Extremal Decomposition Problems in the Euclidean Space

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Abstract

Composition principles for reduced moduli are extended to the case of domains in the n-dimensional Euclidean space, n > 2. As a consequence analogues of extremal decomposition theorems of Kufarev, Dubinin and Kirillova in the planer case are obtained.

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1 Introduction and notations

Extremal decomposition problems have a rich history and go back to M.A. Lavrentiev’s inequality for the product of conformal radii of non-overlapping domains. There exist two methods of their study: the extremal-metric method and the capacitive method. The first one has been systematically developed in papers by G.V. Kuz’mina, E.G. Emel’yanov, A.Yu. Solynin, A. Vasil’ev, and Ch. Pommerenke [9, 14, 6, 11]. The second approach is developed mainly in works of V.N. Dubinin and his students [4, 5, 2, 3]. In particular, a series of well-known results about extremal decomposition follows one way from composition principles for generalized reduced moduli (see [1, p. 56] and [12]). In the present paper we extend the mentioned composition principles to the case of spatial domains. As a consequence we get theorem about extremal decomposition for the harmonic radius [7] obtained earlier in [5].

Throughout the paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space consisting of points $x = (x_1, \ldots, x_n)$, $n \geq 3$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the length of a vector $x \in \mathbb{R}^n$. We introduce the following notations:

- $B(a, r) = \{x \in \mathbb{R}^n : |a - x| < r\}$,
- $S(a, r) = \{x \in \mathbb{R}^n : |a - x| = r\}$, $a \in \mathbb{R}^n$;
- $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $S(0, 1)$;
- $\lambda_n = ((n - 2)\omega_{n-1})^{-1}$.

$D$ is a bounded domain in $\mathbb{R}^n$, $\Gamma$ is a closed subset of $\partial D$. The pair $(D, \Gamma)$ is admissible if there exists the Robin function, $g_\Gamma(z, z_0, D)$ harmonic in $D \setminus \{z_0\}$, continuous in $\overline{D} \setminus \{z_0\}$ and

$$\frac{\partial g_\Gamma}{\partial n} = 0 \text{ on } (\partial D) \setminus \Gamma, \quad (1)$$

$$g_\Gamma = 0 \text{ on } \Gamma, \quad (2)$$

and in a neighborhood of $z_0$ there is an expansion

$$g_\Gamma(z, z_0, D) = \lambda_n \left(|z - z_0|^{2-n} - r(D, z_0, \Gamma)^{2-n} + o(1)\right), \ z \to z_0, \quad (3)$$

where $\partial/\partial n$ means the inward normal derivative on the boundary. In what follows all such pairs are assumed to be admissible.

In the case $\Gamma = \emptyset$ we change the condition (1) by the condition

$$\frac{\partial g_\Gamma}{\partial n} = \frac{1}{\mu_{n-1}(\partial D)} \text{ on } \partial D,$$

where $\mu_{n-1}(\partial D)$ is the area of boundary.

By analogy with the definition of the Robin radius for plain domains from the paper [3] we will call the constant $r(D, z_0, \Gamma)$ the Robin radius of the
domain \( D \) and the set \( \Gamma \). Note that in the case of \( \Gamma = \partial D \) we get the harmonic radius \([7, 10, 5]\).

Let \( \Delta = \{ \delta_k \}_{k=1}^m \) be a collection of real numbers and \( Z = \{ z_k \}_{k=1}^m \) be points of the domain \( D \). For \( \Gamma = \emptyset \) we additionally require

\[
\sum_{k=1}^m \delta_k = 0.
\]

Define the potential function for the domain \( D \), the set \( \Gamma \), the collection of points \( Z \), and numbers \( \Delta \):

\[
u(z) = \nu(z; Z, D, \Gamma, \Delta) = \sum_{k=1}^m \delta_k g_\Gamma(z, z_k, D).
\]

Note that for \( \Gamma = \emptyset \) the function \( g_\Gamma(z, z_k, D) \) is defined up to an additive constant. Nevertheless, the function \( \nu(z) \) is defined uniquely and characterized by the condition

\[
\frac{\partial \nu}{\partial n} = 0 \quad \text{on} \quad \partial D.
\]

It is clear from the definition of the potential function that in a neighborhood of \( z_k \) we have

\[
u(z) = \delta_k \lambda_n |z - z_k|^{2-n} + a_k + o(1), \quad k = 1, \ldots, m,
\]

where

\[
a_k = -\delta_k \lambda_n r(D, z_k, \Gamma)^{2-n} + \sum_{l=1}^m \delta_l g_\Gamma(z_l, z_k, D).
\]

