Non-screenable Half Lightlike Submanifolds
of an Indefinite Kaehler Manifold
of a Quasi-Constant Curvature

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Abstract

We study non-screenable half lightlike submanifolds of an indefinite Kaehler manifold \( \bar{M} \) of quasi-constant curvature. First, we provide a new result for such a non-screenable half lightlike submanifold \( M \). Next, we investigate a statical non-screenable half lightlike submanifold \( M \) of \( \bar{M} \) subject such that (1) the screen distribution \( S(TM) \) is totally umbilical in \( M \), or (2) \( M \) is screen homothetic.

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1 Introduction

In the classical theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a Riemannian manifold of a quasi-constant curvature as a Riemannian manifold \((\bar{M}, \bar{g})\) endowed with a curvature tensor \( \bar{R} \) of the form:

\[
\bar{R}(X,Y)Z = f_1 \{ \bar{g}(Y,Z)X - \bar{g}(X,Z)Y \} \\
+ f_2 \{ \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y \\
+ \bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta \},
\]

(1.1)
for any vector fields $X$, $Y$ and $Z$ on $\tilde{M}$, where $f_1$ and $f_2$ are smooth functions, $\zeta$ is a unit vector field which is called the \textit{characteristic vector field} of $\tilde{M}$, and $\theta$ is a 1-form associated with $\zeta$ by $\theta(X) = \bar{g}(X, \zeta)$.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. The study of such notion was initiated by Duggal-Bejancu [4] and later studied by many authors [6, 7]. Half lightlike submanifold $M$ is a lightlike submanifold of codimension 2 such that $\text{rank}\{\text{Rad}(TM)\} = 1$, where $\text{Rad}(TM) = TM \cap TM^\perp$ is the radical distribution of $M$. It is a special case of general $r$-lightlike submanifolds [4] such that $r = 1$. Its geometry is more general than that of lightlike hypersurfaces or coisotropic submanifolds which are lightlike submanifolds $M$ of codimension 2 such that $\text{rank}\{\text{Rad}(TM)\} = 2$. Much of its theory will be immediately generalized in a formal way to $r$-lightlike submanifolds.

In this paper, we study half lightlike submanifolds of an indefinite Kaehler manifold $\tilde{M}$ of a quasi-constant curvature such that $\zeta$ is non-screenable to $M$, that is, $\zeta$ belongs to the orthogonal complement $S(TM)^\perp$ of the screen distribution $S(TM)$. A half lightlike submanifold $M$ of $\tilde{M}$ with the non-screenable characteristic vector field $\zeta$ is called \textit{non-screenable}. First, we provide a new result for such a half lightlike submanifold. Next, we investigate a statical half lightlike submanifold $M$ of $\tilde{M}$ subject such that (1) the screen distribution $S(TM)$ is totally umbilical, or (2) $M$ is screen homothetic.

2 Preliminaries

Let $(M, g)$ be a codimension 2 half lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ equipped with the tangent bundle $TM$, the normal bundle $TM^\perp$, the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$, a screen distribution $S(TM)$, and a coscreen distribution $S(TM^\perp)$ such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$. Also denote by (2.6)$_1$ the first equation of the two equations in (2.6). We use same notations for any others. Choose $L \in \Gamma(S(TM^\perp))$ as a unit spacelike vector field, i.e., $\tilde{g}(L, L) = 1$, without loss of generality.

Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $TM$, of rank 3. Certainly the vector fields $\xi$ and $L$ belong to $\Gamma(S(TM^\perp))$. Hence we have the following orthogonal decomposition

$$S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$, of rank 2. It is known [5] that, for any null section $\xi$ of $\text{Rad}(TM)$, there exists a
Denote by $\mathcal{N}$ the uniquely defined null vector field $N$ in $S(TM^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})$ locally spanned by $N$. Then we show that $S(TM^{\perp}) = \text{Rad}(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{\text{orth}} ltr(TM)$. Then we call $N$, $ltr(TM)$ and $tr(TM)$ the lightlike transversal vector bundle, lightlike transversal vector bundle and transversal vector bundle of $M$ with respect to the screen distribution $S(TM)$ respectively.

