8, 9, 11, 12, 13, 17, 24, 28, 31] and reference therein) extended, improved and generalized Banach's contraction principle.

Especially, the author of [5] proved the following fixed point result.

Theorem 1.1. *Let (X, d) be a complete metric space. Suppose that a map $T : X \to X$ satisfies the following condition:*

there exists two constant $k \in (0, 1)$ and $L \geq 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y) + Ld(y, Tx).$$

Then

(1) T has a fixed point;

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$, converges to some fixed point x_*, and the following estimates hold:

there exists a constant $k \in (0,1)$ such that

(i) for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_n, x_*) \leq \frac{k^n}{1-k} d(x_0, x_1);$$

(ii) for all $n \in \mathbb{N}$,

$$d(x_n, x_*) \leq \frac{k}{1-k} d(x_{n-1}, x_n).$$

Very recently, the authors of [16] obtained a generalization of Banach's contraction principle. They proved the following theorem.

Theorem 1.2. *Let (X, d) be a complete metric space. Suppose that a map $T : X \to X$ satisfies the following condition:*
for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y).$$

Then, T has a fixed point, and for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point. Moreover, if x_* and y_* are two distinct fixed points of T, then $d(x_*, y_*) \geq \frac{1}{2}$.

Best proximity point theorems is to provide sufficient conditions to solve a minimization problem.

The author of [10] intoduced the concept of best proximity point.

Let (X, d) be a metric space, A and B be nonempty subsets of X, and let $T : A \to B$ be a map.

A point $x \in A$ is called *best proximity point* of the map T if

$$d(x, Tx) = d(A, B).$$

From $d(x, Tx) \geq d(A, B)$ for all $x \in A$, it can be obserbed that the grobal minimum of the map $x \to d(x, Tx)$ is attained from a best proximity point.

Note that if the underying map T is self map, then best proximity point reduce to fixed point.

A lot of authors (for instance, [3, 4, 7, 10, 14, 15, 18, 19, 23, 25, 26, 29, 30] and reference therein) obtained best proximity point theorems for certain contractions.

In this paper, we introduce the concept of generalized Berinde type contraction non-self maps and prove the existence of a best proximity point for such maps in partially odered complete metric spaces.

We recall the following notations and definitions.

Let $(X, \preceq)$ be a partially ordered set.

We say that $(X, \preceq, d)$ is a partially ordered metric space when there exists a metric d on X.

If there exists a metric d on X such that (X, d) is a complete metric space, then we say that $(X, \preceq, d)$ is a partially ordered complete metric space.

Let $(X, \preceq, d)$ be a partially ordered metric space, and let A and B be nonempty subsets of X.

We use the following notations:

$$\begin{aligned}
&d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},\\
&A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},\\
&B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.
\end{aligned}$$

Note that if $A \cap B \neq \emptyset$, then $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. Also, note that [25] if A and B are closed subsets of a normed linear space with $d(A, B) > 0$ then A_0 and B_0 are contained in boundaries of A and B, respectively.

The pair (A, B) said to have *p-property* [27] if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \text{ imply } d(x_1, x_2) = d(y_1, y_2).$$

Obviously, (A, A) has the p-property when A is nonempty subset of X.

Recently, the authors [32] gave the following concept, which is weaker than the p-property.

The pair (A, B) said to have *weak p-property* if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \text{ imply } d(x_1, x_2) \leq d(y_1, y_2).$$

Let $T : A \to B$ be a map.

The map T is said to be *proximally nondecreasing* if it satisfies the condition:

$$x \preceq y, d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B) \text{ imply } u \preceq v$$

for all $x, y, u, v \in A$.

Note that if $A = B$, then T reduces to nondecreasing map, i.e. $x \preceq y$ implies $Tx \preceq Ty$.

X is called *regular* if, for any sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$,

$$x_n \preceq x \text{ for all } n \in \mathbb{N}.$$

X is called *C-regular* if, for any sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.

Note that if X is regular then it is C-regular.

2 Best proximity points

Let $(X, \preceq, d)$ be a partially ordered metric space, and let A and B be nonempty subsets of X.

