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Remarks on Global Solutions to the Cauchy Problem for Semirelativistic Equations with Power Type Nonlinearity

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Abstract

Existence and nonexistence results on global solutions to the Cauchy problem for semirelativistic equations are shown by a simple compactness argument and a test function method, respectively. To obtain the nonexistence of global solutions, semirelativistic equations are transformed into a new equation without nonlocal operators in linear part but with a time derivative in nonlinear part, which is shown to be under control of special choice of test functions.

Mathematics Subject Classification: 35Q40

Keywords: Semirelativistic equation, compactness argument, test function method, nonexistence of weak solutions

1 Introduction

We consider the Cauchy problems for the semirelativistic equations

$$\begin{cases} i\partial_t u \pm (m^2 - \Delta)^{1/2} u = \mu |u|^{p-1} u, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{1}$$

and

$$\begin{cases} i\partial_t u \pm (-\Delta)^{1/2} u = \lambda |u|^{p}, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{2}$$

with $m, \mu \in \mathbb{R}$ and $\lambda \in \mathbb{C} \backslash \{0\}$, where $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}$. Here $(m^2 - \Delta)^{1/2}$ is realized as a Fourier multiplier with symbol $(m^2 + \xi^2)^{1/2}$: $(m^2 - \Delta)^{1/2} = \mathfrak{F}^{-1}(m^2 + \xi^2)^{1/2}\mathfrak{F}$, where $\mathfrak{F}$ is the Fourier transform defined by

$$(\mathfrak{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-1/2} \int \exp(-ix\xi) u(x) dx.$$

We remark that the Cauchy problem such as (1) or (2) arises in various physical settings and accordingly, especially in the massless case, semirelativistic equations are also called half-wave equations, fractional Schrödinger equations, and so on, see [13, 19] and reference therein. Moreover, the semirelativistic equation with Hartree type nonlinearity is used as a model of Boson star. For related subjects, we refer the reader to [1, 5, 8, 10, 11].

In the present paper, we are interested in the global solvability of the Cauchy problems of (1) and (2). It is known that the global solvability of the following Cauchy problems of Schrödinger equations, with and without Hamilton structure, are quite different:

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{p-1} u, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{3}$$

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p}, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}. \end{cases} \tag{4}$$

There is a large literature on the Cauchy problems of (3) and (4). For instance, we refer the reader to [2, 3, 4, 12, 14, 20]. Roughly speaking, the global solvability of (4) is quite worse than that of (3). Since the charge of solutions of (3) is conserved, (3) is global well-posed at least in L^2 with the subcritical power nonlinearity. On the other hand, Ikeda and Wakasugi [17] and Ikeda and Inui [15, 16] showed that with some initial data, which may be small, the corresponding L^2 solution of (4) blows up in finite time for any $p \leq 1 + 4/d$ and $\lambda \in \mathbb{C} \backslash \{0\}$. Both of their proof rely on the blow-up alternative and nonexistence of global weak solutions, where the nonexistence is shown by a test function method. We shall see that a similar situation occurs in (1) and (2).

To state our main results, we introduce the following notation. For $s \in \mathbb{R}$, let $H^s = (1-\Delta)^{-s/2}L^2$ and $\dot{H}^s = (-\Delta)^{-s/2}L^2$ be the usual inhomogeneous and homogeneous Sobolev spaces of order s, respectively. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, let $\dot{B}^s_{p,q}$ be the homogeneous Besov space. For $\rho > 0$, J_ρ denotes an approximation operator of Yosida type defined by $J_\rho = \mathfrak{F}^{-1}\rho^2(\rho^2+\xi^2)^{-1}\mathfrak{F}$. We define function spaces X and D as follows:

$$X = \{\psi \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2);\ \operatorname{supp}\psi \subset [0,T] \times \mathbb{R} \text{ for some } T > 0\},$$

$$D = \{f \in L^1_{\mathrm{loc}} \backslash \{0\};\ \mathrm{Re}(\overline{\lambda} f) = 0,\ \int i\overline{\lambda} f > 0\}.$$

D is a positive cone of L^1_{loc}. Let $\mathcal{S}$ be the set of all rapidly decreasing smooth functions. Let $(\cdot \mid \cdot)$ be the usual L^2 scalar product defined by $(f \mid g) = \int f\overline{g}$. Then we define weak global solutions to (2).

