△-Convergence for Common Fixed Points of Two Asymptotically Nonexpansive Nonself Mappings in Hyperbolic Spaces

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Abstract

The purpose of paper is to study an explicit improved Kuhfitting iterative scheme for common fixed points of two asymptotically nonexpansive nonself mappings in hyperbolic spaces. Under a limit condition, we obtained a △-convergence theorem.

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1 Introduction

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a convex structure. The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure.

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formetric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

In 1976, the concept of $\triangle$-convergence in general metric spaces was coined by Lim [1]. In recent years, Yang and Zhao [2] studied the strong and $\triangle$-convergence theorems for totally asymptotically nonexpansive nonself-mappings in CAT(0) spaces. Wan [3] proved some $\triangle$-convergence theorems in a hyperbolic space, in which a mixed Agarwal-O’Regan-Sahu type iterative scheme for approximating a common fixed point of totally asymptotically nonexpansive mappings was constructed. Li and Liu [4] modified a classical Kuhfittig iteration algorithm in the general setup of hyperbolic space, and prove a $\triangle$-convergence theorem for an implicit iterative scheme.

In this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [5], which is more restrictive than the hyperbolic space introduced in Goebel and Kirk [6] and more general than the hyperbolic space in Reich and Shafrir [7]. Concretely, $(X,d,W)$ is called a hyperbolic space if $(X,d)$ is a metric space and $W : X \times X \times [0,1] \to X$ is a function satisfying

\begin{align*}
(1.1) &\quad d(z,W(x,y,\alpha)) \leq \alpha d(z,x) + (1-\alpha)d(z,y); \\
(1.2) &\quad d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha - \beta|d(x,y); \\
(1.3) &\quad W(x,y,1) = W(y,x,1-\alpha); \\
(1.4) &\quad d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)
\end{align*}

for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$. A nonempty subset $C$ of a hyperbolic space $X$ is convex if $W(x,y,\alpha) \in X (\forall x,y \in X)$ and $\alpha \in [0,1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric [8], Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov [9].

A hyperbolic space $X$ is uniformly convex if for $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that

$$d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r,$$

provided that $d(x,u) \leq r$, $d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$.

A map $\eta : (0, +\infty) \times (0,2] \to (0,1]$ is called modulus of uniform convexity if $\delta = \eta(r,\varepsilon)$ for given $r > 0$. Besides, $\eta$ is monotone if it decreases with $r$, that is,

$$\eta(r_2,\varepsilon) \leq \eta(r_1,\varepsilon), \forall r_2 \geq r_1.$$

Let $C$ be a nonempty subset of a metric space $(X,d)$. A mapping $T : C \to X$ is said to be nonexpansive if

$$d(Tx,Ty) \leq d(x,y), \forall x,y \in C.$$

Recall that $C$ is said to be a retraction of $X$ if there exists a continuous map $P : X \to C$ such that $Px = x, \forall x \in C$. A map $P : X \to C$ is said to be
a retraction if $P^2 = P$. Consequently, if $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$.

Let $C$ be a nonempty and closed subset of a metric space $(X, d)$, A map $P : X \to C$ is a retraction, a mapping $T : C \to X$ is said to be

(1) asymptotically nonexpansive nonself-mapping [10] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, such that

$$d(T(PT)^{-1}x, T(PT)^{-1}y) \leq k_n d(x, y), \quad \forall x, y \in C, \quad n \geq 1.$$ 

(2) uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T(PT)^{-1}x, T(PT)^{-1}y) \leq Ld(x, y), \quad \forall x, y \in C, \quad n \geq 1.$$ 

From the definitions above, we know that each nonexpansive mapping is an asymptotically nonexpansive nonself-mapping, and each asymptotically nonexpansive nonself-mapping is uniformly $L = \sup_{n \geq 1} \{k_n\}$-Lipschitzian.

To study our results in the general setup of hyperbolic spaces, we first collect some basic concepts. Let $\{x_n\}$ be a bounded sequence in hyperbolic space $X$. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}.$$ 

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\}.$$ 

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$ 

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$ 

A sequence $\{x_n\}$ in hyperbolic space $X$ is said to $\Delta$-convergence to $x \in X$, if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we call $x$ the $\Delta$-limit of $\{x_n\}$.

The purpose of paper is to study an explicit improved Kuhfitting iterative scheme for common fixed points of two asymptotically nonexpansive nonself mappings in hyperbolic spaces. Under a limit condition, we obtained a $\Delta$-convergence theorem. This is a development to the results of [4].
2 Preliminary

Lemma 2.1 [11] Let \((X, d, W)\) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity \(\eta\), and let \(C\) be a nonempty, closed and convex subset of \(X\). Then every bounded sequence \(\{x_n\}\) in \(X\) has a unique asymptotic center with respect to \(C\).