A map $T : A \to B$ is called *generalized Berinde type contraction* if there exists $L \geq 0$ such that, for all $x, y \in A$ with $x \preceq y$,

$$\begin{aligned} &d(Tx, Ty) \\ &\leq \frac{d(x, Ty) + d(y, Tx) - 2d(A, B)}{d(x, Tx) + d(y, Ty) + 2d(A, B) + 1} d(x, y) + L\{d(y, Tx) - d(A, B)\} \end{aligned} \tag{2.1}$$

We now present and prove theorem about existence of a best proximity point for generalized Beride type contractions.

Theorem 2.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following conditions are satisfied:*

(1) $T(A_0) \subset B_0$;

(2) the pair (A, B) satisfies the weak p-property;

(3) T is a generalized Berinde type contraction;

(4) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

(5) T is proximally nondecreasing;

(6) T is continuous.

Then, T has a best proximity point. Moreover, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some best proximity point x_*, and the following estimates hold:

there exists a constant $k \in (0, 1)$ such that

(i) for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_n, x_*) \leq \frac{k^n}{1-k} d(x_0, x_1); \tag{2.2}$$

(ii) for $n \in \mathbb{N}$,

$$d(x_n, x_*) \leq \frac{k}{1-k} d(x_{n-1}, x_n). \tag{2.3}$$

Furthermore, if x_* and y_* are two distinct and comparable best proximity points of T, then

$$d(x_*, y_*) \geq \frac{4d(A, B) + 1}{2}(1 - L).$$

Proof. By hypothesis (4), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$. Since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. From (5), we obtain $x_1 \preceq x_2$.

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that, for all $n \in \mathbb{N}$,

$$x_{n-1} \preceq x_n \tag{2.4}$$

and

$$d(x_n, Tx_{n-1}) = d(A, B). \tag{2.5}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0-1} = x_{n_0}$, then $d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0-1}, Tx_{n_0-1})$. Hence, the proof is finished.

Now, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

From (2) we have

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \tag{2.6}$$

for all $n \in \mathbb{N}$. Since T is generalized Berinde type contraction and $x_{n-1} \preceq x_n$

for all $n \in \mathbb{N}$, from (2.1) with $x = x_{n-1}$ and $y = x_n$, and using (2.6) we have

$$\begin{aligned}
&d(x_n, x_{n+1}) \\
\le& d(Tx_{n-1}, Tx_n) \\
\le& \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) - 2d(A,B)}{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + 2d(A,B) + 1} d(x_{n-1}, x_n) \\
&+ L\{d(x_n, Tx_{n-1}) - d(A,B)\} \\
\le& \frac{d(x_{n-1}, Tx_n) - d(A,B)}{d(x_{n-1}, Tx_{n-1}) + d(Tx_{n-1}, x_n) + d(x_n, Tx_n) + d(Tx_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\
\le& \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\
=& \beta_n d(x_{n-1}, x_n)
\end{aligned} \tag{2.7}$$

where $\beta_n = \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+1}$.

Note that $0 \le \beta_{n+1} \le \beta_n < 1$ for all $n \in \mathbb{N}$. Let $k \in (0,1)$ be such that $\beta_1 \le k$.

Then, we have

$$\begin{aligned}
&d(x_n, x_{n+1}) \\
\le& \beta_n d(x_{n-1}, x_n) \\
\le& \beta_n \beta_{n-1} d(x_{n-2}, x_{n-1}) \\
&\cdots\cdots \\
\le& \beta_n \beta_{n-1} \cdots \beta_1 d(x_0, x_1) \\
\le& \beta_1^n d(x_0, x_1) \\
\le& k^n d(x_0, x_1).
\end{aligned} \tag{2.8}$$

For $m > n$, we obtain

$$\begin{aligned}
&d(x_n, x_m) \\
\le& d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
\le& (k^n + k^{n+1} + \cdots k^{m-1}) d(x_0, x_1) \\
\le& \frac{k^n}{1-k} d(x_0, x_1)
\end{aligned}$$

which implies $\{x_n\}$ is a Cauchy sequence in A.

Since A is a closed subset of the complete metric space X, there exists $x_* \in A$ such that

$$\lim_{n\to\infty} x_n = x_*. \tag{2.9}$$

Letting $n \to \infty$ in (2.5), by the continuity of T, it follows that $d(x_*, Tx_*) = d(A,B)$.

