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Note on the Expected Value of a Function of a Fuzzy Variable

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Abstract

Recently, Xue et al. [Computers and Mathematics with Applications 55 (2008) 1215-1224] proposed a formula for the expected value of a function of a fuzzy variable based on the assumption that the fuzzy variable has a continuous membership function. In conclusion, they remained the case where the membership function of the fuzzy variable is discontinuous for the future research, and then expected to get similar results. Thus this note is to propose a new formula for the expected value of a function of a general fuzzy variable which is not restricted on a continuous membership function. Furthermore, we give an example which cannot be applied to the formula that Xue et al. proposed. We also use the same example given by Xue et al. to show how to apply the new formula.

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1 Introduction

Since fuzziness exists in real life, fuzzy set theory has been developed and applied in many scientific fields. Researchers quantified a fuzzy event using a fuzzy variable or a function of a fuzzy variable. The notions of expected value and the expected values of fuzzy numbers were introduced in [4], while the expected value of a fuzzy number is defined as the center of the expected interval. A more general definition was presented in [1] for the mean value of the fuzzy number. The expected value of a fuzzy variable in a credibility space was defined and applied in fuzzy programming processes (see [2]). Recently, Xue et al. [6] proposed a formula for the expected value of a function of a fuzzy variable provided that the function is monotonic, on the assumption that the fuzzy variable has a continuous membership function. In conclusion, they remained the case where the membership function of the fuzzy variable is discontinuous for the future research, and then expected to get similar results. The assumption of the continuity of membership function is very restrictive in theory and in applications in some sense. There are many simple fuzzy variables which have not continuous but piecewise continuous membership function. In this note, we give an example which cannot be applied to the formula that Xue et al. proposed and propose a simple formula for the expected value of a function of a general fuzzy variable which is not restricted on having a continuous membership function. We also use the same example given by Xue et al. to show how to apply the new formula.

2 Fuzzy variables

Let ξ be a real valued fuzzy variable defined on a possibility space $(\Theta, P(\Theta), Pos)$ where Θ is a universe, $P(\Theta)$ is the power set of Θ and Pos is a possibility measure defined on $P(\Theta)$. Its membership function μ is derived from the possibility measure through

$$\mu(x) = Pos\{\theta \in \Theta | \xi(\theta) = x\}, x \in R.$$

Definition 2.1 *([2]) Let ξ be a fuzzy variable with membership function μ. Then for any Borel set B of real numbers,*

$$Pos\{\xi \in B\} = sup_{x \in B}\mu(x),$$

$$Nec\{\xi \in B\} = 1 - sup_{x \in B^c}\mu(x),$$

$$Cr\{\xi \in B\} = (1/2)(Pos\{\xi \in B\} + Nec\{\xi \in B\}).$$

Remark 1 ([2]). Let ξ be a fuzzy variable with membership function . Then for any Borel set B of real numbers,

$$Cr\{\xi = x\} = (1/2)(\mu(x) + 1 - sup_{y \neq x}\mu(x)).$$

Definition 2.2 *([2]) Let ξ be a fuzzy variable on the credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$. The expected value $E[\xi]$ is defined as*

$$E[\xi] = \int_0^{\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite. Especially, if ξ is a nonnegative fuzzy variable (i.e., $\mathrm{Cr}\{\xi < 0\} = 0$), then $E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geq r\}dr$.

Definition 2.3 *([2]) Let ξ be a fuzzy variable on the credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$, $f : R \to R$ a function. The expected value $E[f(\xi)]$ is defined as*

$$E[f(\xi)] = \int_0^{\infty} Cr\{f(\xi) \geq r\}dr - \int_{-\infty}^{0} Cr\{f(\xi) \leq r\}dr$$

provided that at least one of the two integrals is finite.

Definition 2.4 *([3]) Let ξ be a fuzzy variable on the credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$, and $\alpha \in (0, 1]$. Then*

$$\xi'_\alpha = \inf\{x|\mu(x) \geq \alpha\} \quad \textit{and} \quad \xi''_\alpha = \sup\{x|\mu(x) \geq \alpha\}$$

are called the α-pessimistic value and the α-optimistic value of ξ, respectively.

Proposition 2.5 *([3]) Let ξ be a fuzzy variable with finite expected value $E[\xi]$, then we have*

$$E[\xi] = \frac{1}{2}\int_0^1 [\xi'_\alpha + \xi''_\alpha]d\alpha$$

where ξ'_α and ξ''_α are the α-pessimistic value and the α-optimistic value of ξ , respectively.

3 Expected value of a function of a fuzzy variable

Recently, Xue et al. [6] proposed the following formula for the expected value of a function of a fuzzy variable provided that the function is monotonic, on the assumption that the fuzzy variable has a continuous membership function.