Now if we introduce the following notation

\[
g_\Gamma(z_k, z_k, D) = -\lambda_n r(D, z_k, \Gamma)^{2-n},
\]

then the constant in the expansion of the potential function in a neighbourhood of \( z_k \) is

\[
a_k = \sum_{l=1}^m \delta_l g_\Gamma(z_l, z_k, D). \quad (4)
\]

A function \( v(z) \) is admissible for \( D, Z, \Delta, \) and \( \Gamma \) if \( v(z) \in \text{Lip} \) in a neighborhood of each point of \( D \) except maybe finitely many such points, continuous in \( \overline{D} \setminus \bigcup_{k=1}^m \{ z_k \} \), \( v(z) = 0 \) on \( \Gamma \), and in neighborhood of \( z_k \) there is an expansion

\[
v(z) = \delta_k \lambda_n |z - z_k|^{2-n} + b_k + o(1), \quad z \rightarrow z_k.
\]

The Dirichlet integral is the following

\[
I(f, D) = \int_D |\nabla f|^2 \, d\mu,
\]

where \( d\mu = dx_1 \ldots dx_n \).
2 Main results

Lemma 2.1 The asymptotic formula

\[ I(u, D_r) = \left( \sum_{k=1}^{m} \delta_k^2 \right) \lambda_n r^{2-n} + \sum_{k=1}^{m} \delta_k a_k + o(1), \quad r \to 0, \]

is true, where \( u \) is the potential function and \( a_k, k = 1, \ldots, m \) are defined in (4) and \( D_r = D \setminus \bigcup_{k=1}^{m} B(z_k, r) \).

Proof. The Green’s identity

\[ \int_{V} |\nabla u|^2 d\mu = - \int_{\partial V} u \frac{\partial u}{\partial n} ds \]

gives

\[ I(u, D_r) = - \int_{\partial D_r} u \frac{\partial u}{\partial n} ds = - \sum_{k=1}^{m} \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds. \tag{6} \]

The second equality in (6) holds because \( u \frac{\partial u}{\partial n} = 0 \) on \( \partial D \). Note that

\[ u = \delta_k \lambda_n r^{2-n} + a_k + o(1), \quad z \to z_k \]

in a neighbourhood of \( z_k \).

We calculate the integral \( \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds \). Let \( u(z) = h(z) + g(z) \), where \( h(z) = \lambda_n \delta_k |z - z_k|^{2-n} \) and \( g(z) \) is a harmonic function. Note that \( g(z_k) = a_k \).

For \( |z - z_k| = r \) we have the following correlations

\[ r^{n-1} u \frac{\partial u}{\partial n} = r^{n-1} \left( h \frac{\partial h}{\partial n} + h \frac{\partial g}{\partial n} + g \frac{\partial h}{\partial n} + g \frac{\partial g}{\partial n} \right) \]

\[ = (2-n) \lambda_n r^{2-n} \delta_k + \lambda_n \delta_k \frac{\partial g}{\partial n} + (2-n) g \lambda_n \delta_k + g \frac{\partial g}{\partial n} r^{n-1} \]

\[ = - \frac{\lambda_n \delta_k^2}{\omega_{n-1}} (2-n) + g(z_k) \delta_k + o(1), \quad r \to 0. \]

Therefore

\[ \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds = \int_{S(0,1)} u \frac{\partial u}{\partial n} r^{n-1} ds \]

\[ = - \lambda_n \delta_k^2 r^{2-n} - \delta_k a_k + o(1), \quad r \to 0. \]

Substituting it in (6) we get the lemma. \( \square \)
**Lemma 2.2** For an admissible function \( v \) and the potential function \( u \) we have

\[
I(v - u, D_r) = I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{m} \delta_k(b_k - a_k) + o(1), \ r \to 0.
\]

**Proof.** One may observe that

\[
\begin{align*}
I(v - u, D_r) &= \int_{D_r} (|\nabla v|^2 + |\nabla u|^2 - 2 \nabla u \cdot \nabla v) \, d\mu \\
&= \int_{D_r} (|\nabla v|^2 - |\nabla u|^2) \, d\mu + 2 \int_{\partial D_r} (v - u) \frac{\partial u}{\partial n} \, ds \\
&= I(v, D_r) - I(u, D_r) + 2 \sum_{k=1}^{m} \int_{S(z_k, r)} (v - u) \frac{\partial u}{\partial n} \, ds \\
&= I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{m} \delta_k(b_k - a_k) + o(1), \ r \to 0.
\end{align*}
\]

