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Multiparameter Poly-Cauchy and Poly-Bernoulli Numbers and Polynomials

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Abstract

Recently, Komatsu introduced the concept of poly-Cauchy numbers and polynomials which generalize Cauchy numbers and polynomials. In this paper, we introduce new generalization of poly-Cauchy and poly-Bernoulli numbers and polynomials. Also, we introduce new generalizations of Cauchy numbers and polynomials. Moreover, we derive some identities involving the new numbers and polynomials and some types of Stirling numbers. These give generalization of some relations of poly-Cauchy and poly-Bernoulli numbers and polynomials. Furthermore, we obtain some relations between the multiparameter poly-Cauchy numbers and polynomials and new multiparameter poly-Bernoulli numbers and polynomials.

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1 Introduction

Comtet [6] introduced two kinds of Cauchy numbers: The first kind is given by

$$C_n = \int_0^1 (x)_n dx, \quad n \in \mathbb{Z}_{\geq 0} \tag{1}$$

and the second kind is given by

$$\hat{C}_n = \int_0^1 (-x)_n dx, \quad n \in \mathbb{Z}_{\geq 0}, \tag{2}$$

where $(x)_n = x(x-1)...(x-n+1)$. In [12], Komatsu introduced two kinds of poly-Cauchy numbers: The poly-Cauchy numbers of the first kind $C_n^{(k)}$ as a generalization of the Cauchy numbers are given by

$$C_n^{(k)} = \int_0^1 ... \int_0^1 (x_1 x_2 ... x_k)_n \, dx_1 dx_2 ... dx_k, \tag{3}$$

and the poly-Cauchy numbers of the second kind $\hat{C}_n^{(k)}$ are given by

$$\hat{C}_n^{(k)} = \int_0^1 ... \int_0^1 (-x_1 x_2 ... x_k)_n \, dx_1 dx_2 ... dx_k \ . \quad (n \in \mathbb{Z}_{n \geq 0}, k \in \mathbb{N}) \tag{4}$$

The generating function of poly-Cauchy numbers [12] is given by

$$\mathrm{Lif}_k (\ln (1+x)) = \sum_{n=0}^{\infty} C_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k},$$

is the k-th polylogarithm factorial function.
An explicit formula for $C_n^{(k)}$, see [12] is given by

$$C_n^{(k)} = \sum_{m=0}^{n} \frac{s(n,m)}{(m+1)^k}, \quad (n \geq 0, \ k \geq 1) \tag{5}$$

where $s(n,m)$ are Stirling numbers of the first kind, see [9].
Komatsu [14] introduced two kinds of poly-Cauchy numbers with a q parameter: The poly-Cauchy numbers with a q parameter of the first kind $C_{n,q}^{(k)}$ are given by

$$C_{n,q}^{(k)} = \int_0^1 ... \int_0^1 (\prod_{i=0}^{n-1} (x_1 x_2 ... x_k - iq) dx_1 dx_2 ... dx_k \ , \quad (n \geq 0, \ k \geq 1) \tag{6}$$

and the poly-Cauchy numbers with a q parameter of the second kind $\hat{C}_{n,q}^{(k)}$ are given by

$$\hat{C}_{n,q}^{(k)} = \int_0^1 \ldots \int_0^1 (\prod_{i=0}^{n-1}(-x_1 x_2 \ldots x_k - iq) dx_1 dx_2 \ldots dx_k. \, (n \geq 0, \; k \geq 1) \tag{7}$$

On the other hand, in 1997 Kaneko [11] introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k\,(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}, \tag{8}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \tag{9}$$

is the k-th polylogarithm function.
An explicit formula for $B_n^{(k)}$, see [13] is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} S(n,m) \frac{(-1)^m m!}{(m+1)^k} \quad (n \geq 0, \; k \geq 1)\,, \tag{10}$$

where $S(n,m)$ are Stirling numbers of the second kind, see [9].
In Section 2, we present multiparameter poly-Cauchy numbers of the first kind and show that some results given in [12, 14, 15] are special cases of our result. In Section 3, we define multiparameter poly-Cauchy numbers of the second kind and obtain some relationships involving different types of Stirling numbers. In Section 4, we define new generalization of Bernoulli numbers and derive some identites involving the new generalized poly-Cauchy numbers. Finally, in Section 5, we define multiparameter poly-Cauchy polynomials and multiparameter poly-Bernoulli polynomials and derive some relationships between multiparameter poly-Cauchy polynomials and multiparameter poly-Bernoulli polynomials.