Definition 1.1. *Let $u_0 \in L^2$. We say that u is a global weak solution to (2), if u belongs to $L^1_{\mathrm{loc}}([0,\infty); L^2) \cap L^1_{\mathrm{loc}}([0,\infty); L^p)$ and the following identity*

$$\int_0^\infty \big(u(t)\big| i\partial_t\psi(t) \pm (-\Delta)^{1/2}\psi(t)\big)dt = i(u_0|\psi(0)) + \lambda\int_0^\infty \big(|u(t)|^p\big|\psi(t)\big)dt \tag{5}$$

holds for any $\psi \in X$, where the double-sign corresponds to the sign of (2).

Here, we state our main results.

Theorem 1.2. *Let $s = 1/2$ or 1. Let $1 < p < 3$ and let $\mu \in \mathbb{R}$. Then for any $u_0 \in H^s$, there exists a global solution to (1). Moreover, let $u_{0,n}$, $u_0 \in H^{1/2}$ satisfy $u_{0,n} \to u_0$ in $H^{1/2}$ as $n \to \infty$, and let u_n and u be the solutions of (1) with data $u_{0,n}$ and u_0, respectively. Then $u_n \to u$ in $L^\infty([-T,T]; H^{1/2})$ for any $T > 0$ as $n \to \infty$.*

Theorem 1.3. *Let $1 < p \le 2$ and let $\lambda \in \mathbb{C}\backslash\{0\}$. Then, for any $u_0 \in D \cap L^2$, there exist no global weak solutions to (2).*

The following Lemma follows immediately.

Lemma 1.4. *Let $u_0 \in L^2$ and let $u \in C([0,\infty); L^2) \cap C^1([0,\infty); H^{-1})$ be a solution to (2). Then u is also a global weak solution to (2).*

The following Corollary follows from Lemma 1.4 and the standard contraction argument in H^s with $s > 1/2$.

Corollary 1.5. *For any $u_0 \in D \cap L^2$, the corresponding solution to* (2) *does not exist in $C([0,\infty); L^2) \cap C^1([0,\infty); H^{-1})$. In addition, for $u_0 \in D \cap H^s$ with $s > 1/2$ and the solution u to* (2) *with initial data u_0, there exists $T > 0$ such that $\limsup_{t \nearrow T} \|u(t)\|_{H^s} = \infty$.*

We prove Theorem 1.2 by a simple compactness argument for $s = 1/2$. We remark that Theorem 1.2 can be obtained by standard compactness argument and for $p = 3$, $H^{1/2}$ solutions of (1) are obtained by a compactness argument in [19]. However, our construction of $H^{1/2}$ solutions requires only some inequalities for an approximation operator J_ρ and the completeness of L^2. This construction is an improvement of that in [10]. For details, see Section 3.

We prove Theorem 1.3 by a test function method. The test function method works effectively for many kinds of equations to prove the nonexistence of weak solutions, and there is a large literature on this method. We refer [6, 9, 18] for related subjects. The test function method seems to rely on the scaling invariance and locality property of operators at least when one tries to apply it in the total space without any assumption. Then a serious difficulty in proving Theorem 1.3 with this method arises when we try to handle the nonlocal operator $(-\Delta)^{1/2}$. Indeed, it is well known that

$$(-\Delta)^{1/2} f(x) = \left(\int \frac{\cos(y) - 1}{|y|^{3/2}} dy \right)^{-1} \int \frac{f(x-y) - f(x)}{|y|^{3/2}} dy$$

for $f \in \mathcal{S}$. We refer [7] for related subjects. To overcome this difficulty, we transform (2) formally into a wave equation, where we justify this transformation in Section 4. By taking real and imaginary parts of (2) multiplied by $\overline{\lambda}$, we have

$$\partial_t \mathrm{Im}(\overline{\lambda} u) \mp (-\Delta)^{1/2} \mathrm{Re}(\overline{\lambda} u) = -|\lambda|^2 |u|^p, \tag{6}$$

$$\partial_t \mathrm{Re}(\overline{\lambda} u) \pm (-\Delta)^{1/2} \mathrm{Im}(\overline{\lambda} u) = 0. \tag{7}$$

Let $v = \mathrm{Im}(\overline{\lambda} u)$. By combining (7) with $(-\Delta)^{1/2}$ and the time derivative of (6), we obtain

$$\Box v = \partial_t^2 v - \Delta v = -|\lambda|^2 \partial_t |u|^p. \tag{8}$$

Then a difficulty arises in handling the nonlinear term with time derivative. The nonlinear term $\partial_t |u|^p$ seems to prevent ourselves from applying a test function method with a separated test function, which seems to be standard. It is because that for $\psi \in \mathcal{S}$, it is difficult to control $|\psi|$ by $|\psi'|$ from above pointwise. Then we employ a test function of the form $\Phi(t^2 + x^2)$ with $\Phi \in \mathcal{S}$.

