Some New Applications of Modified q-Integral Operators

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Abstract

In this paper we introducing a new sequence of positive linear q-Baskakov Durrmeyer type operators. Korovkin-type theorems for fuzzy continuous functions, an estimate for the rate of convergence and some properties are also obtained for these operators.

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1 Introduction

In the approximation theory the Durrmeyer type integral modifications and q analogues of different Durrmeyer type integral operators are important research area in present time. After the development of quantum calculus A. Lupas gave a new generalized q-Bernstein polynomial. Quantum calculus is
also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solutions of differential equations. Positive linear operators based on q-numbers are quite effective and we could have some different results. In 1997, Phillips introduced another generalization of Bernstein operators based on the q-integers called q-Bernstein operators. In the last decade some new generalizations of well known positive linear operators, based on q-integers were introduced and studied by several authors. For instance q-Meyer-Konig and Zeller operators studied by Trif. [13] and Gupta [2] etc. In 2001, Aral and Gupta [1],[12] introduced a q-generalization of the classical Baskakov operators. In 2012, Honey Sharma [4],[5] introduced the q-Durrmeyer type operators.

Very recently we published a paper based on q-Baskakov-Durrmeyer type operators [11]. In the present paper motivated by H. Sharma we investigate some new applications of q-analogue of the Baskakov-Durrmeyer type and we study better rate of convergence.

First we mention some important definitions of q-Calculus.

**Definition 1.1** For any fixed real number $q > 0$ and $k \in \mathbb{N}$, the q-integers is defined by

$$[k]_q = \begin{cases} k, & \text{if } q = 1, \\ 1 + q + q^2 + \ldots + q^{k-1}, & \text{if } q \neq 1. \end{cases}$$

In this way for a real number $n$ we may write $[n]_q = \frac{1-q^n}{1-q}; q \neq 1$.

**Definition 1.2** The q-factorial is defined by

$$[k]_q! = \begin{cases} 1, & \text{if } k = 0, \\ [1]_q \cdot [2]_q \cdot \ldots \cdot [k]_q, & \text{if } k = 1, 2, \ldots. \end{cases}$$

**Definition 1.3** For any number $k \in (0, n)$, the q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$ 

**Definition 1.4** The q-derivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

$$D_q^0 f := f; D_q^n f := D_q(D_q^{n-1} f), n = 1, 2, 3, \ldots.$$
2 Construction of Operators

N. Deo et al. [8] introduced new version of Bernstein-Durrmeyer-type operators defined as: for \( f \in CI_n \) here \( I_n = [0, \frac{n}{n+1}] \)

\[
(M_nf)(x) = n \left(1 + \frac{1}{n}\right) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{n}{p_{n,k}(t)f(t)dt} \tag{1}
\]

where,

\[ p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^{n} \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}, \]

and established some approximation results on it.

H. Sharma [4] introduced the following q-Durrmeyer type operators defined as: for \( f \in CI_{n,q} \) here \( I_{n,q} = [0, \frac{[n]_q}{[n+1]_q}] \)

\[
(M_n^*f)(x) = \frac{[n+1]_q^2}{[n]_q} \sum_{k=0}^{n} q^{-k} p_{n,k}^*(q;x) \int_{0}^{[n]_q}{p_{n,k}^*(q;qt)f(t)d_qt} \tag{2}
\]

where,

\[ p_{n,k}^*(q;x) = \binom{n}{k}_q \left(\frac{[n+1]_q}{[n]_q} x\right)^k \left(1 - \frac{[n+1]_q}{[n]_q} x\right)^{n-k}_q, \]

and established some approximation results on it.

Very recently we motivated by H. Sharma [4], and N. Deo [7], [8], [9] and introduced a q-analogue of the Baskakov-Durrmeyer type operators [11] defined as: for \( f \in CI_{n,q} \)

\[
(M_{n,q}f)(x) = \frac{[n+1]_q^2}{[n]_q} \sum_{k=0}^{\infty} b_{n,k}(q;x) \int_{0}^{[n]_q}{p_{n,k}(q;qt)f(t)d_qt} \tag{3}
\]

where,

\[ b_{n,k}(q;x) = q^{k^2-k-2} \binom{n+k-1}{k}_q x^k (1+x)^{-(n+k)}_q, x \in CI_{n,q}. \]