2 Multiparameter Poly-Cauchy Numbers of the First Kind

Definition 1. *Let $n \geq 0$, $k \geq 1$ be integers, $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ be a sequence of real numbers and $L = (\ell_1, \ell_2, ..., \ell_k)$ be non-zero real numbers. The multiparameter poly-Cauchy numbers of the first kind $C_{n,L}^{(k)}(\overline{\alpha})$ are defined by*

$$C_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1} \int_0^{\ell_2} \ldots \int_0^{\ell_k} \prod_{i=0}^{n-1} (x_1 x_2 \ldots x_k - \alpha_i)\, dx_1 dx_2 \ldots dx_k. \tag{11}$$

We investigate some special cases:

Case 1 Setting $\alpha_i = i$, $i = 0, 1, ..., n-1$, $L = (1, 1, ..., 1)$ in (11), we have

$$C_{n,L}^{(k)}(\bar{i}) = C_n^{(k)}, \quad \bar{i} = (0, 1, ..., n-1) \tag{12}$$

where $C_n^{(k)}$ are poly-Cauchy numbers of the first kind, see [12].

Case 2 Setting $\alpha_i = iq$, $i = 0, 1, ..., n-1$ in (11), we have

$$C_{n,L}^{(k)}(\bar{i}q) = C_{n,q,L}^{(k)}, \quad \bar{i} = (0, 1, ..., n-1) \tag{13}$$

where $C_{n,q,L}^{(k)}$ are extension of poly-Cauchy numbers with a q parameter, see [15].

Case 3 Setting $\alpha_i = iq$, $i = 0, 1, ..., n-1$, $L = (1, 1, ..., 1)$ in (11), we have

$$C_{n,L}^{(k)}(\bar{i}q) = C_{n,q}^{(k)}, \quad \bar{i} = (0, 1, ..., n-1) \tag{14}$$

where $C_{n,q}^{(k)}$ are the poly-Cauchy numbers with a q parameter, see [14].

Setting $k = 1$ in (11), we define the generalized Cauchy numbers of the first kind associated with $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$, called multiparameter Cauchy numbers of the first kind, by

$$C_{n,\overline{\alpha}} = \int_0^\ell (x - \alpha_0)(x - \alpha_1) ... (x - \alpha_{n-1})\, dx. \tag{15}$$

Case 4 Setting $\alpha_i = i$, $i = 0, 1, ..., n-1$ in (15), we have

$$C_{n,\bar{i}} = C_n, \quad \bar{i} = (0, 1, ..., n-1) \tag{16}$$

where C_n are Cauchy numbers of the first kind, see [16].

Case 5 Setting $\alpha_i = iq$, $i = 0, 1, ..., n-1$ in (15), we obtain

$$C_{n,\bar{i}q} = C_{n,q}, \quad \bar{i} = (0, 1, ..., n-1) \tag{17}$$

where $C_{n,q}$ are Cauchy numbers of the first kind with a parameter q, see [14].

Multiparameter poly-Cauchy numbers of the first kind $C_{n,L}^{(k)}(\overline{\alpha})$ can be expressed in terms of different types of the Stirling numbers as follows:

Theorem 1. *For a sequence of real numbers* $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$,

$$C_{n,L}^{(k)}(\overline{\alpha}) = \sum_{m=0}^{n} \frac{s_{\overline{\alpha}}(n, m)}{(m+1)^k} (\ell_1 \ell_2 ... \ell_k)^{m+1}, \tag{18}$$

where $s_{\overline{\alpha}}(n, m)$ *are the generalized Stirling numbers of the first kind, called Comtet numbers of the first kind, see [5], are defined as*

$$(x; \overline{\alpha})_m = \sum_{i=0}^{m} s_{\overline{\alpha}}(m, i) x^i, \tag{19}$$

where $(x; \overline{\alpha})_m = \prod_{i=0}^{m-1} (x - \alpha_i)$.

Proof. Using equation (11) and (19), hence

$$C_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n} s_{\overline{\alpha}}(n,m)(x_1x_2...x_k)^m dx_1dx_2...dx_k,$$

then we obtain (18). □

Corollary 1. *If $k=1$ in (18), we have the following relationship*

$$C_{n,\overline{\alpha},\ell} = \sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(n,m)}{m+1}\ell^{m+1}, \tag{20}$$

between generalized Cauchy numbers of the first kind and generalized Stirling numbers of first kind.