Here we calculated the integral \( \int_{S(z_k, r)} (v - u) \frac{\partial u}{\partial n} \) a similar way as in the proof of lemma 2.1 and used the Green’s identity

\[
\int_{D_r} (\nabla u \cdot \nabla v) \, d\mu = - \int_{\partial D_r} v \frac{\partial u}{\partial n} \, ds,
\]

where \( n \) is the inner normal vector. □

The quantity

\[
\sum_{k=1}^{m} \delta_k a_k = \sum_{k=1}^{m} \sum_{l=1}^{m} \delta_k \delta_{l} g_{r}(z_l, z_k, D)
\]

we call the reduced modulus and denote it by \( M(D, \Gamma, Z, \Delta) \). According to lemma 2.1

\[
M(D, \Gamma, Z, \Delta) = \lim_{r \to 0} \left( I(u, D_r) - \left( \sum_{k=1}^{n} \delta_k^2 \right) \lambda_k r^{2-n} \right).
\]

**Theorem 2.3** Let sets \( D, \Gamma, \), collections \( Z = \{ z_k \}_{k=1}^{m}, \Delta = \{ \delta_k \}_{k=1}^{m} \), be as in the definition of the reduced modulus \( M = M(D, \Gamma, Z, \Delta) \), \( u(z) \) be the potential function for \( D, \Gamma, Z, \Delta, \) and let \( D_i \subset D \) be pairwise non-overlapping subdomains of \( D, \Gamma_i, Z_i, \Delta_i, \) \( i = 1, \ldots, p \). Assume that the following conditions are fulfilled:

1) \( (D \cap \partial D_i) \subset \Gamma_i, \ i = 1, \ldots, p; \)
2) \( \Gamma \subset \bigcup_{i=1}^{p} \Gamma_i \cup [\mathbb{R}^n \setminus (\bigcup_{i=1}^{p} D_i)]; \)
3) \( Z = \bigcup_{i=1}^{p} Z_i \), that is each point \( z_k \in Z \) coincides with some point \( z_{ij} \in Z_i \) for \( k = k(i, j) \) and vice versa;
4) \( \delta_k = \delta_{ij} \) for \( k = k(i, j) \).
Then the inequality
\[ M \geq \sum_{i=1}^{p} M_i + \sum_{i=1}^{p} I(u - u_i, D_i) \geq \sum_{i=1}^{p} M_i \]
holds.

**Proof.** Consider the function
\[ v(z) = \begin{cases} u_i(z), & z \in D_i, \\ 0, & z \in D \setminus \bigcup_{i=1}^{p} D_i. \end{cases} \]
The condition 1) guarantees that the function \( v(z) \) is continuous in \( D \setminus \bigcup_{k=1}^{m} \{z_k\} \). From the conditions 2) and 3) it follows that \( v(z) = 0 \) for \( z \in \Gamma \) and in a neighbourhood of \( z_k, k = 1, \ldots, m \), there is the expansion (5). Applying lemma 2.2, we get
\[ I(v - u, D) = I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{p} \delta_k (b_k - a_k) + o(1), \quad r \to 0, \quad (7) \]
here \( a_k \) and \( b_k \) from (4) and (5) respectively. By lemma 2.1
\[ I(v, D_r) = \lambda_n r^{2-n} \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij}^2 + \sum_{i=1}^{p} M_i + o(1) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + \sum_{i=1}^{p} M_i + o(1), \]
\[ I(u, D_r) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + M + o(1), \quad r \to 0, \]
taking into account 3), we have
\[ \sum_{k=1}^{m} \delta_k (b_k - a_k) = \sum_{i=1}^{p} M_i - M. \]
Substituting the obtained correlations in (7), we see that the inequality
\[ \sum_{i=1}^{p} I(u - u_i, D_i) \leq I(v - u, D) = M - \sum_{i=1}^{p} M_i + o(1), \quad r \to 0, \]
is true. Theorem is proved. \( \square \)

