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Extremal Decomposition Problems in the Euclidean Space

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Abstract

Composition principles for reduced moduli are extended to the case of domains in the n-dimensional Euclidean space, $n > 2$. As a consequence analogues of extremal decomposition theorems of Kufarev, Dubinin and Kirillova in the planer case are obtained.

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1 Introduction and notations

Extremal decomposition problems have a rich history and go back to M.A. Lavrentiev's inequality for the product of conformal radii of non-overlapping domains. There exist two methods of their study: the extremal-metric method and the capacitive method. The first one has been systematically developed in papers by G.V. Kuz'mina, E.G. Emel'yanov, A.Yu. Solynin, A. Vasil'ev, and Ch. Pommerenke [9, 14, 6, 11]. The second approach is developed mainly in works of V.N. Dubinin and his students [4, 5, 2, 3]. In particular, a series of well-known results about extremal decomposition follows one way from composition principles for generalized reduced moduli (see [1, p. 56] and [12]). In the present paper we extend the mentioned composition principles to the case of spatial domains. As a consequence we get theorem about extremal decomposition for the harmonic radius [7] obtained earlier in [5].

Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space consisting of points $x = (x_1, \dots, x_n)$, $n \geq 3$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the length of a vector $x \in \mathbb{R}^n$. We introduce the following notations:

$B(a,r) = \{x \in \mathbb{R}^n : |a - x| < r\}$,

$S(a,r) = \{x \in \mathbb{R}^n : |a - x| = r\}, \quad a \in \mathbb{R}^n$;

$\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $S(0,1)$;

$\lambda_n = ((n-2)\omega_{n-1})^{-1}$.

D is a bounded domain in $\mathbb{R}^n$, Γ is a closed subset of ∂D. The pair (D, Γ) is admissible if there exists the Robin function, $g_\Gamma(z, z_0, D)$ harmonic in $D \setminus \{z_0\}$, continuous in $\overline{D} \setminus \{z_0\}$ and

$$\frac{\partial g_\Gamma}{\partial n} = 0 \text{ on } (\partial D) \setminus \Gamma, \tag{1}$$

$$g_\Gamma = 0 \text{ on } \Gamma, \tag{2}$$

and in a neighborhood of z_0 there is an expansion

$$g_\Gamma(z, z_0, D) = \lambda_n \left(|z - z_0|^{2-n} - r(D, z_0, \Gamma)^{2-n} + o(1) \right), \ z \to z_0, \tag{3}$$

where $\partial/\partial n$ means the inward normal derivative on the boundary. In what follows all such pairs are assumed to be admissible.

In the case $\Gamma = \emptyset$ we change the condition (1) by the condition

$$\frac{\partial g_\Gamma}{\partial n} = \frac{1}{\mu_{n-1}(\partial D)} \text{ on } \partial D,$$

where $\mu_{n-1}(\partial D)$ is the area of boundary.

By analogy with the definition of the Robin radius for plain domains from the paper [3] we will call the constant $r(D, z_0, \Gamma)$ the Robin radius of the domain D and the set Γ. Note that in the case of $\Gamma = \partial D$ we get the harmonic radius [7, 10, 5].

Let $\Delta = \{\delta_k\}_k^m$ be a collection of real numbers and $Z = \{z_k\}_{k=1}^m$ be points of the domain D. For $\Gamma = \emptyset$ we additionally require

$$\sum_{k=1}^m \delta_k = 0.$$

Define the potential function for the domain D, the set Γ, the collection of points Z, and numbers Δ:

$$u(z) = u(z; Z, D, \Gamma, \Delta) = \sum_{k=1}^m \delta_k g_\Gamma(z, z_k, D).$$

Note that for $\Gamma = \emptyset$ the function $g_\Gamma(z, z_k, D)$ is defined up to an additive constant. Nevertheless, the function $u(z)$ is defined uniquely and characterized by the condition

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial D.$$

It is clear from the definition of the potential function that in a neighborhood of z_k we have

$$u(z) = \delta_k \lambda_n |z - z_k|^{2-n} + a_k + o(1), \ k = 1, \dots, m,$$

where

$$a_k = -\delta_k \lambda_n r(D, z_k, \Gamma)^{2-n} + \sum_{\substack{l=1 \\ l \neq k}}^m \delta_l g_\Gamma(z_l, z_k, D).$$