Theorem 2. *For $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$, we have*

$$C_{n,L}^{(k)}(\overline{\alpha}) = \sum_{j=0}^{n}\sum_{m=j}^{n}\frac{S(n,m;\overline{\alpha})\,s(m,j)}{(j+1)^k}(\ell_1\ell_2...\ell_k)^{j+1}, \tag{21}$$

which gives a relationship of multiparameter poly-Cauchy numbers of the first kind in terms of the multiparameter non-central Stirling numbers of the second kind and Stirling numbers of the first kind, see [3, 8].

Proof. Using equation (11) and from the definition of multiparameter non-central Stirling numbers of the second kind, we obtain
$C_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n} S(n,m;\overline{\alpha})\,(x_1x_2...x_k)_m\, dx_1dx_2...dx_k$,
and from the definition of Stirling numbers of the first kind, we easily obtain (21). □

Corollary 2. *If $k=1$ in (21), then the generalized Cauchy numbers can be expressed in terms of the multiparameter non-central Stirling numbers of the second kind and Stirling numbers of the first kind as*

$$C_{n,\overline{\alpha},\ell} = \sum_{j=0}^{n}\sum_{m=j}^{n}\frac{S(n,m;\overline{\alpha})\,s(m,j)}{j+1}\ell^{j+1}. \tag{22}$$

Theorem 3. *For $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$, we have*

$$C_{n,L}^{(k)}(\overline{\alpha}) = \sum_{m=0}^{n} S(n,m;\overline{\alpha})C_n^{(k)}, \tag{23}$$

where $S(n,m;\overline{\alpha})$ are the multiparameter non-central Stirling numbers of the second kind and $C_n^{(k)}$ are poly-Cauchy numbers of the first kind.

Theorem 4. *An explicit formula of* $C_{n,L}^{(k)}(\overline{\alpha})$ *can be expressed as*

$$C_{n,L}^{(k)}(\overline{\alpha}) = (-1)^n \prod_{i=0}^{n-1} \alpha_i \sum_{m=0}^{n} \frac{P_m\left(-H_{n,\overline{\alpha}}^{(1)}, -H_{n,\overline{\alpha}}^{(2)}, ..., -H_{n,\overline{\alpha}}^{(m)}\right)}{(m+1)^k} (\ell_1\ell_2...\ell_k)^{m+1}, \tag{24}$$

where $P_m(x_1, x_2, ..., x_m) = \sum_{k_1+2k_2+3k_3...=m} \frac{1}{k_1!k_2!...} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} ...$ *is the modified Bell polynomial, see [4, (p.308, Definition 2)] and* $H_{n,\overline{\alpha}}^{(k)} = \sum_{j=0}^{n-1} \frac{1}{(\alpha_j)^k}$ *are the generalized harmonic numbers, see [2].*

Proof. From (11)

$$\begin{aligned} C_{n,L}^{(k)}(\overline{\alpha}) &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}(-1)^n(\alpha_0\alpha_1...\alpha_{n-1})\prod_{i=0}^{n-1}(1-\tfrac{x_1x_2...x_k}{\alpha_i})dx_1dx_2...dx_k \\ &=\int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}(-1)^n(\alpha_0\alpha_1...\alpha_{n-1})\,e^{\sum_{i=0}^{n-1}\ln\left(1-\frac{x_1x_2...x_k}{\alpha_i}\right)}dx_1dx_2...dx_k \\ &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}(-1)^n\prod_{i=0}^{n-1}\alpha_i e^{-\sum_{j=1}^{\infty}\sum_{i=0}^{n-1}\frac{1}{(\alpha_i)^j}\cdot\frac{(x_1x_2...x_k)^j}{j}}dx_1dx_2...dx_k \\ &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}(-1)^n\prod_{i=0}^{n-1}\alpha_i e^{-\sum_{j=1}^{\infty}\frac{H_{n,\overline{\alpha}}^{(j)}}{j}(x_1x_2...x_k)^j}dx_1dx_2...dx_k, \end{aligned}$$

we obtain (24). □

3 Multiparameter Poly-Cauchy Numbers of the Second Kind

Definition 2. *Let* $n \geq 0$, $k \geq 1$ *be integers,* $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ *be a sequence of real numbers and* $L = (\ell_1, \ell_2, ..., \ell_k)$ *be non-zero real numbers. The multiparameter poly-Cauchy numbers of the second kind* $\hat{C}_{n,L}^{(k)}(\overline{\alpha})$ *are defined by*

$$\hat{C}_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\prod_{i=0}^{n-1}(-x_1x_2...x_k - \alpha_i)\,dx_1dx_2...dx_k. \tag{25}$$

We investigate some special cases:

Case 1 Setting $\alpha_i = i$, $i = (0, 1, ..., n-1), L = (1, 1, ..., 1)$ in (25), we have

$$\hat{C}_{n,\bar{i},L}^{(k)} = \hat{C}_n^{(k)}, \quad \bar{i} = (0, 1, ..., n-1), \tag{26}$$

where $\hat{C}_n^{(k)}$ are poly-Cauchy numbers of the the second kind, see [12].