Lemma 2.2 [11,12] Let \((X, d, W)\) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity \(\eta\). Let \(x \in X\) and \(\{\beta_n\}\) be a sequence in \([a, b]\) for some \(a, b \in (0, 1)\). If \(\{x_n\}, \{y_n\}\) are sequences in \(X\) such that \(\lim sup_{n \to \infty} d(x_n, x) \leq c\), \(\lim sup_{n \to \infty} d(y_n, x) \leq c\) and \(\lim_{n \to \infty} d(W(x_n, y_n, \beta_n), x) = c\) for some \(c \geq 0\), then \(\lim_{n \to \infty} d(x_n, y_n) = 0\).

Lemma 2.3 [10] Let \(C\) be a nonempty closed convex subset of a uniformly convex hyperbolic space, and let \(\{x_n\}\) be a bounded sequence in \(C\) such that \(A(\{x_n\}) = \{p\}\) and \(r(\{x_n\}) = \rho\). If \(\{y_m\}\) is another sequence in \(C\) such that \(\lim sup_{m \to \infty} r(y_m, \{x_n\}) = \rho\), then \(\lim_{m \to \infty} y_m = p\).

Lemma 2.4 [13] Let \(\{a_n\}\) and \(\{t_n\}\) be two sequences of nonnegative real numbers satisfying the inequality \(a_{n+1} \leq a_n + t_n\) for all \(n \geq 1\). If \(\sum_{n=1}^{\infty} t_n < +\infty\), then \(\lim_{n \to \infty} a_n\) exists.

3 Main Results

Theorem 3.1 Let \(C\) be a nonempty closed convex subset of a complete uniformly convex hyperbolic space \(X\) with monotone modulus of uniform convexity \(\eta\), and \(P : X \to C\) be the nonexpansive retraction. Let \(S_1, S_2 : C \to X\) be two asymptotically nonexpansive nonself mappings with sequence \(\{k_n\}\), \(\{l_n\} \subset [1, \infty)\) such that \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\), \(\sum_{n=1}^{\infty} (l_n - 1) < \infty\) and \(F = F(S_1) \cap F(S_2) \neq \emptyset\).

Suppose that \(\{\alpha_n\}\) and \(\{\beta_n\}\) are real sequences in \([a, b]\) for some \(a, b \in (0, 1)\). Let \(\{x_n\}\) be a sequence generated by the following manner:

\[
\begin{align*}
x_1 &\in C, \\
y_n &= PW(x_n, S_2(PS_2)^{n-1}x_n, \beta_n), \\
x_{n+1} &= PW(y_n, S_1(PS_1)^{n-1}y_n, \alpha_n).
\end{align*}
\]

Then the sequence \(\{x_n\}\) \(\triangle\)-converges to a point \(q \in F\).

Proof We divide the proof into three steps.

Step 1. We prove that \(\forall p \in F, \lim_{n \to \infty} d(x_n, p)\) and \(\lim_{n \to \infty} d(x_n, F)\) exist.

Setting \(k_n = 1 + u_n, l_n = 1 + v_n, \) so \(\sum_{n=1}^{\infty} u_n < \infty\), \(\sum_{n=1}^{\infty} v_n < \infty\). Using (1.1)
and (3.1), we have that
\[ d(y_n, p) = d(PW(x_n, S_1(PS_1)^{n-1}y_n, \alpha_n), p) \]
\[ \leq d(W(y_n, S_1(PS_1)^{n-1}y_n, \alpha_n), p) \]
\[ \leq (1 - \alpha_n)d(y_n, p) + \alpha_n d(S_1(PS_1)^{n-1}y_n, p) \]
\[ \leq (1 - \alpha_n)d(y_n, p) + \alpha_n k_n d(y_n, p) \]
\[ \leq k_n d(y_n, p) \]
\[ \leq k_n l_n d(x_n, p) \]
\[ = (1 + v_n)(1 + u_n)d(x_n, p) \]
\[ \leq e^{\sum_{n=1}^{\infty}(v_n + u_n + u_nv_v)}d(x_1, p), \]

Since \( \sum_{n=1}^{\infty}(v_n + u_n + u_nv_n) < \infty \), we have \( \{d(x_{n+1}, p)\} \) is bounded, and then \( \{x_n\} \) is also bounded. It implies that there exists a constant \( M > 0 \) such that \( d(x_{n+1}, p) \leq M \) for all \( n \geq 1 \). So

\[ d(x_{n+1}, p) \leq d(x_n, p) + (v_n + u_n + u_nv_n)M. \]

Consequently, it follows from Lemma 2.4 that \( \lim_{n \to \infty} d(x_n, p) \) and \( \lim_{n \to \infty} d(x_n, F) \) exist.