We give a brief outline of this article. In Section 2, we collect some basic estimates for the proof of Theorem 1.2. In Section 3, we give a sketch of proof of Theorem 1.2. In Section 4, we show that global weak solutions of (2) are also those to (8). Then we show Theorem 1.3 in Section 5.

2 Preliminary for the proof of Theorem 1.2

In this section, we collect some basic estimates for the proof of Theorem 1.2.

Lemma 2.1 (Lemma 3.4 in [12]). *Let* $f \in C^1(\mathbb{C};\mathbb{C})$ *with* $f(0) = f'(0) = 0$ *and assume that for some* $p \geq 1$

$$|f'(z_1) - f'(z_2)| \leq C \begin{cases} |z_1 - z_2| \max\{|z_1|, |z_2|\}^{p-2} & \text{if} \quad p \geq 2, \\ |z_1 - z_2|^{p-1} & \text{if} \quad p < 2 \end{cases}$$

for all $z_1, z_2 \in \mathbb{C}$ *with some positive constant* C. *Let* $0 \leq s < \min\{2, p\}$ *and* $1 \leq l, r, q \leq \infty$ *with* $(p-1)/q = 1/l - 1/r$. *Then* f *satisfies the estimate*

$$\|f(u)\|_{\dot{B}^s_{l,2}} \leq C\|u\|_{\dot{B}^s_{r,2}} \|u\|^{p-1}_{L^q}.$$

with some positive constant C.

Lemma 2.2 (Theorem 2 in [21]). *Let* $2 \leq p < \infty$. *There exists* $C > 0$ *such that for any* $\psi \in H^{1/2}(\mathbb{R})$,

$$\|\psi\|_{L^p} \leq C\sqrt{p}\|\psi\|_{H^{1/2}}.$$

Lemma 2.3 (Lemma 2.4 in [10]). *Let* $r > 1$ *and* a, b, $T > 0$. *Let* $f : [0,T] \to [0,\infty)$ *satisfy*

$$f(t) \leq a + b\int_0^t f^{1-1/r}(t')dt'$$

for all $0 \leq t \leq T$. *Then* $f(t) \leq (a^{1/r} + br^{-1}t)^r$ *for all* $0 \leq t \leq T$.

3 Sketch of proof of Theorem 1.2

In this section, we construct $H^{1/2}$ solutions for (1). The properties of $H^{1/2}$ solutions claimed in Theorem 1.2 are obtained by a similar argument in [10].

Let $u_0 \in H^{1/2}$. Here, we consider the corresponding approximation integral equation

$$u_\rho(t) = U(t)J_\rho u_0 - i\mu\int_0^t U(t-t')J_\rho\big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\big)dt', \qquad (9)_\rho$$

where $U(t) = \exp[\pm it(m^2 - \Delta)^{1/2}]$ is a semirelativistic propagator and the double-sign corresponds to the sign of (1). By the standard contraction argument and the energy conservation, for any $\rho > 0$, $(9)_\rho$ has a unique global solution in $C(\mathbb{R};H^1)\cap C^1(\mathbb{R};L^2)$. Moreover, by the conservation of mass and energy, we see that

$$\sup_{\rho>0}\|u_\rho\|_{H^{1/2}}\le CM,$$

where

$$M:=\|u_0\|_{H^{1/2}}+\|u_0\|_{L^2}^{2/(3-p)}+\|u_0\|_{L^2}\|u_0\|_{H^{1/2}}^{(p-1)/2}$$

Then, it suffices to prove that u_ρ is a Cauchy net in $L^\infty([-T,T];L^2)$ for any $T>0$. For $t\in\mathbb{R}$, we estimate

$$\begin{aligned}
&\|u_\rho(t)-u_\sigma(t)\|_{L^2}\\
&\le\|(J_\rho-J_\sigma)u_0\|_{L^2}+|\mu|\int_0^t\big\|(1-J_\rho)\big(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\big)\big\|_{L^2}dt'\\
&+|\mu|\int_0^t\big\|(1-J_\sigma)\big(|J_\sigma u_\sigma(t')|^{p-1}J_\sigma u_\sigma(t')\big)\big\|_{L^2}dt'\\
&+|\mu|\int_0^t\big\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')-|J_\sigma u_\sigma(t')|^{p-1}J_\sigma u_\sigma(t')\big\|_{L^2}dt'.
\end{aligned}\tag{10}$$