Case 2 Setting $\alpha_i = iq$, $i = (0, 1, ..., n-1)$ in (25), we have

$$\hat{C}_{n,\bar{i}q,L}^{(k)} = \hat{C}_{n,q,L}^{(k)}, \quad \bar{i} = (0, 1, ..., n-1), \tag{27}$$

where $\hat{C}_{n,q,L}^{(k)}$ are extension of poly-Cauchy numbers with a q parameter, see [15].

Case 3 Setting $\alpha_i = iq$, $i = (0, 1, ..., n-1)$, $L = (1, 1, ..., 1)$ in (25), we have

$$\hat{C}^{(k)}_{n,\bar{i}q,L} = \hat{C}^{(k)}_{n,q}, \quad \bar{i} = (0, 1, ..., n-1), \tag{28}$$

where $\hat{C}^{(k)}_{n,q}$ are poly-Cauchy numbers with a q parameter, see [14].
Also, we define the generalized Cauchy numbers of the second kind associated with $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$, called multiparameter Cauchy numbers of the second kind, by

$$\hat{C}_{n,\overline{\alpha}} = \int_0^{\ell} (-x-\alpha_0)(-x-\alpha_1) ... (-x-\alpha_{n-1})\, dx. \tag{29}$$

Case 4 Setting $\alpha_i = i$, $i = 0, 1, ..., n-1$ in (29), we have

$$\hat{C}_{n,\bar{i}} = \hat{C}_n, \quad \bar{i} = (0, 1, ..., n-1), \tag{30}$$

where $\hat{C}_n$ are Cauchy numbers of the second kind, see [16].
Case 5 Setting $\alpha_i = iq$, $i = 0, 1, ..., n-1$ in (29), we obtain

$$\hat{C}_{n,\bar{i}q} = \hat{C}_{n,q}, \quad \bar{i} = (0, 1, ..., n-1), \tag{31}$$

where $\hat{C}_{n,q}$ are Cauchy numbers of the second kind with a parameter q, see [14].

Theorem 5. *$\hat{C}^{(k)}_{n,L}(\overline{\alpha})$ can be expressed in terms of the signless generalized Stirling numbers of the first kind as*

$$\hat{C}^{(k)}_{n,L}(\overline{\alpha}) = \sum_{m=0}^{n} \frac{(-1)^n |s_{\overline{\alpha}}(n,m)| \,(\ell_1 \ell_2 ... \ell_k)^{m+1}}{(m+1)^k}, \tag{32}$$

where $|s_{\overline{\alpha}}(n,m)|$ are the signless generalized Stirling numbers of first kind, see [5].

Proof. Using equation (25), from the definition of the signless generalized Stirling numbers of the first kind, we obtain (32). □

Corollary 3. *If $k = 1$ in Theorem 3.1, we have*

$$\hat{C}_{n,\overline{\alpha}} = \sum_{m=0}^{n} \frac{(-1)^n |s_{\overline{\alpha}}(n,m)| \,(\ell)^{m+1}}{m+1}, \tag{33}$$

which gives the generalized Cauchy numbers of the second kind in terms of the signless generalized Stirling numbers of the first kind.

Theorem 6. *For* $\overline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ *, then the multiparameter poly-Cauchy numbers of the second kind can be expressed in terms of the multiparameter non-central Stirling numbers of the first kind, Lah numbers, see [17, p. 5] and poly-Cauchy numbers of the first kind as follows*

$$\hat{C}_{n,L}(\overline{\alpha})^{(k)} = \sum_{\ell=0}^{n} \sum_{m=\ell}^{n} s\,(n, m; \overline{\alpha})\, L(m, \ell) C_{\ell}^{(k)}. \tag{34}$$