In this paper again we modified above equations for \( c > 0 \) so, we get

\[
(M_{n,q,c}f)(x) = \frac{[n+1]_q^2}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}(q;x) \int_{0}^{[n]_q}{p_{n,k}(q;qt)f(t)d_qt} \tag{4}
\]

where,

\[ b_{n,k,c}(q;x) = q^{k^2-k-2} \binom{k+n-c-1}{k}_q (cx)^k (1+cx)^{-(n+k)} \]
H. S. Kasana et. el. [3] obtained a sequence of modified Szász operators for integrable function on \([0, \infty)\) defined as:

\[
(M_{n,x}(f))(x) \equiv M_{n,x}(f(y); t) = n \sum_{k=0}^{\infty} b_{n,k}(t) \int_{0}^{\infty} b_{n,k}(y) f(x + y) dy
\]

(5)

where, \(x\) and \(t\) belong to \([0, \infty)\) and \(x\) is fixed.

In this paper motivated by H. S. Kasana and H. Sharma, we propose a q-Baskakov-Durrmeyer type operators defined as: for \(f \in CI_{n,q}\) and \(c > 0\):

\[
(M_{n,q,c}^*(f))(x) = \frac{[n + 1]_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}(q; x) \int_{0}^{[n]_q} p_{n,k}(q; ty) f(x + ty) d_q y
\]

(6)

The aim of this paper we investigate some new applications of q-analogue of the Baskakov-Durrmeyer type operators. Finally, we give Korovkin-type theorems for fuzzy continuous functions and better error estimations for operators (4) and (6).

3 Calculation of moments

We use the lemma-1 [4] for \(s = 1, 2, \ldots\) and by the definition of q-Beta function, we get

\[
\int_{0}^{[n]_q} p_{n,k}^{*}(q; t) t^s d_q t = \frac{[n+1]_q}{[n+1]_q} q^k \frac{[n]_q!}{[k]_q! q^k \int_{0}^{[n]_q} t^{s+n+1} q^s d_q t}
\]

Theorem 3.1 Let the sequence of positive linear operators \((M_{n,q,c}(f))(x)\) defined by (4). For all \(n \in N; q \in (0, 1), c > 0; f \in CI_{n,q}; x \in I_{n,q}\), we get

\[
(M_{n,q,c}1)(x) = 1
\]

(7)

\[
(M_{n,q,c}t)(x) = \frac{[n]_q}{[n+2]_q[n+1]_q} ([n]_q x + 1)
\]

(8)

\[
(M_{n,q,c}t^2)(x) = \frac{(1 + q)[n]_q^2 + q(1 + q)^2 x [n]_q^3 + q^2 c x^2 [n]_q^4 + q^3 c x^2 [n]_q^4}{[n+3]_q [n+2]_q [n+1]_q^2}
\]

(9)

Proof: For the proof of theorem we set \(f(t) = 1\) in the operators \(M_{n,q,c}\), we get

\[
(M_{n,q,c}1)(x) = \frac{[n+1]_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}^{*}(q; x) \int_{0}^{[n]_q} p_{n,k}(q; ty) d_q t
\]

\[
= \frac{[n+1]_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}^{*}(q; x) \frac{[n]_q}{[n+1]_q} q^k \frac{[n]_q!}{[n+1]_q} q^{&+1} d_q t
\]

\[
= \sum_{k=0}^{\infty} \frac{[n]_q}{[n+1]_q} q^{&+1} \frac{[n]_q!}{[n+1]_q} \left( cx \right)^k (1 + cx)^{&+1} = 1.
\]
Again we set \( f(t) = t \) in the operators \( M_{n,q,c} \), we get

\[
(M_{n,q,c}t)(x) = \frac{[n+1]^2_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}(q;x) \frac{[n]_q^2}{[n+1]^2_q} [k]_q! [k+1]_q! [n+2]_q! \\
= \frac{[n]_q}{[n+2]_q [n+1]_q} \sum_{k=0}^{\infty} b_{n,k,c}(q;x) [k+1]_q \\
= \frac{[n]_q}{[n+2]_q [n+1]_q} ([n]_q x + 1).
\]

Similarly, we set \( f(t) = t^2 \) in the operators \( M_{n,q,c} \), we get

\[
(M_{n,q,c}t^2)(x) = \frac{[n+1]^2_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k,c}(q;x) \frac{[n]_q^3}{[n+1]^3_q} q^k [k]_q! [k+2]_q! [n+3]_q! \\
= \frac{[n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2} \sum_{k=0}^{\infty} b_{n,k,c}(q;x) [k+1]_q [k+2]_q \\
= \frac{[n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2} 1 + q + q(1+q)^2 [n]_q x + q^4 \left( \frac{[n+1]_q x^2}{q} + [n]_q x \right) - [n]_q x \\
= \frac{(1+q)[n]_q^2 + q(1+q)^2 x [n]_q^3 + q^3 c x^2 [n]_q^3 + q^3 x^2 [n]_q^4}{[n+3]_q [n+2]_q [n+1]_q^2}.
\]

This completes the proof of the theorem.

