A Fuzzy Metric on the Space of Fuzzy Sets

Cai-Li Zhou
College of Mathematics and Information Science
Hebei University
Baoding 071002, P.R. China

Guo-Chun Zhang
College of Mathematics and Information Science
Hebei University
Baoding 071002, P.R. China

Copyright © 2015 Cai-Li Zhou and Guo-Chun Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Metric on the space of fuzzy sets plays a very important role in decision making and some other fuzzy application systems. The purpose of this paper is to give a fuzzy metric on the space of fuzzy numbers and investigate some of its properties.

Mathematics Subject Classification: 03E72, 54A40, 54E35

Keywords: Gradual number; Fuzzy number; Fuzzy metric

1 Introduction

How to define fuzzy metric is one of the fundamental problems in fuzzy mathematics which is wildly used in fuzzy optimization and pattern recognition. There are two approaches in this field till now. One is using fuzzy numbers to define metric on ordinary spaces, firstly proposed by Kaleva [13], following which fuzzy normed spaces, fuzzy topology induced by fuzzy metric spaces,
fixed point theorem and other properties of fuzzy metric spaces are studied by a few researchers (see e.g., [8, 11, 15, 19, 20]). The other one is using real numbers to measure the distances between fuzzy sets. The references of this approach can be referred to [4, 5, 12, 18], etc.

Recently, the authors [9] introduced a new concept in fuzzy set theory as “gradual numbers”. Gradual numbers express fuzziness only, without imprecision, which are unique generalization of real numbers and equipped with the same algebraic structures as real numbers (addition is a group, etc.). The authors claimed that the concept of gradual numbers is a missing primitive concept in fuzzy set theory. In the brief time since their introduction, gradual numbers have been researched by many authors (see, e.g., [3, 9, 10, 14, 16, 17]). In particular, a fuzzy number can be denoted as a crisp interval of gradual numbers which can be bounded by two special gradual numbers. By virtue of considering such a structural characterization of fuzzy numbers, Aiche and Dubois [1] gave new methods for ranking random fuzzy intervals. Boukezzoula and Galichet [2] combined the concepts of gradual numbers and the Midpoint-Radius representation to extend the interval proposed operators to fuzzy and gradual intervals. Hence an interesting question arises: does there exist other metric on the space of fuzzy numbers which takes on values in the set of gradual numbers like Hausdorff metric on the space of crisp intervals takes on values in the set of real numbers. This is the motivation of current work. In the present paper, we see a fuzzy number as a crisp interval of gradual numbers and generalize classical Hausdorff metric to the space of fuzzy numbers.

The organization of the paper is as follows. In Section 2, we state some basic concepts about gradual numbers and fuzzy numbers. In Section 3, we introduce the new metric and give some of its properties.

2 Preliminaries

In this section, we state some basic concepts about gradual numbers and fuzzy numbers.

Definition 2.1. [9] A gradual number \( \tilde{r} \) is defined by an assignment function

\[ A_{\tilde{r}} : (0, 1] \rightarrow \mathbb{R}. \]

Naturally a nonnegative gradual number is defined by its assignment function from \( (0, 1] \) to \( [0, +\infty) \).

In the sequel, \( \tilde{r}(\alpha) \) may be substituted for \( A_{\tilde{r}}(\alpha) \). The set of all gradual numbers (resp. nonnegative gradual numbers) is denoted by \( \mathbb{R}(I) \) (resp. \( \mathbb{R}^+(I) \)). A crisp element \( b \in \mathbb{R} \) has its own assignment function \( \tilde{b} : (0, 1] \rightarrow \mathbb{R} \) defined by \( \tilde{b}(\alpha) = b \) for each \( \alpha \in (0, 1] \). We call such elements in \( \mathbb{R}(I) \) constant
gradual numbers. In particular, 0 (resp. 1) denotes constant gradual number defined by \(0(\alpha) = 0\) (resp. \(1(\alpha) = 1\) for all \(\alpha \in (0, 1]\).