From (2.7) we have

$$\begin{aligned} & d(x_n, x_{n+p}) \\ \leq & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ \leq & (\beta_n + \beta_{n+1}\beta_n + \cdots + \beta_{n+p-1}\beta_{n+p-2}\cdots\beta_n) d(x_{n-1}, x_n) \\ \leq & (k + k^2 + \cdots + k^p) d(x_{n-1}, x_n) \\ \leq & \frac{k(1-k^p)}{1-k} d(x_{n-1}, x_n). \end{aligned} \tag{2.10}$$

Letting $p \to \infty$ in (2.10), we have

$$d(x_n, x_*) \leq \frac{k}{1-k} d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$.

From (2.8) and (2.10) we have

$$\begin{aligned} & d(x_n, x_{n+p}) \\ \leq & \frac{k(1-k^p)}{1-k} d(x_{n-1}, x_n) \\ \leq & \frac{k^n(1-k^p)}{1-k} d(x_0, x_1). \end{aligned}$$

Letting $p \to \infty$ in above inequality, we have

$$d(x_n, x_*) \leq \frac{k^n}{1-k} d(x_0, x_1)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Suppose that $d(x_*, Tx_*) = d(A, B) = d(y_*, Ty_*)$ and $x_* \prec y_*$.

Since (A, B) satisfies weak p-property, from (2.1) we have

$$\begin{aligned} & d(x_*, y_*) \\ \leq & d(Tx_*, Ty_*) \\ \leq & \frac{d(x_*, Ty_*) + d(y_*, Tx_*) - 2d(A, B)}{d(x_*, Tx_*) + d(y_*, Ty_*) + 2d(A, B) + 1} d(x_*, y_*) + L\{d(y_*, Tx_*) - d(A, B)\} \\ \leq & \frac{2d(x_*, y_*)}{4d(A, B) + 1} d(x_*, y_*) + L d(x_*, y_*), \end{aligned}$$

which implies

$$d(x_*, y_*) \geq \frac{4d(A, B) + 1}{2}(1 - L).$$

□

Theorem 2.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following conditions are satisfied:*

(1) $T(A_0) \subset B_0$;

(2) the pair (A, B) satisfies the weak p-property;

(3) T is a generalized Berinde type contraction;

(4) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

(5) T is proximally nondecreasing;

(6) A is C-regular.

Then, T has a best proximity point. Moreover, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some best proximity point, and (2.2) and (2.3) hold.

Furthermore, if x_* and y_* are two distinct and comparable best proximity points of T, then

$$d(x_*, y_*) \geq \frac{4d(A, B) + 1}{2}(1 - L).$$

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{x_n\}$, defined by $d(x_n, Tx_{n-1}) = d(A, B)$ for all $n \in \mathbb{N}$, converges to some $x_* \in A$ and $x_{n-1} \preceq x_n$ for all $n \in \mathbb{N}$.

From condition (6), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x_*$ for all k.

Obviously, $d(x_*, Tx_*) \geq d(A, B)$.

Suppose that $d(x_*, Tx_*) > d(A, B)$.

We note that

$$\begin{aligned} &0 < d(x_*, Tx_*) - d(A, B) \\ &= d(x_*, Tx_*) - d(x_{n(k)+1}, Tx_{n(k)}) \\ &\leq d(Tx_{n(k)}, Tx_*) + d(x_*, x_{n(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain

$$0 < d(x_*, Tx_*) - d(A, B) \leq \lim_{k\to\infty} d(Tx_{n(k)}, Tx_*). \tag{2.11}$$

Applying (2.1), for all k, we get that

$$\begin{aligned} &d(Tx_{n(k)}, Tx_*) \\ &\leq \frac{d(x_{n(k)}, Tx_*) + d(x_*, Tx_{n(k)}) - 2d(A, B)}{d(x_{n(k)}, Tx_{n(k)}) + d(x_*, Tx_*) + 2d(A, B) + 1} d(x_{n(k)}, x_*) \\ &+ L\{d(x_*, Tx_{n(k)}) - d(A, B)\}. \end{aligned} \tag{2.12}$$