Proof. From (25) and the definition of the multiparameter non-central Stirling numbers of the first kind, we have

$$\hat{C}_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1} \int_0^{\ell_2} ... \int_0^{\ell_k} \sum_{m=0}^{n} s\,(n, m; \overline{\alpha})\, (-x_1 x_2 ... x_k)_m \, dx_1 dx_2 ... dx_k,$$

from definition of Lah numbers

$$(-x_1 x_2 ... x_k)_m = \sum_{\ell=0}^{m} L(m, \ell)(x_1 x_2 ... x_k)_\ell,$$

hence

$$\hat{C}_{n,L}^{(k)}(\overline{\alpha}) = \int_0^{\ell_1} \int_0^{\ell_2} ... \int_0^{\ell_k} \sum_{m=0}^{n} s\,(n, m; \overline{\alpha}) \sum_{\ell=0}^{m} L(m, \ell)(x_1 x_2 ... x_k)_\ell dx_1 dx_2 ... dx_k,$$

and from the definition of poly-Cauchy numbers of the first kind, yields (34). □

Corollary 4. *If* $k = 1$ *in* (34)*, we have the relationship*

$$\hat{C}_{n,\ell_1}(\overline{\alpha}) = \sum_{\ell=0}^{n} \sum_{m=\ell}^{n} s\,(n, m; \overline{\alpha})\, L(m, \ell) C_{\ell_1}, \tag{35}$$

between the generalized Cauchy numbers of the second kind, the multiparameter non-central Stirling numbers of the first kind, Lah numbers and Cauchy numbers of the first kind.

4 Multiparameter Poly-Bernoulli Numbers

We define the multiparameter poly-Bernoulli numbers $B_{n,\overline{\alpha},L}^{(k)}$ in terms of the generalized Stirling numbers of the second kind as

$$B_{n,\overline{\alpha},L}^{(k)} = \sum_{m=0}^{n} (-1)^{n-m} m! \frac{S_{\overline{\alpha}}(n, m) m!}{(m+1)^k} \left(\ell_1 \ell_2 ... \ell_k\right)^{m+1}, \tag{36}$$

where $S_{\overline{\alpha}}(n,m)$ are the generalized Stirling numbers of the second kind, see [6], are defined as

$$t^n = \sum_{k=0}^{n} S_{\overline{\alpha}}(n,k)(t;\overline{\alpha})_n.$$

Theorem 7. *The generating fuction of* $B^{(k)}_{n,\overline{\alpha},L}$ *is given by*

$$\sum_{n=0}^{\infty} B^{(k)}_{n,\overline{\alpha},L}\frac{t^n}{n!} = \sum_{j=0}^{\infty}\sum_{m=j}^{n}(-1)^m m!\frac{e^{-t\alpha_j}}{(m+1)^k(\alpha_j)_m}\left(\ell_1\ell_2...\ell_k\right)^{m+1}, \tag{37}$$

where $(\alpha_j)_m = \prod_{\substack{i=0\\ i\neq j}}^{m}(\alpha_j-\alpha_i)$.

Proof. From equation (36), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(k)}_{n,\overline{\alpha},L}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m}\frac{S_{\overline{\alpha}}(n,m)m!}{(m+1)^k}\left(\ell_1\ell_2...\ell_k\right)^{m+1}\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty}(-1)^m\frac{m!\left(\ell_1\ell_2...\ell_k\right)^{m+1}}{(m+1)^k}\sum_{n=m}^{\infty}S_{\overline{\alpha}}(n,m)\frac{(-t)^n}{n!},\end{aligned}$$

and from the generating function of the generalized Stirling numbers of the second kind, see ([6], Eq.(9)), yields (37).

□

In addition, there are some relationships between $\hat{C}^{(k)}_{n,L}(\overline{\alpha})$ and $B^{(k)}_{n,\overline{\alpha},L}$

Theorem 8. *For* $n \geq 0$, *we have*

$$\hat{C}^{(k)}_{n,L}(\overline{\alpha}) = \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^n\frac{s_{\overline{\alpha}}(m,j)|s_{\overline{\alpha}}(n,m)|}{m!}B^{(k)}_{j,\overline{\alpha},L}, \tag{38}$$

$$B^{(k)}_{n,\overline{\alpha},L} = \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}\frac{S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)}{m!}\hat{C}^{(k)}_{j,L}(\overline{\alpha}). \tag{39}$$