Theorem 3.1 *([6]) Let ξ be a fuzzy variable with a continuous membership function, and $f : R \to R$ a strictly increasing function. If the Lebesque integrals*

$$\int_0^{\infty} Cr\{f(\xi) \geq r\}dr \quad \textit{and} \quad \int_{-\infty}^{0} Cr\{f(\xi) \leq r\}dr$$

are finite, then

$$E[f(\xi)] = \int_{-\infty}^{\infty} f(r)dCr\{\xi \geq r\}.$$

The assumption of the continuity of membership function is very restrictive in theory and in applications in some sense. We first consider the fuzzy variable with discontinuous membership function for which the formula in Theorem 1 cannot be applied.

Example 3.2 *Let ξ be a fuzzy variable with membership function $\mu(x)$, where*

$$\mu(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise}, \end{cases}$$

and let f be a strictly increasing function $f(x)$, where

$$f(x) = \begin{cases} x & x < 0 \\ x+1 & x \geq 0. \end{cases}$$

Then $\xi'_\alpha = 0$ and $\xi''_\alpha = 1$ and

$$Cr\{\xi \leq x\} = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

But $\int_{-\infty}^{\infty} f(r) dCr\{\xi \geq r\}$ does not exist since $f(x)$ and $Cr\{\xi \leq x\}$ have a common point 0 of discontinuity (see p24 [5]).

Thus this example shows that some fuzzy variable with discontinuous membership function cannot be apply the formula given by Xue et al. [6]. Here, we propose a simple formula for the expected value of functions of a general fuzzy variable. The assumption of continuity of the membership function is not needed.

Theorem 3.3 *Let ξ be a fuzzy variable, and $f : R \to R$ a strictly increasing function. If the Lebesque integrals*

$$\int_0^1 f(\xi'_\alpha) d\alpha \quad \text{and} \quad \int_0^1 f(\xi''_\alpha) d\alpha$$

are finite, then

$$E[f(\xi)] = \frac{1}{2} \int_0^1 [f(\xi'_\alpha) + f(\xi''_\alpha)] d\alpha.$$

Proof. By Proposition 1, it suffices to show that for every $\alpha \in (0,1)$

$$f(\xi'_\alpha) = (f(\xi))'_\alpha \text{ and } f(\xi''_\alpha) = (f(\xi))''_\alpha.$$

We will show that $f(\xi''_\alpha) = (f(\xi))''_\alpha$ and the case for $f(\xi'_\alpha) = (f(\xi))'_\alpha$ is similar. Let s be any number such that $s < (f(\xi))''_\alpha$. Then by the Definition 4, we have $Pos\{f(\xi) \geq s) \geq \alpha$ and

$$\begin{aligned} Pos\{f(\xi) \geq s\} \geq \alpha &\iff Pos\{\xi \geq f^{-1}(s)\} \geq \alpha \\ &\iff \xi''_\alpha \geq f^{-1}(s) \\ &\iff f(\xi''_\alpha) \geq s. \end{aligned}$$

Thus we have $f(\xi''_\alpha) \geq (f(\xi))''_\alpha$ and the reverse inequality is trivial by the same argument and hence we complete the proof.

Note. Similar conclusion can be drawn when f is strictly decreasing.

We consider the same example given by Xue et al. [6].

Example 3.4 *Let ξ be a trapezoidal fuzzy variable (0, 1, 3, 5) and let f be a strictly increasing function $f(x) = x + sinx$. Then*

$$\xi'_\alpha = \alpha \quad and \quad \xi''_\alpha = -2\alpha + 5$$

and by Theorem 3.3

$$\begin{aligned} E[f(\xi)] &= \frac{1}{2}\int_0^1 [f(\xi'_\alpha) + f(\xi''_\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 [\alpha + sin\alpha + -2\alpha + 5 + sin(-2\alpha + 5)]d\alpha \\ &= \frac{1}{2}[11 - 2cos1 + cos3 - cos5]. \end{aligned}$$

We consider Example 3.2 again to apply the formula in Theorem 3.3.

Example 3.5 *Let ξ be a fuzzy variable with membership function $\mu(x)$, where*

$$\mu(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & otherwise, \end{cases}$$

and let f be a strictly increasing function $f(x)$, where

$$f(x) = \begin{cases} x, & x < 0 \\ x + 1, & x \geq 0. \end{cases}$$

Then $\xi'_\alpha = 0$ and $\xi''_\alpha = 1$. Hence we have by Theorem 3.3

$$\begin{aligned} E[f(\xi)] &= \frac{1}{2}\int_0^1 [f(\xi'_\alpha) + f(\xi''_\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 [f(0) + f(1)]d\alpha \\ &= \frac{3}{2}. \end{aligned}$$

Indeed, $f(\xi)$ is a positive fuzzy variable with membership function $\mu_f(x)$, where

$$\mu_f(x) = \begin{cases} 1, & x \in [1,2] \\ 0, & otherwise, \end{cases}$$

and hence we have that

$$Cr\{f(\xi) \geq r\} = \begin{cases} 1, & x \in [0,1] \\ \frac{1}{2}, & x \in (1,2] \\ 0, & x > 2. \end{cases}$$

Thus by Definition 2.3, we immediately have $E[f(\xi)] = \frac{3}{2}$.

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