Step 2. We prove that \( \lim_{n \to \infty} d(x_n, S_1x_n) = d(x_n, S_2x_n) = 0. \)

Assume that \( \lim_{n \to \infty} d(x_n, p) = c \geq 0. \) Using (3.2), we have \( d(y_n, p) \leq l_n d(x_n, p) \). Taking the limsup on both sides in this inequality, we have

\[ \limsup_{n \to \infty} d(y_n, p) \leq c. \]

In addition, \( d(S_1(PS_1)^{n-1}y_n, p) \leq k_n d(y_n, p) \), taking the limsup on both sides in this inequality, we have

\[ \limsup_{n \to \infty} d(S_1(PS_1)^{n-1}y_n, p) \leq c. \]

From (3.3), we have

\[ d(x_{n+1}, p) \leq d(W(y_n, S_1(PS_1)^{n-1}y_n, \alpha_n), p) \leq l_n k_n d(x_n, p) \]

Since \( k_n, l_n \to 1, n \to \infty \) and \( \lim_{n \to \infty} d(x_n, p) = c \), we have

\[ \lim_{n \to \infty} d(W(y_n, S_1(PS_1)^{n-1}y_n, \alpha_n), p) = c. \]
It follows from (3.4)-(3.6) and Lemma 2.2 that
\[
\lim_{n \to \infty} d(y_n, S_1(PS_1)^{n-1}y_n) = 0. \tag{3.7}
\]
In addition, \(d(S_2(PS_2)^{n-1}x_n, p) \leq l_n d(x_n, p)\), and taking the limsup on both sides in this inequality, we have
\[
\limsup_{n \to \infty} d(S_2(PS_2)^{n-1}x_n, p) \leq c. \tag{3.8}
\]
Using (3.3), we have
\[
d(x_{n+1}, p) \leq (1 - \alpha_n)d(y_n, p) + \alpha_n d(S_1(PS_1)^{n-1}y_n, p) \\
\leq (1 - \alpha_n)d(y_n, p) + \alpha_n d(S_1(PS_1)^{n-1}y_n, y_n) + \alpha_n d(y_n, p) \tag{3.9}
\leq d(y_n, p) + d(S_1(PS_1)^{n-1}y_n, y_n).
\]
taking the liminf on both sides in this inequality (3.9), by \(\lim_{n \to \infty} d(x_n, p) = c\) and (3.7), we have
\[
\liminf_{n \to \infty} d(y_n, p) \geq c. \tag{3.10}
\]
It follows from (3.4) and (3.10) that \(\lim_{n \to \infty} d(y_n, p) = c\). Using (3.2), this implies that
\[
d(y_n, p) \leq d(W(x_n, S_2(PS_2)^{n-1}x_n, \beta_n), p) \leq l_n d(x_n, p), \tag{3.11}
\]
and so
\[
\lim_{n \to \infty} d(W(x_n, S_2(PS_2)^{n-1}x_n, \beta_n), p) = 0.
\]
From Lemma 2.2, we obtain
\[
\lim_{n \to \infty} d(x_n, S_2(PS_2)^{n-1}x_n) = 0. \tag{3.12}
\]
From \(y_n = PW(x_n, S_2(PS_2)^{n-1}x_n, \beta_n)\) and (3.12), we have
\[
d(x_n, y_n) = d(x_n, PW(x_n, S_2(PS_2)^{n-1}x_n, \beta_n)) \\
\leq d(x_n, W(x_n, S_2(PS_2)^{n-1}x_n, \beta_n)) \\
\leq (1 - \beta_n)d(x_n, x_n) + \beta_n d(x_n, S_2(PS_2)^{n-1}x_n) \tag{3.13}
\leq d(x_n, S_2(PS_2)^{n-1}x_n) \to 0(n \to \infty).
\]
In addition,
\[
d(x_n, S_1(PS_1)^{n-1}x_n) \leq d(x_n, y_n) + d(y_n, S_1(PS_1)^{n-1}y_n) \\
+d(S_1(PS_1)^{n-1}y_n, S_1(PS_1)^{n-1}x_n) \\
\leq d(x_n, y_n) + d(y_n, S_1(PS_1)^{n-1}y_n) + k_n d(y_n, x_n). \tag{3.14}
\]
Thus, it follows from (3.7) and (3.13) that
\[
\lim_{n \to \infty} d(x_n, S_1(PS_1)^{n-1}x_n) = 0. \tag{3.15}
\]
Using (3.1), we obtain that
\[ d(x_n, x_{n+1}) = d(x_n, PW(y_n, S_1(P S_1)^{y_n-1}y_n, \alpha_n)) \]
\[ \leq d(x_n, W(y_n, S_1(P S_1)^{y_n-1}y_n, \alpha_n)) \]
\[ \leq (1 - \alpha_n)d(x_n, y_n) + \alpha_n d(x_n, S_1(P S_1)^{y_n-1}y_n) \]
\[ \leq (1 - \alpha_n)d(x_n, y_n) + \alpha_n d(x_n, y_n) + \alpha_n d(y_n, S_1(P S_1)^{y_n-1}y_n) \]
\[ \leq d(x_n, y_n) + \alpha_n d(y_n, S_1(P S_1)^{y_n-1}y_n)). \]  
(3.16)