**Definition 2.2.** [22] Let \(\tilde{r}, \tilde{s} \in \mathbb{R}(I)\) and \(\gamma \in \mathbb{R}\). The operations of \(\tilde{r}\) and \(\tilde{s}\) are defined as follows:

1. \((\tilde{r} + \tilde{s})(\alpha) = \tilde{r}(\alpha) + \tilde{s}(\alpha), \forall \alpha \in (0, 1]\);
2. \((\tilde{r} - \tilde{s})(\alpha) = \tilde{r}(\alpha) - \tilde{s}(\alpha), \forall \alpha \in (0, 1]\);
3. \((\tilde{r} \cdot \tilde{s})(\alpha) = \tilde{r}(\alpha) \cdot \tilde{s}(\alpha), \forall \alpha \in (0, 1];
4. \((\tilde{r} \prec \tilde{s})(\alpha) = \tilde{r}(\alpha) \prec \tilde{s}(\alpha), \forall \alpha \in (0, 1];
5. \((\gamma \tilde{r})(\alpha) = \gamma \tilde{r}(\alpha), \forall \alpha \in (0, 1].

**Definition 2.3.** [17] Let \(\tilde{r}, \tilde{s} \in \mathbb{R}(I)\). The relations of \(\tilde{r}\) and \(\tilde{s}\) can be defined as follows:

1. \(\tilde{r} = \tilde{s}\) iff \(\tilde{r}(\alpha) = \tilde{s}(\alpha)\) for all \(\alpha \in (0, 1]\);
2. \(\tilde{r} \geq \tilde{s}\) iff \(\tilde{r}(\alpha) \geq \tilde{s}(\alpha)\) for all \(\alpha \in (0, 1]\);
3. \(\tilde{r} \leq \tilde{s}\) iff \(\tilde{r}(\alpha) \leq \tilde{s}(\alpha)\) for all \(\alpha \in (0, 1]\);
4. \(\tilde{r} \succ \tilde{s}\) iff \(\tilde{r}(\alpha) > \tilde{s}(\alpha)\) for all \(\alpha \in (0, 1]\);
5. \(\tilde{r} < \tilde{s}\) iff \(\tilde{r}(\alpha) < \tilde{s}(\alpha)\) for all \(\alpha \in (0, 1]\).

**Definition 2.4.** [7] Let \(\tilde{r}, \tilde{s} \in \mathbb{R}(I)\). The maximum and minimum operations of \(\tilde{r}\) and \(\tilde{s}\) are defined as follows:

1. \(\max\{\tilde{r}, \tilde{s}\}(\alpha) = \max\{\tilde{r}(\alpha), \tilde{s}(\alpha)\}, \forall \alpha \in (0, 1];
2. \(\min\{\tilde{r}, \tilde{s}\}(\alpha) = \min\{\tilde{r}(\alpha), \tilde{s}(\alpha)\}, \forall \alpha \in (0, 1].

**Definition 2.5.** [22] Let \(\tilde{r}\) be in \(\mathbb{R}(I)\). The mapping \(|\tilde{r}| : (0, 1] \to \mathbb{R}^*(I)\) defined by

\[|\tilde{r}|(\alpha) = |\tilde{r}(\alpha)|, \forall \alpha \in (0, 1]\]

is called the absolute value of \(\tilde{r}\).

**Definition 2.6.** [22] Let \(\{\tilde{r}_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}(I)\) and \(\tilde{r} \in \mathbb{R}(I)\).

1. \(\{\tilde{r}_n\}_{n \in \mathbb{N}}\) is said to converge to \(\tilde{r}\) if for each \(\alpha \in (0, 1]\), \(\lim_{n \to \infty} \tilde{r}_n(\alpha) = \tilde{r}(\alpha)\), and it is denoted as \(\lim_{n \to \infty} \tilde{r}_n = \tilde{r}\).