We obtain

$$\begin{aligned}
&d(x_{n(k)}, Tx_{n(k)}) \\
\leq &d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) \\
= &d(x_{n(k)}, x_{n(k)+1}) + d(A, B) \\
= &d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) \\
\leq &2d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, Tx_{n(k)}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequlity, we have

$$\lim_{k\to\infty} d(x_{n(k)}, Tx_{n(k)}) = d(A, B). \tag{2.13}$$

On the other hand, we get

$$\begin{aligned}
&d(A, B) = d(x_{n(k)+1}, Tx_{n(k)}) \\
\leq &d(x_{n(k)+1}, x_*) + d(x_*, Tx_{n(k)}) \\
\leq &d(x_{n(k)+1}, x_*) + d(x_{n(k)+1}, x_*) + d(x_{n(k)+1}, Tx_{n(k)}) \\
\leq &2d(x_{n(k)+1}, x_*) + d(A, B)
\end{aligned}$$

and so

$$\lim_{k\to\infty} d(x_*, Tx_{n(k)}) = d(A, B). \tag{2.14}$$

Letting $k \to \infty$ in the inequality (2.12), and by using (2.11),(2.13) and (2.14), we have

$$0 < d(x_*, Tx_*) - d(A, B) \leq \frac{d(A, B) + d(A, B) - 2d(A, B)}{d(x_*, Tx_*) + 3d(A, B) + 1} \cdot 0 = 0,$$

which is a contradiction. Thus, $d(x_*, Tx_*) = d(A, B)$. □

By Theorem 2.1 and Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following conditions are satisfied:*

(1) $T(A_0) \subset B_0$;

(2) the pair (A, B) satsfies the weak p-property;

(3) for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx) - 2d(A, B)}{d(x, Tx) + d(y, Ty) + 2d(A, B) + 1} d(x, y);$$

(4) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

(5) T is proximally nondecreasing;

(6) either T is continuous or A is C-regular.

Then, T has a best proximity point. Moreover, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some best proximity point, and (2.2) and (2.3) hold.

Furthermore, if x_* and y_* are two distinct and comparable best proximity points of T, then

$$d(x_*, y_*) \geq \frac{4d(A, B) + 1}{2}.$$

If we take $A = B = X$ in Theorem 2.1 and Theorem 2.2, then we have the following fixed point result.

Corollary 2.4. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) there exists a constant $L \geq 0$ such that, for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y) + Ld(y, Tx);$$

(2) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) T is nondecreasing;

(4) either T is continuous or X is C-regular.

Then, T has a fixed point. Moreover, the Picard iteration $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point, and (2.2) and (2.3) hold.

Furthermore, if x_* and y_* are two distinct and comparable fixed points of T, then

$$d(x_*, y_*) \geq \frac{1}{2}(1 - L).$$

Corollary 2.5. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y);$$

(2) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) T is nondecreasing;

(4) either T is continuous or X is C-regular.

Then, T has a fixed point. Moreover, the Picard iteration $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point, and (2.2) and (2.3) hold.

Furthermore, if x_* and y_* are two distinct and comparable fixed points of T, then

$$d(x_*, y_*) \geq \frac{1}{2}.$$

Remark 2.1. *Corollary 2.5 is a generalization of Theorem 1 of [16] to the case of partially ordered metric spaces.*

Theorem 2.6. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following conditions are satisfied:*

(1) $T(A_0) \subset B_0$;

(2) the pair (A, B) satisfies the weak p-property;

(3) there exist $k \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y) + L\{d(y, Tx) - d(A, B)\} \tag{2.15}$$

for all $x, y \in A$ with $x \preceq y$;

(4) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

(5) T is proximally nondecreasing;

(6) either T is continuous or X is C-regular.

Then, T has a best proximity point. Moreover, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some best proximity point x_*, and (2.2) and (2.3) hold.

Proof. As in the proof of Theorem 2.1, we can find a sequence $\{x_n\}$ in A_0 such that, for all $n \in \mathbb{N}$,

$$x_{n-1} \prec x_n,\ d(x_n, Tx_{n-1}) = d(A, B) \text{ and } d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

From (2.15) we have

$$\begin{aligned} & d(x_n, x_{n+1}) \\ \leq & d(Tx_{n-1}, Tx_n) \\ \leq & kd(x_{n-1}, x_n) + L(d(x_n, Tx_{n-1}) - d(A, B)) \\ = & kd(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$.