From now and in the sequel, let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ the projection morphism of $TM$ on $S(TM)$. Then the local Gauss and Weingarten formulas of $M$ and $S(TM)$ are given by

\begin{align*}
\bar{\nabla}_XY &= \nabla_XY + B(X,Y)N + D(X,Y)L, \\
\bar{\nabla}_XN &= -A_NX + \tau(X)N + \rho(X)L, \\
\bar{\nabla}_XL &= -A_LX + \phi(X)N;
\end{align*}

\begin{align*}
\bar{\nabla}_XPY &= \nabla_XPY + C(X, PY)\xi, \\
\bar{\nabla}_X\xi &= -A_\xi X - \tau(X)\xi,
\end{align*}

respectively, where $\nabla$ and $\nabla^*$ are the induced connections on $TM$ and $S(TM)$ respectively, $B$ and $D$ are called the local second fundamental forms of $M$, $C$ is called the local screen second fundamental form on $S(TM)$, $A_N$, $A_\xi$ and $A_L$ are called the shape operators, and $\tau$, $\rho$ and $\phi$ are 1-forms on $TM$.

Since the connection $\nabla$ is torsion-free, its induced connection $\nabla^*$ is also torsion-free, and $B$ and $D$ are symmetric. The above local second fundamental forms of $M$ and $S(TM)$ are related to their shape operators by

\begin{align*}
B(X,Y) &= g(A_\xi^*X, Y), \\
C(X, PY) &= g(A_NX, PY), \\
D(X, Y) &= g(A_LX, Y) - \phi(X)\eta(Y),
\end{align*}

\begin{align*}
\bar{g}(A_\xi^*X, N) &= 0, \\
\bar{g}(A_NX, N) &= 0, \\
\bar{g}(A_LX, N) &= \rho(X),
\end{align*}

here $\eta$ is a 1-form such that $\eta(X) = \bar{g}(X, N)$. From (2.6)$_1$ and (2.8)$_1$, we get

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X).$$

(2.9)

Both $A_\xi$ and $A_N$ are $S(TM)$-valued, and $A_\xi^*$ is self-adjoint such that

$$A_\xi^*\xi = 0.$$  

(2.10)

The induced connection $\nabla$ of $M$ is not a metric connection and satisfies

$$(\nabla_X\bar{g})(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y).$$

(2.11)
Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of $\nabla$, $\nabla$ and $\nabla^*$ respectively. Using the Gauss-Weingarten formulas, we have Gauss-Codazzi equations:

$$\bar{R}(X, Y)Z = R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X$$
$$+ D(X, Z)A_L Y - D(Y, Z)A_L X$$  \hfill (2.12)
$$+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)$$
$$+ \tau(X)B(Y, Z) - \tau(Y)B(X, Z)$$
$$+ \phi(X)D(Y, Z) - \phi(Y)D(X, Z)\} N$$
$$+ \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + \rho(X)B(Y, Z)$$
$$- \rho(Y)B(X, Z)\} L,$nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X, Y]$$
$$+ \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X$$
$$+ \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X, Y)$$
$$+ \phi(X)\rho(Y) - \phi(Y)\rho(X)\} N$$
$$+ \{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X, Y)$$
$$+ \rho(X)\tau(Y) - \rho(Y)\tau(X)\} L,$$  \hfill (2.13)
$$R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A^*_{\xi} Y - C(Y, PZ)A^*_{\xi} X$$
$$+ \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)$$
$$- \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)\} \xi,$nabla_X(A^*_{\xi} Y) + \nabla_Y(A^*_{\xi} X) + A^*_{\xi}[X, Y]$$
$$- \tau(X)A^*_{\xi} Y + \tau(Y)A^*_{\xi} X$$
$$+ \{C(Y, A^*_{\xi} X) - C(X, A^*_{\xi} Y) - 2d\tau(X, Y)\} \xi.$$  \hfill (2.14)
$$R(X, Y) = -\nabla_X(A^*_{\xi} Y) + \nabla_Y(A^*_{\xi} X) + A^*_{\xi}[X, Y]$$
$$- \tau(X)A^*_{\xi} Y + \tau(Y)A^*_{\xi} X$$
$$+ \{C(Y, A^*_{\xi} X) - C(X, A^*_{\xi} Y) - 2d\tau(X, Y)\} \xi.$$  \hfill (2.15)

In case $R = 0$, we say that $M$ is flat.