2. If \(\lim_{n \to \infty} \sum_{i=1}^{n} \tilde{r}_i\) exists, then the infinite sum of sequence \(\{\tilde{r}_n\}_{n \in \mathbb{N}}\) is defined by \(\sum_{i=1}^{\infty} \tilde{r}_i = \lim_{n \to \infty} \sum_{i=1}^{n} \tilde{r}_i\). If \(\lim_{n \to \infty} \sum_{i=1}^{n} \tilde{r}_i\) does not exist, then the infinite sum of sequence \(\{\tilde{r}_n\}_{n \in \mathbb{N}}\) is said to be divergent.

In the following, we describe some basic results for fuzzy numbers. Let \(\mathcal{K}(\mathbb{R})\) denote the family of all nonempty compact intervals on the real line \(\mathbb{R}\). Interval \(A \in \mathcal{K}(\mathbb{R})\) can be characterized in terms of its infimum \(a^-\) and supremum \(a^+\), i.e., \(A = [a^-, a^+]\). Sum of two nonempty compact intervals \(A = [a^-, a^+]\) and \(B = [b^-, b^+]\) can be defined as follows:

\[A \oplus_{\text{int}} B = [a^- + b^-, a^+ + b^+]\]
The scalar multiplication of $A = [a^-, a^+]$ and arbitrary real number $\gamma$ is defined as follows: $\gamma A = [\gamma a^-, \gamma a^+]$ if $\gamma \geq 0$ and $\gamma A = [\gamma a^+, \gamma a^-]$ if $\gamma < 0$. The Hausdorff metric between $A$ and $B$ is defined as follows:

$$d_H(A, B) = \max \{|a^--b^-|, |a^+--b^+|\}.$$

A fuzzy number is a normal, convex, upper semicontinuous and compactly supported fuzzy set on $\mathbb{R}$. In the sequel, let $\mathcal{F}_c(\mathbb{R})$ denote the family of all fuzzy numbers. According to Fortin and Dubois and Fargier [9], a fuzzy number $\tilde{A}$ can be viewed as a particular gradual interval $\tilde{A} = [\hat{a}^-, \hat{a}^+]$, where $\hat{a}^-$ and $\hat{a}^+$ are defined by $\hat{a}^-(\alpha) = \inf\{x : \hat{A}(x) \geq \alpha\}$ and $\hat{a}^+(\alpha) = \sup\{x : \hat{A}(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$, respectively. A crisp interval $A = [a^-, a^+]$ can be regarded as a degenerate fuzzy number bounded by two constant gradual numbers and a gradual number $\tilde{r}$ as a degenerate fuzzy number $\{\tilde{r}\}$.

Note that the boundaries of conventional intervals are real numbers, the boundaries of fuzzy numbers are gradual numbers. Thus, in the same way, we can define relation, sum and scalar multiplication on the space of fuzzy numbers as follows: Let $\tilde{A} = [\tilde{a}^-, \tilde{a}^+]$ and $\tilde{B} = [\tilde{b}^-, \tilde{b}^+]$ be in $\mathcal{F}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$. Define

1. $\tilde{A} = \tilde{B}$ if and only if $\tilde{a}^- = \tilde{b}^-$ and $\tilde{a}^+ = \tilde{b}^+$;
2. $\tilde{A} \preceq \tilde{B}$ if and only if $\tilde{a}^- \preceq \tilde{b}^-$ and $\tilde{a}^+ \preceq \tilde{b}^+$.
3. $\tilde{A} \oplus \tilde{B} = [\tilde{a}^- + \tilde{b}^-, \tilde{a}^+ + \tilde{b}^+]$;
4. $\gamma \tilde{A} = [\gamma \tilde{a}^-, \gamma \tilde{a}^+]$ if $\gamma \geq 0$ and $\gamma \tilde{A} = [\gamma \tilde{a}^+, \gamma \tilde{a}^-]$ if $\gamma < 0$.

For more details on gradual numbers and their relationships with fuzzy numbers, we refer the reader to [6,9,14].