**Theorem 2.4** Let sets \( D, \Gamma, \) collections \( Z = \{z_k\}_{k=1}^{m}, \Delta = \{\delta_k\}_{k=1}^{m}, \) be as in the definition of the reduced modulus \( M := M(D, \Gamma, Z, \Delta), u(z) \) be the potential function for \( D, \Gamma, Z, \Delta, \) and let \( D_i \subset D, i = 1, \ldots, p, \) be pairwise non-overlapping domains, \( \Gamma_i, Z_i = \{z_{ij}\}_{j=1}^{n_i}, \Delta_i = \{\delta_{ij}\}_{j=1}^{n_i}, \) be from the definition
of the reduced moduli \( M_i = M(D_i, \Gamma_i, Z_i, \Delta_i), u_i(z) \) be the potential function for \( D_i, \Gamma_i, Z_i, \Delta_i, i = 1, \ldots, p \). Assume that \( \Gamma_i \subset \Gamma, i = 1, \ldots, p, Z = \bigcup_{i=1}^{n} Z_i \), (that is each point \( z_k \in Z \) coincides with some point \( z_{ij} \in Z_i \) for \( k = k(i, j) \) and vice versa), \( \delta_k = \delta_{ij} \). Then the inequality

\[
\sum_{i=1}^{p} M_i \geq M + \sum_{i=1}^{p} I(u - u_i, D_i) \geq M
\]

holds.

**Proof.** The function \( u \) is admissible for \( D_i, i = 1, \ldots, p \). Let \( b_k \) be constants from the expansion of the function \( u \) in a neighbourhood of \( z_k, b_{ij} = b_k \) if \( k = k(i, j) \). Applying lemmata 2.1 and 2.2 with the potential functions \( u_k \) for \( D_k \) we get

\[
\sum_{i=1}^{p} \sum_{j=1}^{n_i} (\delta_{ij})^2 r^{2-n} \lambda_n + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} a_{ij} + o(1) = \sum_{i=1}^{p} I(u_i, (D_i)_r) =
\]

\[
= \sum_{i=1}^{p} \left( I(u, (D_i)_r) - 2 \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij}) - I(u - u_i, (D_i)_r) \right) + o(1)
\]

\[
\leq I(u, D_r) - \sum_{i=1}^{p} I(u - u_i, (D_i)_r) - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij}) + o(1)
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (\delta_{ij})^2 r^{2-n} \lambda_n + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} b_{ij} - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij})
\]

\[
- \sum_{i=1}^{p} I(u - u_i, (D_i)_r) + o(1), \; r \to 0.
\]

It implies that

\[
\sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} b_{ij} \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} a_{ij} - \sum_{i=1}^{p} I(u - u_i, D_i)
\]

or equivalently

\[
\sum_{i=1}^{p} I(u - u_i, D_i) + M(D, \Gamma, Z, \Delta) \leq \sum_{i=1}^{p} M(D_i, \Gamma_i, Z_i, \Delta_i).
\]

Here we used the fact that the function \( u - u_i \) has no singularity in \( D_i \). □

Denote by \( r(D_i, x_l) = r(D_i, x_l, \partial D) \) the harmonic radius. Directly from theorem 2.3 we get theorem 2 of the paper [5]
Corollary 2.5 For any non-overlapping domains \( D_l \subset \mathbb{R}^n \), \( n \geq 3 \), points \( x_l \in D_l \) and real numbers \( \delta_l, l = 1, \ldots, m \) the inequality

\[
- \sum_{l=1}^{m} \delta_l^2 r(D_l, x_l)^{2-n} \leq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p |x_l - x_p|^{2-n}
\]

holds true.

Proof. The Green’s function of the ball \( B(0, \rho) \) is

\[
\lambda_n \left( |x - x_0|^{2-n} - \frac{|x_0| x}{\rho} - \frac{\rho x_0}{|x_0|} \right)^{2-n}.
\]

Denote by \( D_l(\rho) \) the intersection \( D_l \cap B(0, \rho) \). By theorem 2.3

\[
M(\rho) \geq \sum_{l=1}^{m} M_l(\rho),
\]

where \( M(\rho) \) is the modulus of the ball \( B(0, \rho) \), the collections \( \{x_l\}_{l=1}^{m}, \Delta = \{\delta_l\}_{l=1}^{m}, \) and \( \Gamma = \partial B \);

\[
M_l(\rho) = -\delta_l^2 r(D_l(\rho), x_l)^{2-n} \lambda_n.
\]

It is sufficient to take a limit as \( \rho \to \infty \). □

Theorems 2.3 and 2.4 imply for \( p = 1 \) monotonicity of the quadratic form

\[
\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D)
\]

under extension of a domain. Following [2] we will say that a domain \( \tilde{D} \) is obtained by extending a domain \( D \) across a part of its boundary \( \gamma \subset \partial D \) if \( D \subset \tilde{D} \) and \( (\partial D) \cap \tilde{D} \) lies in \( \gamma \).