Proof. For the first identity, we have

$$\begin{aligned}RHS &= \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^n\frac{s_{\overline{\alpha}}(m,j)|s_{\overline{\alpha}}(n,m)|}{m!}B^{(k)}_{j,\overline{\alpha},L}\\ &= \sum_{m=0}^{n}(-1)^n\frac{|s_{\overline{\alpha}}(n,m)|}{m!}\sum_{j=0}^{n}s_{\overline{\alpha}}(m,j)\sum_{i=0}^{j}(-1)^{j-i}i!\frac{S_{\overline{\alpha}}(j,i)}{(i+1)^k}\left(\ell_1\ell_2...\ell_k\right)^{i+1}\\ &= \sum_{m=0}^{n}\frac{|s_{\overline{\alpha}}(n,m)|}{m!}\sum_{i=0}^{n}i!\frac{\left(\ell_1\ell_2...\ell_k\right)^{i+1}}{(i+1)^k}\sum_{j=i}^{n}(-1)^{j-i}s_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(j,i),\end{aligned}$$

using
$$\sum_{j=i}^{n}(-1)^{j-i}s_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(j,i)=\begin{cases}1 & \text{if } i=m\\ 0 & \text{if } i\neq m,\end{cases}$$
we obtain (38).
Similarly, we can prove (39).

□

If we put $k=1$ in Theorem 4.2, we have the following Corollary:

Corollary 5. *For $n\geq 0$, we have*
$$\hat{C}_{n,\overline{\alpha}}=\sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)|s_{\overline{\alpha}}(n,m)|}{m!}B_{j,\overline{\alpha}}, \tag{40}$$
$$B_{n,\overline{\alpha}}=\sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}\frac{S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)}{m!}\hat{C}_{j,\overline{\alpha}}. \tag{41}$$

Theorem 9. *For $n\geq 0$, we have*
$$C_{n,L}^{(k)}(\overline{\alpha})=\sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(n,m)}{m!}B_{j,\overline{\alpha},L}^{(k)}, \tag{42}$$
$$B_{n,\overline{\alpha},L}^{(k)}=\sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}\frac{S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)}{m!}C_{j,L}^{(k)}(\overline{\alpha}). \tag{43}$$

Setting $k=1$ in Theorem 4.3 , we obtain the following Corollary

Corollary 6. *For $n\geq 0$, we have*
$$C_{n,\overline{\alpha}}=\sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(n,m)}{m!}B_{j,\overline{\alpha}}, \tag{44}$$
$$B_{n,\overline{\alpha}}=\sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}\frac{S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)}{m!}C_{j,\overline{\alpha}}. \tag{45}$$

5 Multiparameter Poly-Cauchy and Multiparameter Poly-Bernoulli Polynomials

Definition 3. *Multiparameter poly-Cauchy polynomials of the first and the second kinds, respectively, are defined by*
$$C_{n,L}^{(k)}(z;\overline{\alpha})=\int_{0}^{\ell_1}\int_{0}^{\ell_2}...\int_{0}^{\ell_k}\prod_{i=0}^{n-1}(x_1x_2...x_k-\alpha_i-z)\,dx_1dx_2...dx_k, \tag{46}$$

$$\hat{C}_{n,L}^{(k)}(z;\overline{\alpha}) = \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\prod_{i=0}^{n-1}(-x_1x_2...x_k - \alpha_i + z)\,dx_1dx_2...dx_k. \quad (47)$$

Setting $k = 1$ in (46) and (47), we can define the generalized Cauchy polynomials of the first and second kind as follows

Definition 4. *Generalized Cauchy polynomials of the first and the second kind, respectively, are defined by*

$$C_{n,\overline{\alpha}}(z) = \int_0^{\ell}(x-\alpha_0-z)(x-\alpha_1-z)...(x-\alpha_{n-1}-z)\,dx, \quad (48)$$

$$\hat{C}_{n,\overline{\alpha}}(z) = \int_0^{\ell}(-x-\alpha_0+z)(-x-\alpha_1+z)...(-x-\alpha_{n-1}+z)\,dx. \quad (49)$$

Theorem 10. *i) $C_{n,\overline{\alpha},L}^{(k)}(z)$ are expressed in terms of the generalized Stirling numbers of the first kind as*

$$C_{n,L}^{(k)}(z;\overline{\alpha}) = \sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^i\binom{m}{i}\frac{s_{\overline{\alpha}}(n,m)(\ell_1\ell_2...\ell_k)^{m-i+1}}{(m-i+1)^k}(z)^i, \quad (50)$$

ii) $\hat{C}_{n,L}^{(k)}(z;\overline{\alpha})$ are expressed in terms of the signless generalized Stirling numbers of the first kind as

$$\hat{C}_{n,L}^{(k)}(z;\overline{\alpha}) = \sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^{i+n}\binom{m}{i}\frac{|s_{\overline{\alpha}}(n,m)|(\ell_1\ell_2...\ell_k)^{m-i+1}}{(m-i+1)^k}(z)^i. \quad (51)$$