Hence, we deduce

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \cdots \leq k^n d(x_0, x_1).$$

As in the proof of Theorem 2.1, there exists

$$x_* = \lim_{n\to\infty} x_n \in A.$$

If T is continuous, then

$$d(A, B) = \lim_{n\to\infty} d(x_n, Tx_{n-1}) = d(x_*, Tx_*).$$

The rest of proof is similar as proof of Theorem 2.1 and Theorem 2.2. $\square$

If we take $A = B = X$ in Theorem 2.6, then we have the following fixed point result.

Corollary 2.7. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) there exist $k \in (0, 1)$ and $L \geq 0$ such that, for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq kd(x, y) + Ld(y, Tx);$$

(2) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) T is nondecreasing;

(4) either T is continuous or X is C-regular.

Then, T has a fixed point. Moreover, the Picard iteration $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point, and (2.2) and (2.3) hold.

Remark 2.2. *Corollary 2.7 is a generalization of Theorem 1 of [5] to the case of partially ordered metric spaces.*

Corollary 2.8. *[20] Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) there exists $k \in (0, 1)$ such that, for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq kd(x, y);$$

(2) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) T is nondecreasing;

(4) T is continuous.

Then, T has a fixed point. Moreover, the Picard iteration $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point, and (2.2) and (2.3) hold.

For the uniqueness of a fixed point of self map defined on partially ordered metric spaces, the authors [20, 22] considered the following hypothesis.

(B) every pair $x, y \in X$ has a lower and upper bound.

Adding condition (B) to the hypothesis of Corollary 2.8, Corollary 2.8 reduce to Theorem 1 of [22].

Corollary 2.9. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) there exists $k \in (0, 1)$ such that, for all $x, y \in X$ with $x \preceq y$,

$$d(Tx, Ty) \leq kd(x, y);$$

(2) there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) T is nondecreasing;

(4) X is C-regular.

Then, T has a fixed point. Moreover, the Picard iteration $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, converges to some fixed point, and (2.2) and (2.3) hold.

Adding condition (B) to the hypothesis of Corollary 2.9 and replacing C-regularity with regularity, Corollary 2.9 reduce to Theorem 2.3 of [20].

We now give an example to illustrate Theorem 2.1.

Example 2.1. *Let $X = \mathbb{R}^2$ with Euclidean metric and the product order $\preceq$ on $\mathbb{R}^2$, i.e. $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \leq d$. Let $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(x, 1) : 0 \leq x \leq 1\}$, and let $L = 1$.*

Then, $(X, \preceq, d)$ is a partially ordered complete metric space, A and B are closed subsets of X, $A_0 = A, B_0 = B$ and $d(A, B) = 1$. It is easy to see that (A, B) satisfies the weak p-property.

Let $T : A \to B$ be a map defined by $T((x, 0)) = (x, 1)$.

Then, $T(A_0) \subset B_0$.

Let $x_0 = x_1 = (0, 0)$ Then $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d((0, 0), T((0, 0))) = d((0, 0), (0, 1)) = 1 = d(A, B)$, and so the condition (4) of Theorem 2.1 is satisfied.

Let $(x, 0) \preceq (y, 0)$.

Then, $x \leq y$. If $d(A, B) = 1 = d((u, 0), T((x, 0)))$ and $d(A, B) = 1 = d((v, 0), T((y, 0)))$, then $\sqrt{(u - x)^2 + 1} = 1$ and $\sqrt{(v - y)^2 + 1} = 1$. Hence $u = x$ and $v = y$. Thus, $u \leq v$ and so $(u, 0) \preceq (v, 0)$. Hence, T is proximally nondecreasing.

We now show that (2.1) holds.

Let $(x, 0) \preceq (y, 0)$.

If $x = y$, then (2.1) obviously is satisfied.

Assume that $x < y$.