For sufficiently large r, by the Hölder, Gagliardo-Nierenberg, and Lemma 2.2,

$$\begin{aligned}
&\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')-|J_\sigma u_\sigma(t')|^{p-1}J_\sigma u_\sigma(t')\|_{L^2}\\
&\le C\|(|J_\rho u_\rho(t')|^{p-1}+|J_\sigma u_\sigma(t')|^{p-1})(J_\rho u_\rho(t')-J_\sigma u_\sigma(t'))\|_{L^2}\\
&\le C\big(\|J_\rho u_\rho(t')\|_{L^{4(p-1)}}^{p-1}+\|J_\sigma u_\sigma(t')\|_{L^{4(p-1)}}^{p-1}\big)\\
&\cdot\big(\|(1-J_\rho)u_\rho(t')\|_{L^4}+\|(1-J_\sigma)u_\sigma(t')\|_{L^4}\big)\\
&+C\big(\|J_\rho u_\rho(t')\|_{L^{2r(p-1)}}^{p-1}+\|J_\sigma u_\sigma(t')\|_{L^{2r(p-1)}}^{p-1}\big)\|u_\rho(t')-u_\sigma(t')\|_{L^{2r'}}\\
&\le CM^{p-1}\big(\|(1-J_\rho)u_\rho(t')\|_{H^{1/4}}+\|(1-J_\sigma)u_\sigma(t')\|_{H^{1/4}}\big)\\
&+Cr^{(p-1)/2}M^{p-1+1/r}\|u_\rho(t')-u_\sigma(t')\|_{L^2}^{1/r'},
\end{aligned}\tag{11}$$

where C is independent of r. Since

$$\frac{\xi^2}{\rho^2+\xi^2}\le\frac{|\xi|^{1/4}}{\rho^{1/4}},$$

we have

$$\|(1-J_\rho)u_\rho(t')\|_{H^{1/4}}\le\rho^{-1/4}\|u_\rho(t')\|_{H^{1/2}}\le M\rho^{-1/4}.\tag{12}$$

Similarly,

$$\begin{aligned}
&\|(1-J_\rho)(|J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t'))\|_{L^2}\\
&\le\rho^{-1/4}\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\|_{\dot H^{1/4}}\\
&\le C\rho^{-1/4}\||J_\rho u_\rho(t')|^{p-1}J_\rho u_\rho(t')\|_{\dot B^{1/4}_{2,2}}.
\end{aligned}$$

By Lemma 2.1 and the embedding inequality,

$$\| |J_\rho u_\rho(t')|^{p-1} J_\rho u_\rho(t')\|_{\dot{B}^{1/4}_{2,2}} \leq C\|J_\rho u_\rho(t')\|_{\dot{B}^{1/4}_{4,2}} \|J_\rho u_\rho(t')\|^{p-1}_{L^{4(p-1)}} \leq CM^p \tag{13}$$

with some positive constant C. Combining (10), (11), (12), and (13),

$$\begin{aligned} &\|u_\rho(t) - u_\sigma(t)\|_{L^2} \\ &\leq C(\rho^{-1/4} + \sigma^{-1/4}) + Cr^{(p-1)/2} \int_0^t \|u_\rho(t') - u_\sigma(t')\|^{1-1/r}_{L^2} dt'. \end{aligned} \tag{14}$$

By (14) and Lemma 2.3,

$$\|u_\rho(t) - u_\sigma(t)\|_{L^2} \leq C\big((\rho^{-1/4} + \sigma^{-1/4})^{1/r} + r^{(p-3)/2} t\big)^r$$

and, by taking the limits $\rho, \sigma \to \infty$ and then $r \to \infty$, this shows that u_ρ is a Cauchy net in $L^\infty([-T,T]; L^2)$ for any $T > 0$.