**Lemma 3.2** For the special case \( q = 1 \) we have

\[
(M_{n,1,c}1)(x) = 1; \\
(M_{n,1,c}t)(x) = \frac{n^2 x + n}{(n+2)(n+1)}; \\
(M_{n,1,c}t^2)(x) = \frac{n^2 [n^2 x^2 + nx(1+cx) + 3nx + 2]}{(n+3)(n+2)(n+1)^2}.
\]

**Lemma 3.3** The sequence of positive linear operators \( M_{n,q,c} \), we get following central moments: let \( \phi^i = (t-x)^i, i = 1, 2, \ldots \)

\[
(M_{n,q,c}\phi^1)(x) = (M_{n,q,c}t)(x) - x(M_{n,q,c}1)(x) \\
= \frac{[n]_q}{[n+2]_q [n+1]_q} ([n]_q x + 1) - x \cdot 1 = \frac{(1-3x)[n]_q - 2x}{[n+2]_q [n+1]_q}; \\
(M_{n,q,c}\phi^2)(x) = (M_{n,q,c}t^2)(x) - 2x(M_{n,q,c}t)(x) + x^2(M_{n,q,c}1)(x)
\]
\[ \begin{align*}
&= \frac{(1 + q)[n]^2 + q(1 + q)^2x[n]_q^3 + q^3cx^2[n]_q^3 + q^3x^2[n]_q^4}{[n + 3]_q[n + 2]_q[n + 1]_q^2} \\
&\quad - 2x\frac{[n]_q([n]_qx + 1)}{[n + 2]_q[n + 1]_q} + x^2 \cdot 1 \\
&= x^2 \left(1 - \frac{2[n]_q^2}{[n + 2]_q[n + 1]_q} + \frac{q^3(c - 1)[n]_q^3}{[n + 3]_q[n + 2]_q[n + 1]_q} \right) \\
&\quad + x \left(\frac{q(1 + q)^2[n]_q^3}{[n + 3]_q[n + 2]_q[n + 1]_q^2} - \frac{2[n]_q}{[n + 2]_q[n + 1]_q} \right) \\
&\quad + \frac{(1 + q)[n]_q^2}{[n + 3]_q[n + 2]_q[n + 1]_q^2}.
\end{align*} \]

**Lemma 3.4** For the special case \( q = 1 \) we have the following central moment

\[
(M_{n,q,c}\phi^1)(x) = \frac{n(1 - 3x) - 2x}{(n + 2)(n + 1)}
\]

\[
(M_{n,q,c}\phi^2)(x) = \frac{n^3[(c - 1)x + 2x] + n^2[11x^2 - 8x + 2] + n[17x^2 - 8x + 2] + 6x^2}{(n + 3)(n + 2)(n + 1)^2}.
\]

### 4 Korovkin-type theorems for fuzzy continuous functions

In this section we mention some important definitions given by M. Burgin [6].

**Definition 4.1** A number \( a \) is called an \( r \)-limit of a sequence \( S \) (it is denoted by \( a = r - \lim S \)) if for any \( \epsilon \in \mathbb{R} \), the inequality \( |a - a_i| < r + \epsilon \) is valid for almost all \( a_i \), i.e. there is such \( n \) that for any \( i > n \), we have \( |a - a_i| < r + \epsilon \).

**Definition 4.2** A sequence \( S \) that has an \( r \)-limit is called \( r \)-convergent and it is said that \( S \), \( r \)-converges to its \( r \)-limit \( a \). It is denoted by \( S \to ra \).