Now if we introduce the following notation

$$g_\Gamma(z_k, z_k, D) = -\lambda_n r(D, z_k, \Gamma)^{2-n},$$

then the constant in the expansion of the potential function in a neighbourhood of z_k is

$$a_k = \sum_{l=1}^m \delta_l g_\Gamma(z_l, z_k, D). \tag{4}$$

A function $v(z)$ is admissible for D, Z, Δ, and Γ if $v(z) \in \mathrm{Lip}$ in a neighbourhood of each point of D except maybe finitely many such points, continuous in $\overline{D} \setminus \bigcup_{k=1}^m \{z_k\}$, $v(z) = 0$ on Γ, and in neighbourhood of z_k there is an expansion

$$v(z) = \delta_k \lambda_n |z - z_k|^{2-n} + b_k + o(1), \ z \to z_k. \tag{5}$$

The Dirichlet integral is the following

$$I(f, D) = \int_D |\nabla f|^2 \, d\mu,$$

where $d\mu = dx_1 \dots dx_n$.

2 Main results

Lemma 2.1 *The asymptotic formula*

$$I(u, D_r) = \left(\sum_{k=1}^{m} \delta_k^2\right) \lambda_n r^{2-n} + \sum_{k=1}^{m} \delta_k a_k + o(1),\ r \to 0,$$

is true, where u *is the potential function and* a_k, $k = 1, \dots, m$ *are defined in* (4) *and* $D_r = D \setminus \bigcup_{k=1}^{m} B(z_k, r)$.

Proof. The Green's identity

$$\int_V |\nabla u|^2 d\mu = -\int_{\partial V} u \frac{\partial u}{\partial n} ds$$

gives

$$I\left(u, D_r\right) = -\int_{\partial D_r} u \frac{\partial u}{\partial n} ds = -\sum_{k=1}^{m} \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds. \tag{6}$$

The second equality in (6) holds because $u \dfrac{\partial u}{\partial n} = 0$ on ∂D. Note that

$$u = \delta_k \lambda_n r^{2-n} + a_k + o(1),\ z \to z_k$$

in a neighbourhood of z_k.

We calculate the integral $\int_{S(z_k,r)} u \frac{\partial u}{\partial n} ds$. Let $u(z) = h(z) + g(z)$, where $h(z) = \lambda_n \delta_k |z - z_k|^{2-n}$ and $g(z)$ is a harmonic function. Note that $g(z_k) = a_k$. For $|z - z_k| = r$ we have the following correlations

$$r^{n-1} u \frac{\partial u}{\partial n} = r^{n-1}\left(h \frac{\partial h}{\partial n} + h \frac{\partial g}{\partial n} + g \frac{\partial h}{\partial n} + g \frac{\partial g}{\partial n}\right)$$

$$= (2-n)\lambda_n^2 \delta_k^2 r^{2-n} + r\lambda_n \delta_k \frac{\partial g}{\partial n} + (2-n) g \lambda_n \delta_k + g \frac{\partial g}{\partial n} r^{n-1}$$

$$= -\frac{\lambda_n \delta_k^2}{\omega_{n-1}} r^{2-n} - \frac{g(z_k)\delta_k}{\omega_{n-1}} + o(1), r \to 0.$$

Therefore

$$\int_{S(z_k,r)} u \frac{\partial u}{\partial n} ds = \int_{S(0,1)} u \frac{\partial u}{\partial n} r^{n-1} ds$$

$$= -\lambda_n \delta_k^2 r^{2-n} - \delta_k a_k + o(1), \quad r \to 0.$$

Substituting it in (6) we get the lemma. $\square$

Lemma 2.2 *For an admissible function v and the potential function u we have*

$$I(v-u,D_r)=I(v,D_r)-I(u,D_r)-2\sum_{k=1}^{m}\delta_k(b_k-a_k)+o(1),\ r\to 0.$$

Proof. One may observe that

$$\begin{aligned}I(v-u,D_r)&=\int_{D_r}\left(|\nabla v|^2+|\nabla u|^2-2\nabla u\nabla v\right)d\mu\\&=\int_{D_r}\left(|\nabla v|^2-|\nabla u|^2\right)d\mu+2\int_{\partial D_r}(v-u)\frac{\partial u}{\partial n}ds\\&=I(v,D_r)-I(u,D_r)+2\sum_{k=1}^{m}\int_{S(z_k,r)}(v-u)\frac{\partial u}{\partial n}ds\\&=I(v,D_r)-I(u,D_r)-2\sum_{k=1}^{m}\delta_k(b_k-a_k)+o(1),\ r\to 0.\end{aligned}$$