3 Main Results

In order to generalize the Hausdorff metric to $\mathcal{F}_c(\mathbb{R})$, we naturally define as follows:

**Definition 3.1.** Let $\tilde{A} = [\tilde{a}^-, \tilde{a}^+]$ and $\tilde{B} = [\tilde{b}^-, \tilde{b}^+]$ be in $\mathcal{F}_c(\mathbb{R})$. Define

$$d_H(\tilde{A}, \tilde{B}) = \max \left\{|\tilde{a}^--\tilde{b}^-|, |\tilde{a}^+--\tilde{b}^+|\right\}.$$

We call $d_H$ gradual Hausdorff metric on $\mathcal{F}_c(\mathbb{R})$. In particular, we define $\|\tilde{A}\| = d_H(\tilde{A}, 0) = \max\{|\tilde{a}^-|, |\tilde{a}^+|\}$, where 0 is the fuzzy number $\{0\}$.

Obviously, if $A$ and $B$ are two crisp intervals, then $d_H(A, B)$ is nothing else but $d_H(A, B)$. If $\tilde{r}$ and $\tilde{s}$ are two gradual numbers, then $d_H(\tilde{r}, \tilde{s}) = |\tilde{r} - \tilde{s}|$. If $\tilde{r}$ and $\tilde{s}$ are constant gradual number, then $d_H(\tilde{r}, \tilde{s})$ is the Euclidean distance on $\mathbb{R}$.
Theorem 3.2. Let $\tilde{A}, \tilde{B}$ and $\tilde{C}$ be in $\mathcal{F}_c(\mathbb{R})$. Then

1. $\tilde{d}_H(\tilde{A}, \tilde{B}) \geq 0$;
2. $\tilde{d}_H(\tilde{A}, \tilde{B}) = 0$ if and only if $\tilde{A} = \tilde{B}$;
3. $\tilde{d}_H(\tilde{A}, \tilde{B}) = \tilde{d}_H(\tilde{B}, \tilde{A})$;
4. $\tilde{d}_H(\tilde{A}, \tilde{B}) \leq \tilde{d}_H(\tilde{A}, \tilde{C}) + \tilde{d}_H(\tilde{C}, \tilde{B})$.

Proof. (1) and (3) are obvious. We prove (2) and (4) as follows:

2. If $\tilde{d}_H(\tilde{A}, \tilde{B}) = 0$, then we have $|\tilde{a}^- - \tilde{b}^-| = 0$ and $|\tilde{a}^+ - \tilde{b}^+| = 0$. It follows that $\tilde{a}^- = \tilde{b}^-$ and $\tilde{a}^+ = \tilde{b}^+$, which implies that $\tilde{A} = \tilde{B}$. The converse is obvious.

4. According to Proposition 2.8 in [21], we have $|\tilde{a}^- - \tilde{b}^-| \leq |\tilde{a}^- - \tilde{c}^-| + |\tilde{c}^- - \tilde{b}^-|$ and $|\tilde{a}^+ - \tilde{b}^+| \leq |\tilde{a}^+ - \tilde{c}^+| + |\tilde{c}^+ - \tilde{b}^+|$. Then

$$\max\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\} \leq \max\{|\tilde{a}^- - \tilde{c}^-| + |\tilde{c}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{c}^+| + |\tilde{c}^+ - \tilde{b}^+|\} \leq \max\{|\tilde{a}^- - \tilde{c}^-|, |\tilde{a}^+ - \tilde{c}^+|\} + \max\{|\tilde{c}^- - \tilde{b}^-|, |\tilde{c}^+ - \tilde{b}^+|\},$$

i.e., $\tilde{d}_H(\tilde{A}, \tilde{B}) \leq \tilde{d}_H(\tilde{A}, \tilde{C}) + \tilde{d}_H(\tilde{C}, \tilde{B})$. This completes the proof.