The Ricci tensor of $\bar{M}$, denote it by $\bar{Ric}$, is defined by

$$\bar{Ric}(X, Y) = trace\{Z \rightarrow \bar{R}(X, Z)Y\}, \forall X, Y, Z \in \Gamma(TM).$$

Denote by $R^{(0, 2)}$ the induced tensor of type $(0, 2)$ on $M$ such that

$$R^{(0, 2)}(X, Y) = trace\{Z \rightarrow R(X, Z)Y\}, \forall X, Y, Z \in \Gamma(TM).$$  \hfill (2.16)

Using (2.6)~(2.8) and the Gauss equation (2.12), we get

$$R^{(0, 2)}(X, Y) = \bar{Ric}(X, Y) + B(X, Y)tr A_N + D(X, Y)tr A_L$$
$$- g(A_N X, A^*_{\xi} Y) - g(A_L X, A^*_{\xi} Y) + \rho(X)\phi(Y)$$
$$- g(\bar{R}(\xi, Y)X, N) - g(\bar{R}(L, Y)X, L).$$  \hfill (2.17)

Using the lightlike transversal part of (2.13) and the Bianchi’s identity, we get

$$R^{(0, 2)}(X, Y) - R^{(0, 2)}(Y, X) = 2d\tau(X, Y).$$
This shows that, in general, \( R^{(0,2)} \) is not symmetric. A tensor field \( R^{(0,2)} \) of \( M \), given by (2.16), is called its \textit{induced Ricci tensor} and denote it by \( Ric \) if it is symmetric. In this case, \( M \) is called \textit{Ricci flat} if \( Ric = 0 \). \( M \) is called an \textit{Einstein manifold} if there exists a smooth function \( \kappa \) such that

\[
Ric = \kappa g.
\]  

(2.18)

Let \( \nabla^\ell X_N = \pi(\nabla_X N) \), where \( \pi \) is the projection morphism of \( T\bar{M} \) on \( ltr(TM) \). Then \( \nabla^\ell \) is a linear connection on \( ltr(TM) \). We say that \( \nabla^\ell \) is the \textit{lightlike transversal connection} of \( M \). We define a curvature tensor \( \tensor{R}{^\ell} \) by

\[
\tensor{R}{^\ell}(X,Y)N = \nabla^\ell_X \nabla^\ell_Y N - \nabla^\ell_Y \nabla^\ell_X N - \nabla^\ell_{[X,Y]} N.
\]

If \( \tensor{R}{^\ell} \) vanishes identically, then the lightlike transversal connection \( \nabla^\ell \) is said to be \textit{flat}. We quote the following result (see [10, 11]).

**Theorem 2.1.** Let \( M \) be a half lightlike submanifold of a semi-Riemannian manifold \( (\bar{M}, \bar{g}) \). The following statements are equivalent:

(i) The lightlike transversal connection of \( M \) is flat, i.e., \( \tensor{R}{^\ell} = 0 \).

(ii) The 1-form \( \tau \) is closed, i.e., \( d\tau = 0 \), on any \( U \subset M \).

(iii) The tensor field \( R^{(0,2)} \) of \( M \) is an induced Ricci tensor of \( M \).

**Note 1.** \( d\tau \) is independent to the choice of the section \( \xi \in \Gamma(TM^\bot) \). Indeed, suppose \( \tau \) and \( \bar{\tau} \) are 1-forms with respect to the sections \( \xi \) and \( \bar{\xi} \), respectively, by directed calculation, we show that \( d\tau = d\bar{\tau} \) [5]. In case \( d\tau = 0 \), by the cohomology theory, there exists a smooth function \( f \) such that \( \tau = df \). Thus \( \tau(X) = X(f) \). If we take \( \bar{\xi} = \lambda \xi \), it follows that \( \tau(X) = \bar{\tau}(X) + X(\ln \lambda) \). Setting \( \lambda = \exp(f) \) in this equation, we get \( \bar{\tau}(X) = 0 \). Thus if \( d\tau = 0 \), then we can take a 1-form \( \tau \) such that \( \tau = 0 \). We call the pair \( \{\xi, N\} \) such that the corresponding 1-form \( \tau \) vanishes the \textit{canonical null pair} of \( M \).