Here we calculated the integral $\int_{S(z_k,r)}(v-u)\frac{\partial u}{\partial n}$ a similar way as in the proof of lemma 2.1 and used the Green's identity

$$\int_{D_r}(\nabla u\cdot\nabla v)d\mu=-\int_{\partial D_r}v\frac{\partial u}{\partial n}ds,$$

where n is the inner normal vector. □

The quantity

$$\sum_{k=1}^{m}\delta_k a_k=\sum_{k=1}^{m}\sum_{l=1}^{m}\delta_k\delta_l g_\Gamma(z_l,z_k,D)$$

we call the reduced modulus and denote it by $M(D,\Gamma,Z,\Delta)$. According to lemma 2.1

$$M(D,\Gamma,Z,\Delta)=\lim_{r\to 0}\left(I(u,D_r)-\left(\sum_{k=1}^{n}\delta_k^2\right)\lambda_n r^{2-n}\right).$$

Theorem 2.3 *Let sets D, Γ, collections $Z=\{z_k\}_{k=1}^m$, $\Delta=\{\delta_k\}_{k=1}^m$, be as in the definition of the reduced modulus $M=M(D,\Gamma,\ Z,\Delta)$, $u(z)$ be the potential function for D, Γ, Z, Δ, and let $D_i\subset D$ be pairwise non-overlapping subdomains of D, Γ_i, $Z_i=\{z_{ij}\}_{j=1}^{n_i}$, $\Delta_i=\{\delta_{ij}\}_{j=1}^{n_i}$, be from the definition of the reduced moduli $M_i=M(B_i,\Gamma_i,\ Z_i,\ \Delta_i)$, $u_i(z)$ be the potential function for D_i, Γ_i, Z_i, Δ_i, $i=1,...,p$. Assume that the following conditions are fulfilled:*

1)$(D\cap\partial D_i)\subset\Gamma_i,\ i=1,...,p$;

2) $\Gamma\subset(\bigcup_{i=1}^{p}\Gamma_i)\bigcup\left[\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{p}\overline{D}_i\right)\right]$;

3)$Z=\bigcup_{i=1}^{p}Z_i$, that is each point $z_k\in Z$ coincides with some point $z_{ij}\in Z_i$ for $k=k(i,j)$ and vice versa;

4) $\delta_k=\delta_{ij}$ *for* $k=k(i,j)$.

Then the inequality

$$M \geq \sum_{i=1}^{p} M_i + \sum_{i=1}^{p} I(u - u_i, D_i) \geq \sum_{i=1}^{p} M_i$$

holds.

Proof. Consider the function

$$v(z) = \begin{cases} u_i(z), & z \in D_i, \\ 0, & z \in D \setminus \left(\bigcup_{i=1}^{p} D_i\right). \end{cases}$$

The condition 1) guarantees that the function $v(z)$ is continuous in $\overline{D} \setminus \bigcup_{k=1}^{m} \{z_k\}$. From the conditions 2) and 3) it follows that $v(z) = 0$ for $z \in \Gamma$ and in a neighbourhood of z_k, $k = 1, ..., m$, there is the expansion (5). Applying lemma 2.2, we get

$$I(v - u, D) = I(v, D_r) - I(u, D_r) - 2\sum_{k=1}^{p} \delta_k (b_k - a_k) + o(1), \; r \to 0, \qquad (7)$$

here a_k and b_k from (4) and (5) respectively. By lemma 2.1

$$I(v, D_r) = \lambda_n r^{2-n} \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij}^2 + \sum_{i=1}^{p} M_i + o(1) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + \sum_{i=1}^{p} M_i + o(1),$$

$$I(u, D_r) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + M + o(1), \qquad r \to 0,$$

taking into account 3), we have

$$\sum_{k=1}^{m} \delta_k (b_k - a_k) = \sum_{i=1}^{p} M_i - M.$$

Substituting the obtained correlations in (7), we see that the inequality

$$\sum_{i=1}^{p} I(u - u_i, D_i) \leq I(v - u, D) = M - \sum_{i=1}^{p} M_i + o(1), \qquad r \to 0,$$