Proof. For the first identity, from (46) and the definition of the generalized Stirling numbers of the first kind, we obtain

$$\begin{aligned} C_{n,L}^{(k)}(z;\overline{\alpha}) &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n}s_{\overline{\alpha}}(n,m)(x_1x_2...x_k-z)^m\,dx_1dx_2...dx_k \\ &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n}s_{\overline{\alpha}}(n,m)\sum_{i=0}^{m}\binom{m}{i}(-z)^i(x_1x_2...x_k)^{m-i}\,dx_1dx_2...dx_k \\ &= \sum_{i=0}^{n}\sum_{m=i}^{n}\binom{m}{i}(-z)^i\frac{s_{\overline{\alpha}}(n,m)(\ell_1\ell_2...\ell_k)^{m-i+1}}{(m-i+1)^k}. \end{aligned}$$

For the second identity, from (47) and the definition of the signless generalized Stirling numbers of the first kind, we obtain

$$\begin{aligned} \hat{C}_{n,\overline{\alpha},L}^{(k)}(z) &= \int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n}(-1)^n|s_{\overline{\alpha}}(n,m)|(x_1x_2...x_k-z)^m\,dx_1dx_2...dx_k \\ &= (-1)^n\int_0^{\ell_1}\int_0^{\ell_2}...\int_0^{\ell_k}\sum_{m=0}^{n}|s_{\overline{\alpha}}(n,m)|\sum_{i=0}^{m}\binom{m}{i}(-z)^i(x_1x_2...x_k)^{m-i}\,dx_1dx_2...dx_k \\ &= (-1)^n\sum_{i=0}^{n}\sum_{m=i}^{n}\binom{m}{i}(-z)^i\frac{|s_{\overline{\alpha}}(n,m)|(\ell_1\ell_2...\ell_k)^{m-i+1}}{(m-i+1)^k}. \end{aligned}$$

□

Setting $k=1$ in (50) and (51), we obtain the following Corollary.

Corollary 7. *Generalized Cauchy polynomials of the first kind $C_{n,\overline{\alpha}}(z)$ are expressed in terms of the generalized Stirling numbers of the first kind as*

$$C_{n,\overline{\alpha}}(z)=\sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^{i}\binom{m}{i}\frac{s_{\overline{\alpha}}\left(n,m\right)\left(\ell\right)^{m-i+1}}{m-i+1}z^{i}, \tag{52}$$

generalized Cauchy polynomials of first kind $\hat{C}_{n,\overline{\alpha}}(z)$ are expressed in terms of the signless generalized Stirling numbers of the first kind as

$$\hat{C}_{n,\overline{\alpha}}(z)=\sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^{i+n}\binom{m}{i}\frac{\left|s_{\bar{\alpha}}\left(n,m\right)\right|\left(\ell\right)^{m-i+1}}{m-i+1}z^{i}. \tag{53}$$

Coppo and Candelpergher [7] and Bayad and Hamahata [1] introduced the poly-Bernoulli polynomial $B_n^{(k)}(z)$ by

$$\frac{\mathrm{Li}_k\left(1-e^{-x}\right)}{1-e^{-x}}e^{-xz}=\sum_{n=0}^{\infty}B_n^{(k)}(z)\frac{t^n}{n!}$$

and

$$\frac{\mathrm{Li}_k\left(1-e^{-x}\right)}{1-e^{-x}}e^{xz}=\sum_{n=0}^{\infty}B_n^{(k)}(z)\frac{t^n}{n!}.$$

Also, Komastu [10] introduced poly-Bernoulli polynomials $B_n^{(k)}(z)$ by

$$B_n^{(k)}(z)=(-1)^n\sum_{m=0}^{n}S(n,m)(-1)^m m!\sum_{i=0}^{m}\binom{m}{i}\frac{(-z)^i}{(m-i+1)^k}.$$

Next, we introduce the multiparameter Bernoulli polynomials in terms of the generalized Stirling numbers of the second kind as

$$B_{n,\overline{\alpha},L}^{(k)}(z)=(-1)^n\sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^m m!\binom{m}{i}\frac{S_{\overline{\alpha}}(n,m)\left(\ell_1\ell_2\ldots\ell_k\right)^{m-i+1}}{(m-i+1)^k}(-z)^i. \tag{54}$$