### 3 Non-screenable half lightlike submanifolds

Let \( \bar{M} = (\bar{M}, \bar{g}, J) \) be an indefinite Kaehler manifold, where \( \bar{g} \) is a semi-Riemannian metric and \( J \) is an almost complex structure such that

\[
J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\nabla_X J)Y = 0,
\]

(3.1)

for all \( X, Y \in \Gamma(T\bar{M}) \). Let \( (M, g) \) be a half lightlike submanifold of \( \bar{M} \), where \( g \) is a degenerate metric on \( M \) induced by \( \bar{g} \). Due to [8, 9], we choose a screen distribution \( S(TM) \) such that \( J(Rad(TM)), J(ltr(TM)) \) and \( J(S(TM^\bot)) \) are vector subbundles of \( S(TM) \). In this case, \( S(TM) \) is expressed as follow:

\[
S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{\text{orth}} J(S(TM^\bot)) \oplus_{\text{orth}} H_o,
\]
where \( H_0 \) is a non-degenerate and almost complex distribution with respect to \( J \), i.e., \( J(H_0) = H_0 \). The tangent bundle \( TM \) is decomposed as follow:

\[
TM = H \oplus J(ltr(TM)) \oplus_{\text{orth}} J(S(TM^\perp)),
\]

(3.2)

where \( H \) is a 2-lightlike almost complex distribution on \( M \) such that

\[
H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_0.
\]

Consider two null and one spacelike vector fields \( \{U, V\} \) and \( W \) such that

\[
U = -JN, \quad V = -J\xi, \quad W = -JL,
\]

(3.3)

respectively. Denote by \( S \) the projection morphism of \( TM \) on \( H \). By (3.2), for any vector field \( X \) on \( M \), the vector field \( JX \) is decomposed as

\[
JX = FX + u(X)N + w(X)L,
\]

(3.4)

where \( u, v \) and \( w \) are 1-forms locally defined on \( M \) by

\[
u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W),
\]

(3.5)

and \( F \) is a tensor field of type \((1, 1)\) globally defined on \( M \) by

\[
F = J \circ S.
\]

Applying \( \nabla_X \) to (3.3) and using the Gauss-Weingarten formulas, we have

\[
B(X, U) = C(X, V), \quad C(X, W) = D(X, U),
\]

(3.6)

\[
D(X, V) = B(X, W),
\]

\[
\nabla_X U = F(A_\xi X) + \tau(X)U + \rho(X)W, \quad \nabla_X V = F(A_\xi X) - \tau(X)V - \phi(X)W,
\]

(3.7)

\[
\nabla_X W = F(A_L X) + \phi(X)U.
\]

(3.8)

\[
\nabla_X W = F(A_L X) + \phi(X)U.
\]

(3.9)

**Definition 1.** We say that \( \zeta \) is *non-screenable* to \( M \) if it belongs to the orthogonal complement \( S(TM)^\perp \) of the screen distribution \( S(TM) \). In this case, \( M \) is called an non-screenable half lightlike submanifold.

As \( S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} \text{Rad}(TM) \oplus ltr(TM) \), \( \zeta \) is decomposed as

\[
\zeta = eL + \alpha \xi + \beta N,
\]

(3.10)

where \( e, \alpha \) and \( \beta \) are smooth functions such that \( e = \theta(L) \), \( \alpha = \theta(N) \) and \( \beta = \theta(\xi) \). From (3.10) and the fact that \( \bar{g}(\zeta, \zeta) = 1 \), we see that

\[
e^2 + 2\alpha \beta = 1, \quad \theta(X) = \beta \eta(X), \quad \theta(PX) = 0
\]

and \( (e, \alpha, \beta) \neq (0, 0, 0) \). In case \( e = 0 \), \( M \) is called an ascreenable half lightlike submanifold. In case \( \alpha = 0 \), \( M \) is called a transversal half lightlike submanifold.
Theorem 3.1. Let \( M \) be an non-screenable half lightlike submanifold of an indefinite Kaehler manifold \( \bar{M} \) of a quasi-constant curvature. Then \( f_1 = 0 \) and \( f_2\theta = 0 \). Therefore the curvature tensor \( \bar{R} \) of \( \bar{M} \) satisfies \( \bar{R} = 0 \) on \( \bar{M} \).