Theorem 3.3. Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ be in $\mathcal{F}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$. Then

1. $\tilde{d}_H(\gamma \tilde{A}, \gamma \tilde{B}) = |\gamma|\tilde{d}_H(\tilde{A}, \tilde{B})$;
2. $\tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}) = \tilde{d}_H(\tilde{A}, \tilde{B})$;
3. $\tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D}) \leq \tilde{d}_H(\tilde{A}, \tilde{B}) + \tilde{d}_H(\tilde{C}, \tilde{D})$;
4. $\tilde{d}_H(\tilde{A}, \tilde{B}) \leq \|\tilde{A}\| + \|\tilde{B}\|$.

Proof. (1) If $\gamma \geq 0$, then for each $\alpha \in (0, 1]$, we have

$$\tilde{d}_H(\gamma \tilde{A}, \gamma \tilde{B})(\alpha) = \max\{|(\gamma \tilde{a}^-)(\alpha) - (\gamma \tilde{b}^-)(\alpha)|, |(\gamma \tilde{a}^+)(\alpha) - (\gamma \tilde{b}^+)(\alpha)|\} = \max\{|\gamma \tilde{a}^- - \gamma \tilde{b}^-|, |\gamma \tilde{a}^+ - \gamma \tilde{b}^+|\} = \gamma \max\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\} = \gamma \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha).$$

This implies that $\tilde{d}_H(\gamma \tilde{A}, \gamma \tilde{B}) = \gamma \tilde{d}_H(\tilde{A}, \tilde{B})$. If $\gamma < 0$, then for each $\alpha \in (0, 1]$, using the same method, we have $\tilde{d}_H(\gamma \tilde{A}, \gamma \tilde{B})(\alpha) = (-\gamma)\tilde{d}_H(\tilde{A}, \tilde{B})(\alpha)$, which implies that $d_H(\gamma \tilde{A}, \gamma \tilde{B}) = (-\gamma)\tilde{d}_H(\tilde{A}, \tilde{B})$. Hence, $\tilde{d}_H(\gamma \tilde{A}, \gamma \tilde{B}) = |\gamma|\tilde{d}_H(\tilde{A}, \tilde{B})$.

2. It is easy to see that $\tilde{A} \oplus \tilde{C} = [\tilde{a}^- + \tilde{c}^- + \tilde{a}^+ + \tilde{c}^+]$ and $\tilde{B} \oplus \tilde{C} = [\tilde{b}^- + \tilde{c}^- + \tilde{b}^+ + \tilde{c}^+]$. Then, for each $\alpha \in (0, 1]$, we have

$$\tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C})(\alpha) = \max\{|(\tilde{a}^- + \tilde{c}^-)(\alpha) - (\tilde{b}^- + \tilde{c}^-)(\alpha)|, |(\tilde{a}^+ + \tilde{c}^+)(\alpha) - (\tilde{b}^+ + \tilde{c}^+)(\alpha)|\} = \max\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\} = \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha).$$
It follows that \( \tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D}) = \tilde{d}_H(\tilde{A}, \tilde{B}) \).

(3) Since \( \tilde{A} \oplus \tilde{C} = [\tilde{a}^- + \tilde{c}^-, \tilde{a}^+ + \tilde{c}^+] \) and \( \tilde{B} \oplus \tilde{D} = [\tilde{b}^- + \tilde{d}^-, \tilde{b}^+ + \tilde{d}^+] \), by properties of classical Hausdorff metric, for each \( \alpha \in (0, 1] \), we have

\[
\tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D})(\alpha) = \max \{ |(\tilde{a}^- + \tilde{c}^-)(\alpha) - (\tilde{b}^- + \tilde{d}^-)(\alpha)|, |(\tilde{a}^+ + \tilde{c}^+)(\alpha) - (\tilde{b}^+ + \tilde{d}^+)(\alpha)| \} \\
\leq \max \{ |\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+| \} + \max \{ |\tilde{c}^- - \tilde{d}^-|, |\tilde{c}^+ - \tilde{d}^+| \} \\
= \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha) + \tilde{d}_H(\tilde{C}, \tilde{D})(\alpha) \\
= \left( \tilde{d}_H(\tilde{A}, \tilde{B}) + \tilde{d}_H(\tilde{C}, \tilde{D}) \right)(\alpha).
\]

This implies that \( \tilde{d}_H(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D}) \leq \tilde{d}_H(\tilde{A}, \tilde{B}) + \tilde{d}_H(\tilde{C}, \tilde{D}) \).