Corollary 2.6 If \( \tilde{D} \) is obtained by extending \( D \) across \( \Gamma, \tilde{\Gamma} \subset (\Gamma \cup (\mathbb{R}^n \setminus D)) \), then for any real numbers \( \delta_l \) and points \( z_l \in D \)

\[
\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, \tilde{D}) \geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D) + I(u - \tilde{u}, D)
\]

\[
\geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D).
\]
If \( \tilde{D} \) is obtained by extending \( D \) across the part of \((\partial D) \setminus \Gamma, \tilde{\Gamma} = \Gamma\), then
\[
\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_F \left( z_l, z_p, \tilde{D} \right) \leq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_F \left( z_l, z_p, D \right) - I(u - \bar{u}, D)
\]
\[
\leq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_F \left( z_l, z_p, D \right),
\]
here \( u \) and \( \bar{u} \) are the potential functions for \( D, \Gamma \), \( Z = \{z_i\}_{i=1}^{m}, \Delta = \{\delta_l\}_{l=1}^{m} \) and \( \tilde{D}, \tilde{\Gamma}, Z = \{z_i\}_{i=1}^{m}, \Delta = \{\delta_l\}_{l=1}^{m} \), respectively.

In [3] the notion of the Robin radius
\[
r(D, z_0, \Gamma) = \exp \lim_{z \to z_0} (g_D(z, z_0, \Gamma) + \log |z - z_0|)
\]
was introduced. This quantity generalized the notion of the conformal radius. An analogue of Kufarev’s theorem (see [8]) for non-overlapping domains \( D_1, D_2 \) lying in the unit disk \( U \) under the condition \(((\partial D_k) \cap U) \subset \Gamma_k \subset \partial D_k, a_k \in D_k, k = 1, 2 \) is the inequality
\[
r(D_1, a_1, \Gamma_1) r(D_2, a_2, \Gamma_2) \leq |a_2 - a_1|^2 \left[ 1 - \frac{a_2 - a_1}{1 - \overline{a_1}a_2} \right]^{-1}.
\]
By setting in theorem 2.3 \( p = 2, \Gamma = \emptyset \), we obtain in \( \mathbb{R}^n \) the following inequality.

**Corollary 2.7** Let \( D_1 \) and \( D_2 \) be non-overlapping and lie in the ball \( U = B(0, 1) \), \( a_k \in D_k, (\partial D_k) \cap U \subset \Gamma_k \subset \partial D_k, k = 1, 2 \). Then
\[
-\lambda_n r(D_1, a_1, \Gamma_1)^{2-n} - \lambda_n r(D_2, a_2, \Gamma_2)^{2-n} \leq M(U, \emptyset, \{a_1, a_2\}, \{1, -1\}). \quad (8)
\]

To calculate \( M(U, \emptyset, \{a_1, a_2\}, \{1, -1\}) \) we need to know the Neumann function of the unit ball. Note that it is a quite complicated problem in \( \mathbb{R}^n \). In particular, for \( n = 3 \) (see [13])
\[
g_\emptyset(x, y, U) = \frac{1}{4\pi} \left( \frac{1}{|x - y|} + \frac{|y|}{|x|^2 - y} - \log \left| 1 - (x, y) + \frac{|x|^2 - |y|^2}{|y|} \right| \right).
\]
In [13] there is an analytic view of \( g_\emptyset(D, x, y) \) for \( n = 4, 5 \). So, for \( n = 3 \) the inequality (8) has the following form
\[
- r(D_1, a_1, \Gamma_1)^{-1} - r(D_2, a_2, \Gamma_2)^{-1} \leq \frac{2}{|a_1 - a_2|} - \frac{2|a_2|}{|a_1||a_2|^2 - a_2|}
\]
\[
+ 2 \log \left| 1 - (a_1, a_2) + \frac{|a_1|a_2^2 - a_2}{|a_2|} \right| + \frac{1}{1 - |a_1|^2} + \frac{1}{1 - |a_2|^2} - \log(4(1 - |a_1|^2)(1 - |a_2|^2)).
\]

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