Proof. Comparing the tangential, lightlike transversal and co-screen components of the two equations (1.1) and (2.12), we obtain

\[
R(X, Y)Z = \frac{f_1}{f_2}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + f_2\{[\theta(Y)X - \theta(X)Y]\theta(Z) + \alpha[g(Y, Z)\theta(X) - g(X, Z)\theta(Y)]\xi\}
\]

\((\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\) (3.11)

\[
+ \phi(X)D(Y, Z) - \phi(Y)D(X, Z)
= \beta f_2\{g(Y, Z)\theta(X) - g(X, Z)\theta(Y)\}.
\]

\((\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + \rho(X)B(Y, Z) - \rho(Y)B(X, Z)\) (3.12)

\[
= \epsilon f_2\{g(Y, Z)\theta(X) - g(X, Z)\theta(Y)\}.
\]

Taking the scalar product with \( N \) to (2.14), we have

\[
g(R(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ).
\]

Substituting (3.11) into the last equation and using (2.7) and (2.8), we get

\[
(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \quad (3.13)
- \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)
- \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ)
= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}
+ \alpha f_2\{\theta(X)g(Y, PZ) - \theta(Y)g(X, PZ)\}.
\]

Applying \( \nabla_X \) to (3.6) \( _1 \): \( B(Y, U) = C(Y, V) \), we have

\[
(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) + g(A_N Y, \nabla_X V) - g(A_N Y, \nabla_X U).
\]

Using (3.1), (3.4) and (3.6)~(3.8), the last equation is reduced to

\[
(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) - \phi(X)D(Y, U) - \rho(X)D(Y, V)
- g(A_N^2 X, F(A_N Y)) - g(A_N^2 Y, F(A_N X)).
\]
Substituting this equation into (3.12) such that $Z = U$, we get
\[
(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) + \tau(Y)C(X, V) \\
- \rho(X)D(Y, V) + \rho(Y)D(X, V)
\]
\[= \beta f_2\{\theta(X)v(Y) - \theta(Y)v(X)\}.
\]
Comparing this equation with (3.14) such that $PZ = V$, we get
\[\beta f_2\{\theta(X)v(Y) - \theta(Y)v(X)\} = f_1\{\eta(X)u(Y) - \eta(Y)u(X)\} + \alpha f_2\{\theta(X)u(Y) - \theta(Y)u(X)\}.
\]
Taking $X = \xi$, $Y = V$ and $X = \xi$, $Y = U$ to (3.15) by turns, we get
\[\beta f_2 = 0, \quad f_1 + \alpha \beta f_2 = 0.
\]
From these two equations, we get $f_1 = 0$ and $\beta f_2 = 0$.

Applying $\nabla_X$ to (3.6)$_2$: $D(Y, U) = C(Y, W)$, we have
\[
(\nabla_X D)(Y, U) = (\nabla_X C)(Y, W) + \phi(Y)C(X, U) \\
+ g(A_\gamma Y, \nabla_X W) - g(A_\gamma Y, \nabla_X U).
\]
Using (3.1), (3.4) and (3.6)~(3.8), the last equation is reduced to
\[
(\nabla_X D)(Y, U) = (\nabla_X C)(Y, W) - \tau(X)C(Y, W) - \rho(X)D(Y, W) \\
- \rho(X)B(Y, U) + \phi(X)C(Y, U) - \phi(Y)C(X, U) \\
- g(A_\gamma X, F(A_\gamma Y)) - g(A_\gamma Y, F(A_\gamma X)).
\]
Substituting this equation into (3.13) such that $Z = U$, we get
\[
(\nabla_X C)(Y, W) - (\nabla_Y C)(X, W) - \tau(X)C(Y, W) + \tau(Y)C(X, W) \\
- \rho(X)D(Y, W) + \rho(Y)D(X, W)
\]
\[= ef_2\{v(Y)\theta(X) - v(X)\theta(Y)\}.
\]
Comparing this equation with (3.14) such that $f_1 = 0$ and $PZ = V$, we get
\[ef_2\{\theta(X)v(Y) - \theta(Y)v(X)\} = \alpha f_2\{\theta(X)w(Y) - \theta(Y)w(X)\}.
\]
Taking $Y = V$ and $Y = W$ to (3.16) by turns, we have
\[ef_2\theta(X) = 0, \quad \alpha f_2\theta(X) = 0, \quad \forall X \in \Gamma(TM).
\]
In case $\alpha = 0$: As $\bar{g}(\zeta, \zeta) = 1$, we have $e^2 = 1$. We may assume that $e = 1$, without loss of generality. In this case, we have $f_2\theta(X) = 0$.