(4) According to conclusion (3), it is obvious that

\[
\tilde{d}_H(\tilde{A}, \tilde{B}) \leq \tilde{d}_H(\tilde{A}, \tilde{0}) + \tilde{d}_H(\tilde{0}, \tilde{B}),
\]

i.e., \( \tilde{d}_H(\tilde{A}, \tilde{B}) \leq \|\tilde{A}\| + \|\tilde{B}\| \). This completes the proof. \( \square \)

**Definition 3.4.** Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \) and \( \tilde{A} \in \mathcal{F}_c(\mathbb{R}) \). We call that \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) converges to \( \tilde{A} \) with respect to the gradual Hausdorff metric \( \tilde{d}_H \) if and only if

\[
\lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{A}) = 0.
\]

We denote it by \( \lim_{n \to \infty} \tilde{A}_n = \tilde{A} \) or \( \tilde{A}_n \xrightarrow{\tilde{d}_H} \tilde{A} \).

**Theorem 3.5.** Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \) and \( \tilde{A}, \tilde{B} \in \mathcal{F}_c(\mathbb{R}) \). If

\[
\lim_{n \to \infty} \tilde{A}_n = \tilde{A}, \quad \lim_{n \to \infty} \tilde{A}_n = \tilde{B},
\]

then \( \tilde{A} = \tilde{B} \).

**Proof.** If \( \lim_{n \to \infty} \tilde{A}_n = \tilde{A} \) and \( \lim_{n \to \infty} \tilde{A}_n = \tilde{B} \), then, according to Definition 2.6, for each \( \alpha \in (0, 1] \), we have

\[
\lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = 0 \quad \text{and} \quad \lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{B})(\alpha) = 0.
\]

Then for arbitrary \( \varepsilon > 0 \), there exist \( N_1 > 0 \) and \( N_2 > 0 \) such that

\[
\tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) < \frac{\varepsilon}{2} \quad \text{as} \quad n > N_1, \quad \tilde{d}_H(\tilde{A}_n, \tilde{B})(\alpha) < \frac{\varepsilon}{2} \quad \text{as} \quad n > N_2.
\]

Let \( N = \max\{N_1, N_2\} \). According to conclusion (4) in Theorem 3.2, we have

\[
0 \leq \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha) \leq \tilde{d}_H(\tilde{A}, \tilde{A})(\alpha) + \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha) < \varepsilon
\]

as \( n > N \). Because of arbitrariness of \( \varepsilon \), we have \( \tilde{d}_H(\tilde{A}, \tilde{B})(\alpha) = 0 \). It follows that \( \tilde{d}_H(\tilde{A}, \tilde{B}) = 0 \). According to conclusion (2) in Theorem 3.2, we have \( \tilde{A} = \tilde{B} \). This completes the proof. \( \square \)
Theorem 3.6. Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \). \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) converges to fuzzy number \( \tilde{A} \) with respect to \( \tilde{d}_H \) if and only if \( \tilde{a}_n^- \) converges to \( \tilde{a}^- \) and \( \tilde{a}_n^+ \) converges to \( \tilde{a}^+ \) simultaneously.

Proof. Necessity. Suppose that \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) converges to fuzzy number \( \tilde{A} \). By Definition 2.6 and Definition 3.4, for each \( \alpha \in (0, 1] \), we have
\[
\lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = 0.
\]
Since
\[
\tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = \max\{|\tilde{a}_n^- (\alpha) - \tilde{a}^- (\alpha)|, |\tilde{a}_n^+ (\alpha) - \tilde{a}^+ (\alpha)|\},
\]
then for arbitrary \( \varepsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that
\[
\tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = \max\{|\tilde{a}_n^- (\alpha) - \tilde{a}^- (\alpha)|, |\tilde{a}_n^+ (\alpha) - \tilde{a}^+ (\alpha)|\} < \varepsilon
\]
as \( n > N_0 \), which implies that \( |\tilde{a}_n^- (\alpha) - \tilde{a}^- (\alpha)| < \varepsilon \) and \( |\tilde{a}_n^+ (\alpha) - \tilde{a}^+ (\alpha)| < \varepsilon \) simultaneously as \( n > N_0 \). It follows that \( \lim_{n \to \infty} \tilde{a}_n^- = \tilde{a}^- \) and \( \lim_{n \to \infty} \tilde{a}_n^+ = \tilde{a}^+ \).