4 Preliminary for the proof of Theorem 1.3

In this section, we see that global weak solutions to (2) are also those to (8), which are defined below:

Definition 4.1. *Let $u_0 \in L^2$. We say that u is a global weak solution to (8), if $v = \mathrm{Im}(\overline{\lambda} u)$ and $|u|^p$ belong to $L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$ and the following identity*

$$\begin{aligned} &\int_0^\infty \big(v(t)\big|\Box\varphi(t)\big) dt \\ &= \pm\big(\mathrm{Re}(\overline{\lambda} u_0)\big|(-\Delta)^{1/2}\varphi(0)\big) - \big(v(0)\big|\partial_t\varphi(0)\big) + |\lambda|^2 \int_0^\infty \big(|u(t)|^p\big|\partial_t\varphi(t)\big) dt \end{aligned} \tag{15}$$

holds for $\varphi \in C_0^2(\mathbb{R}^2; \mathbb{R})$, where the double-sign corresponds to the sign of (2).

Lemma 4.2. *Let $u_0 \in L^2$. Then, global weak solutions to (2) are those to (8).*

Proof. Let $\varphi \in C_0^2(\mathbb{R}^2; \mathbb{R})$. Then $(-\Delta)^{1/2}\varphi$ and $\partial_t\varphi$ belong to X. By taking real and imaginary parts of (5) with ψ replaced by $\lambda(-\Delta)^{1/2}\varphi$ and $\lambda\partial_t\varphi$, respectively, we obtain

$$\begin{aligned}
&\mathrm{Re}\int_0^\infty \big(\overline{\lambda}u(t)\big|i\partial_t^2\varphi(t) \pm \partial_t(-\Delta)^{1/2}\varphi(t)\big)dt\\
&= \int_0^\infty \big(v(t)\big|\partial_t^2\varphi(t)\big)dt \pm \int_0^\infty \big(\mathrm{Re}(\overline{\lambda}u(t))\big|\partial_t(-\Delta)^{1/2}\varphi(t)\big)dt\\
&= -(v(0)|\partial_t\varphi(0)) + |\lambda|^2\int_0^\infty \big(|u(t)|^p\big|\partial_t\varphi(t)\big)dt,\\
&\mathrm{Im}\int_0^\infty \big(\overline{\lambda}u(t)\big|i\partial_t(-\Delta)^{1/2}\varphi(t) \mp \Delta\varphi(t)\big)dt\\
&= -\int_0^\infty \big(\mathrm{Re}(\overline{\lambda}u(t))\big|\partial_t(-\Delta)^{1/2}\varphi(t)\big)dt \mp \int_0^\infty \big(v(t)\big|\Delta\varphi(t)\big)dt\\
&= \big(\mathrm{Re}(\overline{\lambda}u_0)\big|(-\Delta)^{1/2}\varphi(0)\big).
\end{aligned}$$

By combining those identities, we obtain (15). □

Remark 4.3. *By combining Lemmas 1.4 and 4.2, solutions to* (2) *which belong to* $C([0,\infty);L^2)\cap C^1([0,\infty),H^{-1})$ *are also shown to be global weak solutions to* (8).

5 Proof of Theorem 1.3

Here we prove Theorem 1.3 by showing the nonexistence of global weak solutions to (8) for any initial data in $D\cap L^2$. Let $u_0\in D\cap L^2$ and $R>0$. Let $\eta\in C^\infty([0,\infty))$ satisfy

$$\eta(y)\begin{cases} = 1 & \text{if} \quad y\le 2,\\ \searrow & \text{if} \quad 2<y<4,\\ = 0 & \text{if} \quad y\ge 4,\end{cases}$$

and let $\eta_R(y)=\eta(y/R)$ and $(\eta')_R(y)=\eta'(y/R)$. We put

$$\begin{aligned}
\Phi_R(y) &= \int_0^y \eta_R(z)^{\frac{p}{p-1}}dz - \int_0^\infty \eta_R(z)^{\frac{p}{p-1}}dz,\\
\varphi^R(t,x) &= \Phi_{R^2}\big((t+R)^2+x^2\big).
\end{aligned}$$

Then, $\Phi_R \in C^2([0,\infty);\mathbb{R})$, $\operatorname{supp}\Phi_R \subset [0,4R]$, and

$$
\begin{aligned}
\partial_t \varphi^R(t,x) &= 2(t+R)\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{p}{p-1}},\\
\partial_t^2 \varphi^R(t,x) &= 2\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{p}{p-1}}\\
&\quad + \frac{4p}{p-1}\frac{(t+R)^2}{R^2}\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}(\eta')_{R^2}\big((t+R)^2+x^2\big),\\
\partial_x^2 \varphi^R(t,x) &= 2\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{p}{p-1}}\\
&\quad + \frac{4p}{p-1}\frac{x^2}{R^2}\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}(\eta')_{R^2}\big((t+R)^2+x^2\big).
\end{aligned}
$$