is true. Theorem is proved. □

Theorem 2.4 *Let sets D, Γ, collections $Z = \{z_k\}_{k=1}^{m}$, $\Delta = \{\delta_k\}_{k=1}^{m}$, be as in the definition of the reduced modulus $M := M(D, \Gamma, Z, \Delta)$, $u(z)$ be the potential function for D, Γ, Z, Δ, and let $D_i \subset D$, $i = 1, ..., p$, be pairwise non-overlapping domains, Γ_i, $Z_i = \{z_{ij}\}_{j=1}^{n_i}$, $\Delta_i = \{\delta_{ij}\}_{j=1}^{n_i}$, be from the definition* *of the reduced moduli* $M_i = M(D_i, \Gamma_i, Z_i, \Delta_i)$, $u_i(z)$ *be the potential function for* D_i, Γ_i, Z_i, Δ_i, $i = 1, ..., p$. *Assume that* $\Gamma_i \subset \Gamma$, $i = 1, ..., p$, $Z = \bigcup_{i=1}^{m} Z_i$, *(that is each point* $z_k \in Z$ *coincides with some point* $z_{ij} \in Z_i$ *for* $k = k(i, j)$ *and vice versa),* $\delta_k = \delta_{ij}$. *Then the inequality*

$$\sum_{i=1}^{p} M_i \geq M + \sum_{i=1}^{p} I(u - u_i, D_i) \geq M$$

holds.

Proof. The function u is admissible for D_i, $i = 1, \ldots, p$. Let b_k be constants from the expansion of the function u in a neighbourhood of z_k, $b_{ij} = b_k$ if $k = k(i, j)$. Applying lemmata 2.1 and 2.2 with the potential functions u_k for D_k we get

$$\sum_{i=1}^{p}\sum_{j=1}^{n_i}(\delta_{ij})^2 r^{2-n}\lambda_n + \sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}a_{ij} + o(1) = \sum_{i=1}^{p} I(u_i, (D_i)_r) =$$

$$= \sum_{i=1}^{p}\left(I(u, (D_i)_r) - 2\sum_{j=1}^{n_i}\delta_{ij}(b_{ij} - a_{ij}) - I(u - u_i, (D_i)_r)\right) + o(1)$$

$$\leq I(u, D_r) - \sum_{i=1}^{p} I(u - u_i, (D_i)_r) - 2\sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}(b_{ij} - a_{ij}) + o(1)$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{n_i}(\delta_{ij})^2 r^{2-n}\lambda_n + \sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}b_{ij} - 2\sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}(b_{ij} - a_{ij})$$

$$- \sum_{i=1}^{p} I(u - u_i, (D_i)_r) + o(1), \ r \to 0.$$

It implies that

$$\sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}b_{ij} \leq \sum_{i=1}^{p}\sum_{j=1}^{n_i}\delta_{ij}a_{ij} - \sum_{i=1}^{p} I(u - u_i, D_i)$$

or equivalently

$$\sum_{i=1}^{p} I(u - u_i, D_i) + M(D, \Gamma, Z, \Delta) \leq \sum_{i=1}^{p} M(D_i, \Gamma_i, Z_i, \Delta_i).$$

Here we used the fact that the function $u - u_i$ has no singularity in D_i. □

Denote by $r(D_l, x_l) = r(D_l, x_l, \partial D)$ the harmonic radius. Directly from theorem 2.3 we get theorem 2 of the paper [5]

Corollary 2.5 *For any non-overlapping domains $D_l \subset \mathbb{R}^n$, $n \geq 3$, points $x_l \in D_l$ and real numbers δ_l, $l = 1, \ldots, m$ the inequality*

$$-\sum_{l=1}^{m} \delta_l^2 r\left(D_l, x_l\right)^{2-n} \leq \sum_{l=1}^{m} \sum_{\substack{p=1 \\ p \neq l}}^{m} \delta_l \delta_p |x_l - x_p|^{2-n}$$

holds true.

Proof. The Green's function of the ball $B(0, \rho)$ is

$$\lambda_n \left(|x - x_0|^{2-n} - \left| \frac{|x_0| x}{\rho} - \frac{\rho x_0}{|x_0|} \right|^{2-n} \right).$$

Denote by $D_l(\rho)$ the intersection $D_l \cap B(0, \rho)$. By theorem 2.3

$$M(\rho) \geq \sum_{l=1}^{m} M_l(\rho),$$

where $M(\rho)$ is the modulus of the ball $B(0, \rho)$, the collections $\{x_l\}_{l=1}^m$, $\Delta = \{\delta_l\}_{l=1}^m$, and $\Gamma = \partial B$,

$$M_l(\rho) = -\delta_l^2 r\left(D_l(\rho), x_l\right)^{2-n} \lambda_n.$$

It is sufficient to take a limit as $\rho \to \infty$. $\square$

Theorems 2.3 and 2.4 imply for $p = 1$ monotonicity of the quadratic form

$$\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_\Gamma(z_l, z_p, D)$$

under extension of a domain. Following [2] we will say that a domain $\tilde{D}$ is obtained by extending a domain D across a part of its boundary $\gamma \subset \partial D$ if $D \subset \tilde{D}$ and $(\partial D) \cap \tilde{D}$ lies in γ.