In the following, we establish the sufficiency. Suppose that \( \tilde{a}_n^- \) converges to \( \tilde{a}^- \) and \( \tilde{a}_n^+ \) converges to \( \tilde{a}^+ \) simultaneously. Then for arbitrary \( \varepsilon > 0 \) and arbitrary \( \alpha \in (0, 1] \), there exist \( N_0^- \) and \( N_0^+ \) in \( \mathbb{N} \) such that
\[
|\tilde{a}_n^- (\alpha) - \tilde{a}^- (\alpha)| < \varepsilon \text{ as } n > N_0^-
\]
and
\[
|\tilde{a}_n^+ (\alpha) - \tilde{a}^+ (\alpha)| < \varepsilon \text{ as } n > N_0^+,
\]
respectively. Let \( N_0 = \max\{N_0^-, N_0^+\} \). Then we have
\[
\tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = \max\{|\tilde{a}_n^- (\alpha) - \tilde{a}^- (\alpha)|, |\tilde{a}_n^+ (\alpha) - \tilde{a}^+ (\alpha)|\} < \varepsilon
\]
as \( n > N_0 \). It follows that \( \lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{A})(\alpha) = 0 \) for each \( \alpha \in (0, 1] \), which implies that \( \lim_{n \to \infty} \tilde{d}_H(\tilde{A}_n, \tilde{A}) = 0 \). This completes the proof. \( \square \)

Corollary 3.7. Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \). If \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) converges to fuzzy number \( \tilde{A} \) with respect to \( \tilde{d}_H \), then
\[
\lim_{n \to \infty} \tilde{A}_n = \left[ \lim_{n \to \infty} \tilde{a}_n^-, \lim_{n \to \infty} \tilde{a}_n^+ \right].
\]

Proof. The result follows from Theorem 3.6 immediately. \( \square \)

In the following, we introduce the concepts of sums for a series of fuzzy numbers.

Definition 3.8. Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \) and \( \tilde{S}_n = \bigoplus_{i=1}^n \tilde{A}_i \) the partial sum of sequence \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \). If the sequence \( \{\tilde{S}_n\}_{n \in \mathbb{N}} \) converges with respect to \( \tilde{d}_H \), then the infinite sum \( \bigoplus_{n=1}^{\infty} \tilde{A}_n \) of sequence \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) is said to be convergent and we also write
\[
\bigoplus_{n=1}^{\infty} \tilde{A}_n = \lim_{n \to \infty} \tilde{S}_n = \lim_{n \to \infty} \bigoplus_{i=1}^n \tilde{A}_i,
\]
i.e., if \( \bigoplus_{n=1}^{\infty} \tilde{A}_n = \tilde{A} \), then \( \lim_{n \to \infty} \tilde{d}_H(\bigoplus_{i=1}^n \tilde{A}_i, \tilde{A}) = 0 \).
Theorem 3.9. Let \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \). If the infinite sum of \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \) exists, then we have
\[
\bigoplus_{n=1}^{\infty} \tilde{A}_n = \left[ \sum_{n=1}^{\infty} \tilde{a}_n^-, \sum_{n=1}^{\infty} \tilde{a}_n^+ \right].
\]

Proof. The result follows from Definition 2.6, Definition 3.8 and Corollary 3.7 immediately.