Then, we deduce

$$
\begin{aligned}
&d(T((x, 0)), T((y, 0)))\{d((x, 0), T((x, 0))) + d((y, 0), T((y, 0))) + 2d(A, B) + 1\} \\
=&d((x, 1), (y, 1))\{d((x, 0), (x, 1)) + d((y, 0), (y, 1)) + 2d(A, B) + 1\} \\
=&|x - y|\{3 + 2d(A, B)\} \\
=&|x - y|\{2 - 2d(A, B)\} + |x - y|\{4d(A, B) + 1\} \\
\leq&d((x, 0), (y, 0))\{d((x, 0), (y, 1)) + d((y, 0), (x, 1)) - 2d(A, B)\} + 5d((y, 0), (x, 1)) \\
=&d((x, 0), (y, 0))\{d((x, 0), T((y, 0))) + d((y, 0), T((x, 0))) - 2d(A, B)\} \\
+&5d((y, 0), T((x, 0))).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&d(T((x, 0)), T((y, 0))) \\
\leq&\frac{d((x, 0), T((y, 0))) + d((y, 0), T((x, 0))) - 2d(A, B)}{d((x, 0), T((x, 0))) + d((y, 0), T((y, 0))) + 2d(A, B) + 1} + Ld((y, 0), T((x, 0)))
\end{aligned}
$$

and hence, T is a generalized Berinde type contraction.

Thus, all condition of Theorem 2.1 are satisfied. By Theorem 2.1, there exists $(x_*, 0) \in A$ such that $d((x_*, 0), T((x_*, 0))) = 1 = d(A, B)$. More precisely, the point $(0, 0) \in A$ is the best proximity point of T.

Note that condition (2.15) of Theorem 2.6 is not safisfied.

In fact, if there exists a constant $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) + L(d(y,Tx) - d(A,B))$$

for all $x, y \in A$ with $x \preceq y$, where $L \geq 0$, then we have, for $x = (0,0), y = (1,0)$ and $L = 0$,

$$\begin{aligned} 1 =& d(Tx,Ty) \\ \leq& kd(x,y) + L(d(y,Tx) - d(A,B)) \\ =& kd((0,0),(1,0)) \\ =& k. \end{aligned}$$

However, it is not possible. Thus, condition (2.15) of Theorem 2.6 is not satisfied.

References

[1] A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Analysis:Theory, Methods and Applications*, **72** (2010), 2238-2242. http://dx.doi.org/10.1016/j.na.2009.10.023

[2] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87** (2008), 109-116. http://dx.doi.org/10.1080/00036810701556151

[3] H. Aydi, E. Karapinar, I. Erhan, P. Salim, Best proximity points of generalized almost ψ-Geraghty contractive non-self-mappings, *Fixed Point Theory and Applications*, **2014** (2014), 32. http://dx.doi.org/10.1186/1687-1812-2014-32

[4] C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir-Keeler contractions, *Nonlinear Analysis: Theory, Methods and Applications*, **69** (2008), 3790-3794. http://dx.doi.org/10.1016/j.na.2007.10.014

[5] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Analysis Forum*, **9** (2004), 43-53.

[6] K.C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Newyork, 1985. http://dx.doi.org/10.1017/cbo9780511625756

[7] J. Caballero, J. Harjani and K. Sadarangani, A best proximity point theorem for Geraghty-contractions, *Fixed Point Theory and Applications*, **2012** (2012), 231. http://dx.doi.org/10.1186/1687-1812-2012-231

[8] S.H. Cho, J.S. Bae, Common fixed point theorems for mappings satisfying property ($E.A$) on cone metric spaces, *Mathematical and Computer Modelling*, **53** (2011), 945-951. http://dx.doi.org/10.1016/j.mcm.2010.11.002

[9] B.S. Choudhury, Unique fixed point theorem for weakly C-contractive mappings, *Kathmandu Univ. J. Sci. Eng. Technol.*, **5** (2009), 6-13. http://dx.doi.org/10.3126/kuset.v5i1.2842

[10] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323** (2006), 1001-1006. http://dx.doi.org/10.1016/j.jmaa.2005.10.081

[11] M. Geraghty, On contractive mappings, *Proc. Amer. Math. Soc.*, **40** (1973), 604-608. http://dx.doi.org/10.1090/s0002-9939-1973-0334176-5

[12] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), no. 2, 1468-1476. http://dx.doi.org/10.1016/j.jmaa.2005.03.087

[13] M.A. Khamsi, V.Y. Kreinovich, Fixed point theorems for dissipative mappings in complete probabilistic metric spaces, *Math. Jap.*, **44** (1996), 513-520.