In case $e = 0$: As $\bar{g}(\zeta, \zeta) = 1$, we have $2\alpha \beta = 1$. Thus we show that $\alpha \neq 0$ and $\beta \neq 0$. Taking the product with $2\beta$ to $\alpha f_2\theta(X) = 0$, we have $f_2\theta(X) = 0$.

In case $e \neq 0$ and $\alpha \neq 0$: Taking the product with $e$ to $ef_2\theta(X) = 0$, we have $e^2 f_2\theta(X) = 0$. Also taking the product with $2\beta$ to $\alpha f_2\theta(X) = 0$, we get $2\alpha \beta f_2\theta(X) = 0$. Adding the resulting two equations and using the fact that $e^2 + 2\alpha \beta = 1$, we obtain $f_2\theta(X) = 0$. Therefore, by (1.1), the curvature tensor $\bar{R}$ of $M$ satisfies $\bar{R} = 0$ on $M$. 

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4 Totally umbilical screen distribution

As \( \bar{R} = 0 \) on \( M \) by Theorem 3.1, (2.17) is reduced to

\[
R^{(0,2)}(X,Y) = B(X,Y)\text{tr} A_N + D(X,Y)\text{tr} A_L + \rho(X)\phi(Y) \tag{4.1}
\]

\[
- g(A_N X, A^*_N Y) - g(A_L X, A^*_L Y).
\]

**Definition 2.** A half lightlike submanifold \( M \) of a semi-Riemannian manifold \( (\bar{M}, \bar{g}) \) is called *statical* \([12, 13]\) if \( \bar{\nabla}_X \xi \in \Gamma(S(TM)) \) for any \( X \in \Gamma(TM) \).

From (2.3) and (2.8), we show that the above definition is equivalent to the following two conditions: \( \phi = 0 \) and \( \rho = 0 \). Note that the first condition \( \phi = 0 \) is equivalent to the conception that \( M \) is *irrotational*, i.e., \( \bar{\nabla}_X \xi \in \Gamma(TM) \) \([15]\). The second condition \( \rho = 0 \) is equivalent to the conception that \( M \) is *solenoidal*, i.e., \( A_L X \in \Gamma(S(TM)) \) \([14]\).

**Definition 3.** A screen distribution \( S(TM) \) is called *totally umbilical* \([4, 9]\) in \( M \) if there exists a smooth function \( \gamma \) such that \( A_N X = \gamma P X \), or equivalently,

\[
C(X, PY) = \gamma g(X, Y). \tag{4.2}
\]

In case \( \gamma = 0 \), we say that \( S(TM) \) is *totally geodesic* in \( M \).

**Theorem 4.1.** Let \( M \) be an irrotational non-screenable half lightlike submanifold of an indefinite Kähler manifold \( \bar{M} \) of a quasi-constant curvature. If \( S(TM) \) is totally umbilical, then the following properties are satisfied

1. \( S(TM) \) is totally geodesic and parallel distribution,
2. \( M \) is locally a product manifold \( C_\xi \times M^* \), where \( C_\xi \) is a null geodesic tangent to \( \text{Rad}(TM) \) and \( M^* \) is a leaf of \( S(TM) \),
3. the curvature tensor \( R \) of \( M \) is of the form
   \[
   R(X,Y)Z = D(Y,Z)A_L X - D(X,Z)A_L Y,
   \]
4. \( d\tau = 0 \), \( R^{(0,2)} \) is symmetric and the transversal connection is flat.
5. Moreover, if \( M \) is an Einstein manifold, then \( M \) is Ricci flat.