In the following we generalize support function of classical intervals to fuzzy numbers as follows:

Definition 3.10. Let \( \tilde{A} = [\tilde{a}^-, \tilde{a}^+] \) be in \( \mathcal{F}_c(\mathbb{R}) \). The gradual number-valued support function \( \tilde{s}(x, \tilde{A}) \) of \( \tilde{A} \) is defined as follows:
\[
\tilde{s}(p, \tilde{A}) = \sup\{ p \tilde{r} : \tilde{r} \in \tilde{A} \}, \forall p \in \mathbb{R}.
\]
Obviously,
\[
\tilde{s}(p, \tilde{A}) = \begin{cases} 
p \tilde{a}^+ & \text{if } p \geq 0, 
p \tilde{a}^- & \text{if } p < 0.
\end{cases}
\]
In particular, \( \tilde{s}(p, \tilde{A}) = \tilde{a}^+ \) if \( x = 1 \) and \( \tilde{s}(p, \tilde{A}) = -\tilde{a}^- \) if \( x = -1 \). Thus, we have
\[
d_H(\tilde{A}, \tilde{B}) = \max \left\{ |\tilde{s}(p, \tilde{A}) - \tilde{s}(p, \tilde{B})| : |p| = 1 \right\}, \forall \tilde{A}, \tilde{B} \in \mathcal{F}_c(\mathbb{R}).
\]

Theorem 3.11. If \( \tilde{A}, \tilde{B} \in \mathcal{F}_c(\mathbb{R}) \), \( \{\tilde{A}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_c(\mathbb{R}) \) and \( \bigoplus_{n=1}^{\infty} \tilde{A}_n \) exists, then
\[
(1) \quad \tilde{s}(p, \tilde{A} + \tilde{B}) = \tilde{s}(p, \tilde{A}) + \tilde{s}(p, \tilde{B});
(2) \quad \tilde{s}(p, \gamma \tilde{A}) = \gamma \tilde{s}(p, \tilde{A}), \forall \gamma > 0;
(3) \quad \tilde{s}(p, \bigoplus_{n=1}^{\infty} \tilde{A}_n) = \sum_{n=1}^{\infty} \tilde{s}(p, \tilde{A}_n).
\]

Proof. (1) and (2) are obvious, we only prove (3). If \( \bigoplus_{n=1}^{\infty} \tilde{A}_n \) exists, then, by Theorem 3.9, we have
\[
\bigoplus_{n=1}^{\infty} \tilde{A}_n = \left[ \sum_{n=1}^{\infty} \tilde{a}_n^-, \sum_{n=1}^{\infty} \tilde{a}_n^+ \right].
\]
It follows that
\[
\tilde{s} \left( p, \bigoplus_{n=1}^{\infty} \tilde{A}_n \right) = \begin{cases} 
p \sum_{n=1}^{\infty} \tilde{a}_n^+ & \text{if } p \geq 0, 
p \sum_{n=1}^{\infty} \tilde{a}_n^- & \text{if } p < 0.
\end{cases}
\]
For the right part-side of equality, we have
\[
\sum_{n=1}^{\infty} \tilde{s}(p, \tilde{A}_n) = \sum_{n=1}^{\infty} p \tilde{a}_n^+ = p \sum_{n=1}^{\infty} \tilde{a}_n^+
\]
if \( p \geq 0 \) and
\[
\sum_{n=1}^{\infty} \tilde{s}(p, \tilde{A}_n) = \sum_{n=1}^{\infty} p \tilde{a}_n^- = p \sum_{n=1}^{\infty} \tilde{a}_n^-
\]
if $p < 0$. Hence, we have $\tilde{s}(p, \bigoplus_{n=1}^{\infty} \tilde{A}_n) = \sum_{n=1}^{\infty} \tilde{s}(p, \tilde{A}_n)$. This completes the proof. \qed

**Acknowledgements.** The project is supported by the National Natural Science Foundation of China (41201327), Natural Science Foundation of Hebei Province (A2013201119) and Science and Technology Bureau of Baoding City (14ZF058). Their financial support is gratefully acknowledged.

**References**


Received: January 7, 2015; Published: January 25, 2015