[14] S. Karpagam, S. Agrawal, Best proximity point theorems for p-cyclic Meir-Keeler contractions, *Fixed Point Theory and Applications*, **2009** (2009), 1-10, Article ID 197308. http://dx.doi.org/10.1155/2009/197308

[15] E. Karapinar, On best proximity point of ψ-Geraghty contractions, *Fixed Point Theory and Applications* **2013** (2013), 200. http://dx.doi.org/10.1186/1687-1812-2013-200

[16] F. Khojasteh, M. Abbas, S. Costache, Two new types of fixed point theorems in complete metric spaces, *Abstract and Applied Analysis*, **2014** (2014), 1-5. Article ID, 325840. http://dx.doi.org/10.1155/2014/325840

[17] E, Karapinar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler contractive mappings, *Fixed Point Theory and Applications*, **2013** (2013), 94. http://dx.doi.org/10.1186/1687-1812-2013-94

[18] C. Mongkolkeha, Y.J. Cho, P. Kumam, Best proximity points for generalized proximal C-contraction mappings in metric spaces with partial orders, *Journal of Inequalities and Applications*, **2013** (2013), 94. http://dx.doi.org/10.1186/1029-242x-2013-94

[19] C. Mongkolkeha, P. Kumam, Best proximity point theorems for generalized cyclic contractions in ordered metric spaces, *J. Optim. Theory Appl.*, **155** (2012), 215-226. http://dx.doi.org/10.1007/s10957-012-9991-y

[20] J.J. Netro, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22** (2005), 223-239. http://dx.doi.org/10.1007/s11083-005-9018-5

[21] Efe A. Ok, *Real analysis with Economics Applications*, Princeton University Press, 2007.

[22] A.C.M. Rang, M.C. Reuring, A fixed point theorem in partially ordered sets and some applications to matric equations, *Proc. Amer. Math. Sco.*, **132** (2004), 1435-1443.

[23] S. Reich, Approximate selections, best approximations, fixed points and invariant sets, *J. Math. Anal. Appl.*, **62** (1978), 104-113. http://dx.doi.org/10.1016/0022-247x(78)90222-6

[24] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Analysis: Theory, Methods and Applications*, **75** (2012), 2154-2165. http://dx.doi.org/10.1016/j.na.2011.10.014

[25] S. Sadiq Basha, P. Veeramani, Best proximity pait theorems for multifunctions with open fibres, *J. A pprox. Theory*, **103** (2000), 119-129. http://dx.doi.org/10.1006/jath.1999.3415

[26] S. Sadiq Basha, Best proximity point theorems on partially ordered sets, *Optim. Lett.*, **7** (2012), 1035-1043. http://dx.doi.org/10.1007/s11590-012-0489-1

[27] V. Sankar Raj, *Banachs contraction principle for non-self mappings*, Preprint.

[28] W. Sintunavarat, Y.J. Cho, P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, *Comput. Math. Appl.*, **62** (2011), 1969-1978. http://dx.doi.org/10.1016/j.camwa.2011.06.040

[29] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 93, Article ID 93. http://dx.doi.org/10.1186/1687-1812-2012-93

[30] M.A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Analysis: Theory, Methods and Applications*, **70** (2009), 3665-3671. http://dx.doi.org/10.1016/j.na.2008.07.022

[31] S.K. Yang, J.S. Bae, S.H. Cho, Coincidence and common fixed and periodic point theorems in cone metric spaces, *Computers and Mathematics with Applications*, **61** (2011), 170-177. http://dx.doi.org/10.1016/j.camwa.2010.10.031

[32] J. Zhang, Y. Su, Q. Cheng, A note on "A best proximity point theorem for Geraghty-contractions", *Fixed Ponit Theory and Applications*, **2013** (2013), 99. http://dx.doi.org/10.1186/1687-1812-2013-99

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