**Proof.** Applying \( \nabla_X \) to \( C(Y, PZ) = \gamma g(Y, PZ) \) and using (2.11), we have

\[
(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).
\]

Substituting this and (4.2) into (3.14) such that \( f_1 = 0 \), we obtain

\[
\{X\gamma - \gamma\tau(X)\}g(Y, PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X, PZ)
+ \gamma\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\}
- \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ) = 0.
\]
Replacing $Y$ by $\xi$ to this and using (2.9) and the fact that $\phi = 0$, we see that
\[
\gamma B(X, Y) + \rho(\xi) D(X, Y) = \{\xi \gamma - \gamma \tau(\xi)\} g(X, Y).
\]
Taking $Y = U$ to this equation and using (3.5), (3.6)$_1$, and (4.2), we have
\[
\gamma^2 u(X) + \gamma \rho(\xi) w(X) = \{\xi \gamma - \gamma \tau(\xi)\} v(X). \tag{4.3}
\]
(1) Replacing $X$ by $U$ to (4.3), we get $\gamma = 0$. Thus $S(TM)$ is totally geodesic. As $C = 0$, from (2.3) we see that $S(TM)$ is a parallel distribution.

(2) As $S(TM)$ is a parallel distribution, $\text{Rad}(TM)$ is also an auto-parallel distribution by (2.5) and (2.10), and $TM = \text{Rad}(TM) \oplus S(TM)$, by the decomposition theorem [3], $M$ is locally a product manifold $C_{\xi} \times M^*$, where $C_{\xi}$ is a null geodesic tangent to $\text{Rad}(TM)$ and $M^*$ is a leaf of $S(TM)$.

(3) As $f_1 = f_2 \theta = A_N = 0$, from (3.11), the curvature tensor $R$ is given by
\[
R(X, Y)Z = D(Y, Z)A_LX - D(X, Z)A_LY.
\]

(4) As $A_N = \phi = 0$, (4.1) is reduced to
\[
R^{(0, 2)}(X, Y) = D(X, Y)tr A_L - g(A_L X, A_L Y). \tag{4.4}
\]
Thus $R^{(0, 2)}$ is symmetric induced Ricci tensor of $M$. By Theorem 2.1, $d\tau = 0$ and the transversal connection is flat.

(5) As $C = 0$, using (2.8) and (3.6)$_2$, we have
\[
D(X, U) = 0, \quad A_L X = \rho(X) \xi. \tag{4.5}
\]
Substituting (2.18) into (4.4) such that $X = V$ and $Y = U$ and using (4.5), we obtain $\kappa = 0$. Therefore, $M$ is Ricci flat.

Denote by $G = J(\text{Rad}(TM)) \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_0$. Then $G$ is a complementary vector subbundle to $J(ltr(TM))$ in $S(TM)$ and we have
\[
S(TM) = J(ltr(TM)) \oplus G.
\]

**Theorem 4.2.** Let $M$ be a statical non-screenable half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature. If $S(TM)$ is totally umbilical, then $M$ is locally a product manifold $C_{\xi} \times C_{\nu} \times M^2$, where $C_{\xi}$ and $C_{\nu}$ are null geodesics tangent to $\text{Rad}(TM)$ and $J(ltr(TM))$ respectively and $M^2$ is a leaf of the distribution $G$ of $M$.

**Proof.** By (4) of Theorem 4.1, we get $d\tau = 0$. Thus we can take $\tau = 0$ by Note 1, without loss generality. Also as $A_N = \rho = 0$, from (3.7), we have
\[
\nabla_X U = 0. \tag{4.6}
\]
Thus $J(ltr(TM))$ is parallel. From (2.5) and (2.10), $Rad(TM)$ is also parallel. For any $X \in \Gamma(G)$ and $Y \in \Gamma(H_o)$, using (4.6), we derive
\[ g(\nabla_X Y, U) = 0, \quad g(\nabla_X V, U) = 0, \quad g(\nabla_X W, U) = 0. \]
Thus $G$ is also parallel. By the decomposition theorem of de Rham [3], $M$ is locally a product manifold $C_\xi \times C_\mu \times M^2$, where $C_\xi$ and $C_\mu$ are null geodesics tangent to $Rad(TM)$ and $J(ltr(TM))$ respectively and $M^2$ is a leaf of $G$.

