Commutators of Almost Normal Operators that Have Trace Zero

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This is written to the memory of my Father, Atanasie.

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Abstract

Kittaneh [1] extended a result of G. Weiss concerning commutators of trace zero. In this note we show that the operators that Kittaneh proved to satisfy such an extension are almost normal operators that satisfy a conjecture of Voiculescu. We prove that a trace class commutator $[S, K]$ of an operator $S$ that satisfies Voiculescu’s conjecture and a Hilbert-Schmidt operator $K$ has trace zero.

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1 Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_2(\mathcal{H})$ (or simply $\mathcal{C}_1$ and $\mathcal{C}_2$) the trace class and the Hilbert-Schmidt class, respectively. For arbitrary operators $S, T \in L(\mathcal{H})$, $[S, T]$ will denote the
commutator $ST - TS$ and $D_S$ will denote self-commutator of $S$, that is $[S^*, S]$. An operator $S \in L(H)$ for which $D_S \in C_1(H)$ is called almost normal. The class of operators defined on $H$ which are almost normal will be denoted by $AN(H)$.

In [5] proved that if $N$ is a normal operator and $K \in C_2$ such that $[N, K] \in C_1$, then $\text{tr} ([N, K]) = 0$. Kittaneh [1] extended this result to some non-normal operators as follows. If $K \in C_2$ and $S \in L(H)$ is an operator such that $[S, K] \in C_1$, then $\text{tr} ([S, K]) = 0$ provided that one of the following conditions holds: (a) $S$ is a subnormal operator with $D_S \in C_1$; (b) $S$ is a hyponormal contraction and $I - SS^* \in C_1$; or (c) $S^2$ is a normal operator with $D_S \in C_1$.

Voiculescu’s Conjecture 4 $(C_4)$, (cf. [3] or [4]) states that if $S \in AN(H)$, then there exists $T \in AN(H)$ such that $S \oplus T = N + K$, where $N$ is a normal operator and $K$ is a Hilbert-Schmidt operator. For an operator $S \in AN(H)$ that satisfies $(C_4)$, we will write $S \in (C_4)$. We mention that $(C_4)$ has remained unsolved even for arbitrary weighted shifts.

2 Main results

The purpose of this note is to show that all non-normal operators that satisfy one of the conditions (a), (b), or (c) above are included into a wider class of operators that satisfy $(C_4)$ and that such operators satisfy a Weiss type of result.

**Proposition 1.** If $S$ satisfies one of the conditions (a), (b) or (c) above, then $S \in (C_4)$.

**Proof.** If $S$ is a subnormal operator with $D_S \in C_1$ and $egin{pmatrix} S & A \\ 0 & T \end{pmatrix}$ is a normal extension of $S$, then $D_S = A^*A$, thus $A \in C_2$ and $D_T = -AA^*$, that is $S \in (C_4)$.

If $S$ is a hyponormal contraction with $I - SS^* \in C_1$, then $0 \leq I - S^*S \in C_1$ and the operator

$$
\begin{pmatrix}
S & \sqrt{I - SS^*} \\
\sqrt{I - SS^*} & -S^*
\end{pmatrix}
$$

is a unitary and again $S \in (C_4)$.

If $S^2$ is normal and $D_S \in C_1$, then according to [2], $S$ has the following matrix representation

$$
S = \begin{pmatrix}
A & 0 & 0 \\
0 & B & C \\
0 & 0 & -B
\end{pmatrix},
$$

where $A$ and $B$ are normal operators and $0 \leq C \in C_2$, and once again $S \in (C_4)$.

**Theorem 2.** If $S \in AN(H)$ satisfies $(C_4)$ and $K$ is a Hilbert-Schmidt operator such that $[S, K] \in C_1(H)$, then $\text{tr} ([S, K]) = 0$. 


Proof. Let $S \in \text{AN} (\mathcal{H})$ satisfy (C$_4$) and let $T \in \text{AN} (\mathcal{H})$ such that $S \oplus T = N + K'$, where $N$ is a normal operator and $K'$ is a Hilbert-Schmidt operator. Let $K \in \mathcal{C}_2$ such that $[S, K] \in \mathcal{C}_1$ and set $\tilde{K} = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$. On one hand

$$[S \oplus T, \tilde{K}] = \begin{pmatrix} [S, K] & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_1 (\mathcal{H} \oplus \mathcal{H})$$

and $\text{tr} ([S, K]) = \text{tr} ([S \oplus T, \tilde{K}])$. On other hand

$$[S \oplus T, \tilde{K}] = [N + K', \tilde{K}] = [N, \tilde{K}] + [K', \tilde{K}].$$

Since $K', \tilde{K} \in \mathcal{C}_2$, we obtain $[K', \tilde{K}] \in \mathcal{C}_1$ and $[N, \tilde{K}] \in \mathcal{C}_1$, and according to Weiss’ theorem, $\text{tr} ([N, \tilde{K}]) = 0$. It is well known $\text{tr} ([K', \tilde{K}]) = 0$ and thus $\text{tr} ([S, K]) = 0$. □

We conclude this note with the natural question.

**Question 3.** If $S \in \text{AN} (\mathcal{H})$ and $K \in \mathcal{C}_2 (\mathcal{H})$ such that $[S, K] \in \mathcal{C}_1 (\mathcal{H})$, is it true that $\text{tr} ([S, K]) = 0$?

**References**


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