From the last equation and the definition of the generating function of generalized Stirling numbers of the second kind, see [6], the generating function of $B_{n,\overline{\alpha},L}^{(k)}(z)$ is defined as

$$\sum_{n=0}^{\infty}B_{n,\overline{\alpha},L}^{(k)}(z)\frac{t^n}{n!}=\sum_{i=0}^{n}\sum_{m=i}^{n}\ (-1)^m\binom{m}{i}\frac{\left(\ell_1\ell_2\ldots\ell_k\right)^{m-i+1}(-z)^i}{(m-i+1)^k}\sum_{j=0}^{\infty}\frac{e^{-t\alpha_j}}{\prod_{\substack{i=0\\ i\neq j}}(\alpha_j-\alpha_i)}. \tag{55}$$

Theorem 11. *For $n \geq 0$, we have*

$$B^{(k)}_{n,\overline{\alpha},L}(z) = \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)m!C^{k}_{j,L}(z;\overline{\alpha}), \tag{56}$$

$$B^{(k)}_{n,\overline{\alpha},L}(z) = \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)m!\hat{C}^{(k)}_{j,L}(z;\overline{\alpha}). \tag{57}$$

and

$$C^{(k)}_{n,L}(z;\overline{\alpha}) = \sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(n,m)}{m!}B^{(k)}_{j,\overline{\alpha},L}(z), \tag{58}$$

$$\hat{C}^{(k)}_{n,L}(z;\overline{\alpha}) = \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n}\frac{s_{\overline{\alpha}}(m,j)|s_{\overline{\alpha}}(n,m)|}{m!}B^{(k)}_{j,\overline{\alpha},L}(z). \tag{59}$$

Proof. We prove (56) as follows. From (50), we have

$$\begin{aligned} RHS\ of\ (56) &= \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)m!C^{k}_{j,L}(z;\overline{\alpha}) \\ &= \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}S_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(n,m)m!\sum_{i=0}^{j}\sum_{l=i}^{j}\binom{l}{i}s_{\overline{\alpha}}(j,l)\frac{(\ell_1\ell_2...\ell_k)^{l-i+1}}{(l-i+1)^k}(-z)^i \\ &= \sum_{m=0}^{n}(-1)^{n-m}S_{\overline{\alpha}}(n,m)m!\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i}\frac{(\ell_1\ell_2...\ell_k)^{l-i+1}}{(l-i+1)^k}(-z)^i\sum_{j=l}^{n}S_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(j,l), \end{aligned}$$

since

$$\sum_{j=l}^{n}s_{\overline{\alpha}}(j,l)S_{\overline{\alpha}}(m,j) = \begin{cases}1 & \text{if } l=m\\ 0 & \text{if } l\neq m,\end{cases}$$

hence by (54) we obtain (56).
Similarly, we can prove (57).
We prove (58) as follows. From (54), we have

$$\begin{aligned} RHS\ of\ (57) &= \sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(n,m)}{m!}B^{(k)}_{j,\overline{\alpha},L}(z) \\ &= \sum_{j=0}^{n}\sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(m,j)s_{\overline{\alpha}}(n,m)}{m!}\sum_{i=0}^{j}\sum_{l=i}^{j}l!(-1)^{l+j}\binom{l}{i}S_{\overline{\alpha}}(j,l)\frac{(\ell_1\ell_2...\ell_k)^{l-i+1}}{(l-i+1)^k}(-z)^i \\ &= \sum_{m=0}^{n}\frac{s_{\overline{\alpha}}(n,m)}{m!}\sum_{l=0}^{n}\sum_{i=0}^{l}l!\binom{l}{i}\frac{(\ell_1\ell_2...\ell_k)^{l-i+1}}{(l-i+1)^k}(-z)^i\sum_{j=l}^{n}(-1^{l+j})s_{\overline{\alpha}}(m,j)S_{\overline{\alpha}}(j,l), \end{aligned}$$

since

$$\sum_{j=l}^{n}(-1)^{j+l}S_{\overline{\alpha}}(j,l)s_{\overline{\alpha}}(m,j)=\begin{cases}1 & \text{if } l=m\\ 0 & \text{if } l\neq m,\end{cases}$$

hence by (50) we obtain (58).
Similarly, we can prove (59). □

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