**Definition 4.3** A function \( f : \mathbb{R} \to \mathbb{R} \) is called \( r \)-continuous in \( X \subset \mathbb{R} \) if \( \gamma(f,X) \leq r \) and is called fuzzy continuous in \( X \) if \( \gamma(f,X) \leq \infty \) where \( \gamma(f,X) \) defined as,

\[
\gamma(f,X) \geq \inf\{\sup\{|f(x) - g(x)| : x \in X\} : g(x) \in C(X)\}.
\]
For example the functions \( f(x) = x^n \) when \( x \in [n, n+1), n \in \mathbb{Z} \) and \( g(x) = [x]^n \) are fuzzy continuous in each finite interval of the real line \( \mathbb{R} \), but they are not continuous in any interval with the length larger than 1. To define the Riemann integral for a continuous function \( f(x) \), step functions are utilized. If the integral of \( f(x) \) exists, then any such step function is fuzzy continuous.

**Theorem 4.4** Let a sequence \((q_n)_n; q_n \in (0, 1)\) such that \( r - \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( M_{n, q_n, c}; n \in N \) be defined by (4). If \( r_i - \lim_{n \to \infty} \left| (M_{n, q_n, c} e_i)(x) - e_i \right| = 0 \) for \( i = 0, 1, 2 \). Then for all functions \( f \in C(I_n) \), we get
\[
  r - \lim_{n \to \infty} \left| (M_{n, q_n, c} f)(x) - f \right| = 0
\]
where, \( r \) is any real number such that \( r \geq K_3(r_0 + r_1 + r_2) \) for some \( K_3 > 0 \).

**Proof:** Let the functions \( e_i \) defined as; \( e_i(x) = t^i \) for all \( x \in I_n \). Now, for each \( \epsilon > 0 \), there corresponds \( \delta > 0 \) such that \( \|\lambda(t - x)\| \leq \epsilon \) whenever \( |t - x| \leq \delta \). Again for \( |t - x| > \delta \), there exist a positive number \( M \) such that \( \|\lambda(t - x)\| \leq M \leq M \frac{(t-x)^2}{\delta^2} \). Thus for all \( t \) and \( x \in I_n \), we get
\[
  |\lambda(t - x)| \leq \epsilon + M \frac{(t-x)^2}{\delta^2}.
\]
Applying \( M_{n, q_n, c} \) on (10), we get
\[
  |(M_{n, q_n, c} f)(x) - f(x)| \leq \epsilon(M_{n, q_n, c} e_0)(x) + M \frac{M}{\delta^2} (M_{n, q_n, c} (t-x)^2)(x)
\]
\[
  |(M_{n, q_n, c} f)(x) - f(x)| \leq \epsilon e + \epsilon |(M_{n, q_n, c} e_0)(x) - e_0(x)| + K_3 \sum_{i=0}^{2} |(M_{n, q_n, c} e_i)(x) - e_i(x)|
\]
where, \( K_3 = \max\{\frac{M}{\delta^2}, \frac{2Mx}{\delta^2}, \frac{Mx^2}{\delta^2}\} \). Then for every \( \epsilon > 0 \) there exist \( N = N(\epsilon) > 0 \) such that for all \( n \in N \), we get
\[
  |(M_{n, q_n, c} f)(x) - f(x)| \leq \epsilon + \epsilon (r_0 + \epsilon) + K_3 (3\epsilon + r_0 + r_1 + r_2) \leq r + \epsilon \_1
\]
here, \( \epsilon_1 = \epsilon(1 + r_0 + \epsilon + 3K_3) \). Since \( \epsilon \) is arbitrary and small, \( r - \lim_{n \to \infty} q_n = 1 \), we get
\[
  r - \lim_{n \to \infty} |(M_{n, q_n, c} f)(x) - f| = 0.
\]
This completes the proof.

**Theorem 4.5** Let a sequence \((q_n)_n; q_n \in (0, 1)\) such that \( r - \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( M_{n, q_n, c}^*; n \in N \) be defined by (6). If \( r_i - \lim_{n \to \infty} \left| (M_{n, q_n, c}^* e_i)(x) - e_i \right| = 0 \) for \( i = 0, 1, 2 \). Then for all functions \( f \in C(I_n) \), we get
\[
  r - \lim_{n \to \infty} |(M_{n, q_n, c}^* f)(x) - f| = 0
\]
where, \( r \) is any real number such that \( r \geq K_4(r_0 + r_1 + r_2) \) for some \( K_4 > 0 \).
The proof of the theorem is analogous as theorem 4.4.