Corollary 2.6 *If $\tilde{D}$ is obtained by extending D across Γ, $\tilde{\Gamma} \subset \left(\Gamma \bigcup (\mathbb{R}^n \setminus \overline{D})\right)$, then for any real numbers δ_l and points $z_l \in D$*

$$\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\tilde{\Gamma}} \left(z_l, z_p, \tilde{D}\right) \geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_\Gamma \left(z_l, z_p, D\right) + I(u - \tilde{u}, D)$$

$$\geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_\Gamma \left(z_l, z_p, D\right).$$

If $\tilde{D}$ is obtained by extending D across the part of $(\partial D) \setminus \Gamma$, $\tilde{\Gamma} = \Gamma$, then

$$\sum_{l=1}^{m}\sum_{p=1}^{m}\delta_l\delta_p g_{\tilde{\Gamma}}\left(z_l, z_p, \tilde{D}\right) \leq \sum_{l=1}^{m}\sum_{p=1}^{m}\delta_l\delta_p g_{\Gamma}\left(z_l, z_p, D\right) - I(u-\tilde{u}, D)$$
$$\leq \sum_{l=1}^{m}\sum_{p=1}^{m}\delta_l\delta_p g_{\Gamma}\left(z_l, z_p, D\right),$$

here u and $\tilde{u}$ are the potential functions for D, Γ, $Z = \{z_l\}_{l=1}^m$, $\Delta = \{\delta_l\}_{l=1}^m$ and $\tilde{D}$, $\tilde{\Gamma}$, $Z = \{z_l\}_{l=1}^m$, $\Delta = \{\delta_l\}_{l=1}^m$, respectively.

In [3] the notion of the Robin radius

$$r\left(D, z_0, \Gamma\right) = \exp \lim_{z\to z_0}\left(g_D\left(z, z_0, \Gamma\right) + \log|z - z_0|\right)$$

was introduced. This quantity generalized the notion of the conformal radius. An analogue of Kufarev's theorem (see [8]) for non-overlapping domains D_1, D_2 lying in the unit disk U under the condition $((\partial D_k) \cap U) \subset \Gamma_k \subset \partial D_k$, $a_k \in D_k$, $k = 1, 2$ is the inequality

$$r\left(D_1, a_1, \Gamma_1\right) r\left(D_2, a_2, \Gamma_2\right) \leq |a_2 - a_1|^2\left[1 - \left|\frac{a_2 - a_1}{1 - \overline{a_1}a_2}\right|^2\right]^{-1}.$$

By setting in theorem 2.3 $p = 2$, $\Gamma = \emptyset$, we obtain in $\mathbb{R}^n$ the following inequality.

Corollary 2.7 *Let D_1 and D_2 be non-overlapping and lie in the ball $U = B(0, 1)$, $a_k \in D_k$, $(\partial D_k) \cap U \subset \Gamma_k \subset \partial D_k$, $k = 1, 2$. Then*

$$-\lambda_n r\left(D_1, a_1, \Gamma_1\right)^{2-n} - \lambda_n r\left(D_2, a_2, \Gamma_2\right)^{2-n} \leq M\left(U, \emptyset, \{a_1, a_2\}, \{1, -1\}\right). \quad (8)$$

To calculate $M\left(U, \emptyset, \{a_1, a_2\}, \{1, -1\}\right)$ we need to know the Neumann function of the unit ball. Note that it is a quite comlicated problem in $\mathbb{R}^n$. In particular, for $n = 3$ (see [13])

$$g_{\emptyset}\left(x, y, U\right) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} + \frac{|y|}{|x|y|^2 - y|} - \log\left|1 - (x, y) + \frac{|x|y|^2 - y|}{|y|}\right|\right).$$

In [13] there is an analytic view of $g_{\emptyset}(D, x, y)$ for $n = 4, 5$. So, for $n = 3$ the inequality (8) has the following form

$$-r\left(D_1, a_1, \Gamma_1\right)^{-1} - r\left(D_2, a_2, \Gamma_2\right)^{-1} \leq -\frac{2}{|a_1 - a_2|} - \frac{2|a_2|}{|a_1|a_2|^2 - a_2|}$$
$$+ 2\log\left|1 - (a_1, a_2) + \frac{|a_1|a_2|^2 - a_2|}{|a_2|}\right| + \frac{1}{1 - |a_1|^2} + \frac{1}{1 - |a_2|^2}$$
$$- \log(4(1 - |a_1|^2)(1 - |a_2|^2)).$$

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