With the test function φ^R, the third term on the right hand side of (15) is calculated as

$$
\begin{aligned}
&\int_0^\infty \big(|u(t)|^p\big|\partial_t\varphi^R(t)\big)dt\\
&= 2\int_0^\infty \Big(|u(t)|^p\Big|(t+R)\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{p}{p-1}}\Big)dt\\
&\geq 2R\|u\cdot\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|^p_{L^p([0,\infty)\times\mathbb{R})}.
\end{aligned}
$$

Since the minimum value of t^2+x^2 on

$$\{(t,x)\in[0,\infty)\times\mathbb{R};\ 2R^2\leq(t+R)^2+x^2\leq 4R^2\}$$

is obtained when $(t,x)=((\sqrt{2}-1)R,0)$ and $(\sqrt{2}-1)^2\geq 1/7$,

$$
\begin{aligned}
\operatorname{supp}\Big((\eta')_{R^2}\big((t+R)^2+x^2\big)\Big) &\subset \{(t,x)\in[0,\infty)\times\mathbb{R};\ 2R^2\leq(t+R)^2+x^2\leq 4R^2\}\\
&\subset \{(t,x)\in[0,\infty)\times\mathbb{R};\ R^2/7\leq t^2+x^2\leq 3R^2\}\\
&=: A_R,
\end{aligned}
$$

the left hand side of (15) is estimated as

$$
\begin{aligned}
&\left|\int_0^\infty \big(v(t)\big|\Box\varphi^R(t)\big)dt\right|\\
&\leq \frac{4p}{p-1}\int_0^\infty \left(|v(t)|\,\left|\,\left|\frac{(t+R)^2-x^2}{R^2}\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}(\eta')_{R^2}\big((t+R)^2+x^2\big)\right|\right)dt\\
&\leq \frac{16p}{p-1}\|(\eta')_{R^2}\big((t+R)^2+x^2\big)\|_{L^{p'}([0,\infty)\times\mathbb{R})}\|v\cdot\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|_{L^p(A_R)}\\
&\leq |\lambda|\frac{16p}{p-1}\|\eta'\big((t+1)^2+x^2\big)\|_{L^{p'}([0,\infty)\times\mathbb{R})}R^{\frac{2}{p'}}\|u\cdot\eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|_{L^p(A_R)}.
\end{aligned}
$$

Since $\eta_{R^2}\big((t+R)^2+x^2\big) \nearrow 1$ as $R \to \infty$, by $\int v(0)dx < 0$, $\mathrm{Re}(\overline{\lambda} u_0) = 0$, and (15),

$$\begin{aligned}
&\|u \cdot \eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|^p_{L^p([0,\infty)\times\mathbb{R})} \\
&\leq \frac{1}{2R|\lambda|^2}\Big(2R|\lambda|^2\|u \cdot \eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|^p_{L^p([0,\infty)\times\mathbb{R})} \\
&- \big(v(0)\big|2R\eta_{R^2}(R^2+x^2)^{\frac{p}{p-1}}\big)\Big) \\
&\leq \frac{1}{2R|\lambda|^2}\left(|\lambda|^2\int_0^\infty \big(|u(t)|^p\big|\partial_t\varphi^R(t)\big)dt - \big(v(0)\big|\partial_t\varphi^R(0)\big)\right) \\
&\leq \frac{8p}{|\lambda|(p-1)}\|\eta'\big((t+1)^2+x^2\big)\|_{L^{p'}([0,\infty)\times\mathbb{R})} \\
&\cdot R^{\frac{2}{p'}-1}\|u \cdot \eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|_{L^p(A_R)},
\end{aligned}$$

for sufficiently large R. Then we see that $\|u\|_{L^p([0,\infty)\times\mathbb{R})} = 0$ for $1 < p \leq 2$. In fact, by

$$\|u\|^{p-1}_{L^p([0,\infty)\times\mathbb{R})} \leq \frac{8p}{|\lambda|(p-1)}\|\eta'\big((t+1)^2+x^2\big)\|_{L^{p'}([0,\infty)\times\mathbb{R})}$$

and the dominated convergence theorem,

$$\|u\|^p_{L^p([0,\infty)\times\mathbb{R})} \leq C\lim_{R\to\infty}\|u \cdot \eta_{R^2}\big((t+R)^2+x^2\big)^{\frac{1}{p-1}}\|_{L^p(A_R)} = 0.$$

This is a contradiction to the assumption that $u_0 \in D$.

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