**Theorem 4.6** Let $f$ be the integrable and bounded in the interval $I_n$ and let if $f''$ exists at a point $x \in I_n$. Let a sequence $(q_n)n; q_n \in (0,1)$ such that \lim_{n \to \infty} q_n = 1$ and let the sequence of positive linear operators $M_{n,q_n,c}; n \in N$ be defined by (4). Then, one gets that

$$\lim_{n \to \infty} [n]_{q_n} |(M_{n,q_n,c}f)(x) - f(x)| = (1 - 3x)f'(x) + \frac{(c - 1)x^2 + 2x}{2} f''(x)$$

Proof: Let if $f''$ exists at a point $x \in I_n$, then by using Taylors expansion, we write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \lambda(t-x) \quad (11)$$

where, $\lambda(t-x) \to 0$ as $t \to x$. Applying $M_{n,q_n,c}$, we get

$$(M_{n,q_n,c}f)(x) = f(x)(M_{n,q_n,c}1)(x) + f'(x)(M_{n,q_n,c}(t-x))(x) + \frac{f''(x)}{2} (M_{n,q_n,c}(t-x)^2)(x) + (M_{n,q_n,c}(t-x)^2 \lambda(t-x))(x).$$

By using theorem 1 and Multiplying $[n]_{q_n}$ both sides, we get

$$[n]_{q_n}[(M_{n,q_n,c}f) - f] = f'(x)[n]_{q_n} \left( \frac{(1-3x)[n]_{q_n} - 2x}{[n+2]_{q_n}[n+1]_{q_n}} \right) \ldots$$

$$\ldots + \frac{f''(x)[n]_{q_n}}{2} M_{n,q_n,c} \phi^2(x) + [n]_{q_n} R_{[n]_{q_n}}(t,x). \quad (12)$$

Here we write, $[n]_{q_n} R_{[n]_{q_n}}(t,x) = \left[ n+1 \right]_{q_n}^2 \sum_{k=0}^{\infty} b_{n,k,c}(q;x) \int_0^{[n]_{q_n}} p_{n,k}(q;\delta) \phi^2 \delta d\delta dq t$

$$\left| [n]_{q_n} R_{[n]_{q_n}}(t,x) \right| \leq \left[ n+1 \right]_{q_n}^2 \sum_{k=0}^{\infty} b_{n,k,c}(q;x) \int_0^{[n]_{q_n}} p_{n,k}(q;\delta) \phi^2 \delta d\delta dq t$$

$$\leq [n]_{q_n} \epsilon (M_{n,q_n,c}(t-x)^2)(x) + \frac{[n]_{q_n} M \delta^2}{\delta^2} (M_{n,q_n,c}(t-x)^4)(x)$$

$$\leq [n]_{q_n} \epsilon o \left( \frac{1}{[n]_{q_n}} \right) + \frac{[n]_{q_n} M \delta^2}{\delta^2} o \left( \frac{1}{[n]_{q_n}^2} \right)$$

$$\leq \epsilon + \frac{M}{\left( [n]_{q_n} \right)^{\frac{1}{4}}} o \left( \frac{1}{[n]_{q_n}} \right) \leq \epsilon + Mo \left( \frac{1}{\sqrt{[n]_{q_n}}} \right); \text{ for } \delta = ([n]_{q_n})^{\frac{1}{4}}.$$

Since $\epsilon$ is arbitrary and small, \lim_{n \to \infty} q_n = 1$ and whenever $n \to \infty$, we get

$$\left| [n]_{q_n} R_{[n]_{q_n}}(t,x) \right| \to 0. \quad (13)$$
By using (12) in equation (13), we get
\[
\lim_{n \to \infty} [n]_{q_n} |(M_{n,q_n,c}f)(x) - f(x)| = (1 - 3x)f'(x) + \frac{(c-1)x^2 + 2x}{2} f''(x)
\]
This completes the proof.

**Theorem 4.7** Let \( f \) be the integrable and bounded in the interval \( I_n \) and let if \( f'' \) exists at a point \( x; t \in I_n \). Let a sequence \((q_n)_n; q_n \in (0,1)\) such that \( \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( M_{n,q_n,c}; n \in N \) be defined by (6). Then, one gets that
\[
\lim_{n \to \infty} [n]_{q_n} |(M_{n,q_n,c}^*f)(t) - f(t)| = (1 - 3t)f'(x + t) + \frac{(c-1)t^2 + 2t}{2} f''(x + t)
\]
The proof of the theorem is analogous as theorem 4.6.

## 5 Conclusion

We conclude that modified operators (4) and (6) improve the approximation process when the value of \( n \) is very large i.e. when \( n \) tends to infinity.

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## References


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