5 Screen homothetic submanifolds

**Definition 4.** A half lightlike submanifold $M$ is called *screen homothetic* [1, 6] if there exists a non-zero constant $\varphi$ such that $A_N = \varphi A^*_\xi$, or equivalently,
\[ C(X, PY) = \varphi B(X, Y). \tag{5.1} \]

**Note 2.** As $A_N = \varphi A^*_\xi$, the form (4.1) of the tensor field $R^{0,2}$ is reduced to
\[ R^{0,2}(X, Y) = B(X, Y) tr A_N + D(X, Y) tr A_L + \rho(X) \phi(Y) \tag{5.2} \]
\[- \varphi(g(A^*_\xi X, A^*_\xi Y)) - g(A_L X, A_L Y). \]

It follows that if $M$ is statical, then $R^{0,2}$ is symmetric. Thus $d\tau = 0$ and the transversal connection is flat. As $d\tau = 0$, we can take $\tau = 0$ by Note 1.

As $\{U, V\}$ is a null basis of $J(Rad(TM)) \oplus J(ltr(TM))$, let
\[ \mu = U - \varphi V, \quad \nu = U + \varphi V, \]
$\{\mu, \nu\}$ form an orthogonal basis of $J(Rad(TM)) \oplus J(ltr(TM))$. From (2.6), (2.8), (3.6), (5.1) and the fact that $\rho = 0$, we see that
\[ B(X, \mu) = 0, \quad D(X, \mu) = 0, \quad A^*_\xi \mu = 0, \quad A_L \mu = 0. \tag{5.3} \]

Let $\mathcal{H}' = Span\{\mu\}$. Then $\mathcal{H} = H_o \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} Span\{\nu\}$ is a complementary vector subbundle to $\mathcal{H}'$ in $S(TM)$ and we have
\[ S(TM) = \mathcal{H}' \oplus_{\text{orth}} \mathcal{H}. \tag{5.4} \]

**Theorem 5.1.** Let $M$ be a statical non-screenable screen homothetic half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature. Then $M$ is locally a product manifold $C_\xi \times C_\mu \times M^2$, where $C_\xi$ and $C_\mu$ are null and non-null geodesics tangent to $Rad(TM)$ and $\mathcal{H}'$, respectively and $M^2$ is a leaf of the distribution $\mathcal{H}$ of $M$. 
Proof. Using (3.7), (3.8) and the fact that $F$ is linear operator, we have

$$\nabla_{X} \mu = 0.$$  \hfill (5.5)

This implies that $\mathcal{H}'$ is parallel. From (2.5) and (2.10), $\text{Rad}(TM)$ is also parallel. For any $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(D_{0})$, using (5.5), we derive

$$g(\nabla_{X} Y, \mu) = 0, \quad g(\nabla_{X} V, \mu) = 0, \quad g(\nabla_{X} W, \mu) = 0.$$

Thus $\mathcal{H}$ is also a parallel distribution. By the decomposition theorem [3], $M$ is locally a product manifold $C_{\xi} \times C_{\mu} \times M^{3}$, where $C_{\xi}$ and $C_{\mu}$ are null and non-null geodesics tangent to $\text{Rad}(TM)$ and $\mathcal{H}'$ respectively and $M^{3}$ is a leaf of $\mathcal{H}$.

**Theorem 5.2.** Let $M$ be a statical non-screenable screen homothetic Einstein half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. Then $M$ is Ricci flat, i.e., $\kappa = 0$.

Proof. Since $M$ is Einstein manifold, (5.2) is reduced to

$$g(A_{L} X, A_{L} Y) + \varphi g(A_{\xi}^{*} X, A_{\xi}^{*} Y) - g(A_{\xi}^{*} X, Y) tr A_{N} - g(A_{L} X, Y) tr A_{L} + \kappa g(X, Y) = 0.$$

Put $X = Y = \mu$ and using (5.3)$_{3,4}$, we have $\kappa = 0$. Thus $M$ is Ricci flat.

**References**


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