It follows from (3.7) and (3.13) that
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{3.17} \]

Let \( L = \sup\{k_n : n \geq 1\}, S_1 \) is uniformly \( L \)-Lipschitzian. Denote as \((P S_1)^{1-1}\)
the identity maps from \( C \) onto itself. Thus by the inequality (3.15) and (3.17), we have
\[ d(x_n, S_1x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, S_1(P S_1)^{x_n}x_n+1) \]
\[ + d(S_1(P S_1)^{x_n+1}, S_1(P S_1)^{x_n}x_n) + d(S_1(P S_1)^{x_n}x_n, S_1x_n) \]
\[ \leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(P S_1)^{x_n}x_n+1) \]
\[ + d(S_1(P S_1)^{1-1}(P S_1)^{x_n}x_n, S_1(P S_1)^{1-1}x_n) \]
\[ \leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(P S_1)^{x_n}x_n+1) + Ld((P S_1)^{x_n}x_n, x_n) \]
\[ \leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(P S_1)^{x_n}x_n+1) \]
\[ + Ld(S_1(P S_1)^{x_n}x_n, x_n) \to 0(n \to \infty). \]  
(3.18)

Similarly, we may show that \( \lim_{n \to \infty} d(x_n, S_2x_n) = 0. \)

Step 3. We prove that \( \{x_n\} \) \( \Delta \)-converges to a point \( q \in F \). Since \( \{x_n\} \) is bounded, by Lemma 2.1, it has a unique asymptotic center \( A_C(\{x_n\}) = \{q\}. \)

If \( \{w_n\} \) is any sequence of \( \{x_n\} \) with \( A_C(\{w_n\}) = \{w\} \), then by (3.18), we have
\[ \lim_{n \to \infty} d(w_n, S_1w_n) = 0. \tag{3.19} \]

We claim that \( w \in F \). In fact, for all \( m, n \geq 1, \)
\[ d(S_1(P S_1)^{m-1}w, w_n) \leq d(S_1(P S_1)^{m-1}w, S_1(P S_1)^{m-1}w_n) \]
\[ + d(S_1(P S_1)^{m-1}w_n, S_1(P S_1)^{m-2}w_n) + \cdots + d(S_1w_n, w_n) \]
\[ \leq k_m d(w, w_n) + Ld(S_1w_n, w_n) + \cdots + d(S_1w_n, w_n) \]
\[ \to 0(n \to \infty). \]

Taking \( \lim \sup \) on both sides of the above estimate and using and using (3.19), we obtain that
\[ r(S_1(P S_1)^{m-1}w, \{w_n\}) = \limsup_{n \to \infty} d(S_1(P S_1)^{m-1}w, w_n) \]
\[ \leq \limsup_{n \to \infty} d(w, w_n) = r(w, \{w_n\}). \]

This implies that
\[ \lim_{m \to \infty} r(S_1(P S_1)^{m-1}w, \{w_n\}) = d(w, x_n). \]
By Lemma 2.3, we have
\[
\lim_{m \to \infty} S_1(PS_1)^{m-1} w = w.
\]
Because \( S_1 \) is uniformly continuous, we have
\[
S_1 w = S_1 \lim_{m \to \infty} S_1(PS_1)^{m-1} w = S_1 \lim_{m \to \infty} PS_1(PS_1)^{m-1} w = \lim_{m \to \infty} S_1(PS_1)^m w = w.
\]
Consequently, \( w \in F(S_1) \). Using the same method, we have prove that \( w \in F(S_2) \) and \( w \in F \). By the uniqueness of asymptotic center, we have \( w = q \). It implies that \( q \) is the unique asymptotic center of \( \{w_n\} \) for each subsequence \( \{w_n\} \) of \( \{x_n\} \), that is \( \{x_n\} \) \( \triangle \)-converges to a point \( q \in F \).

**Remark 3.1**

(a) Theorem 3.1 removes the assumption about \( 0 < b(1 - a) < \frac{1}{2} \) in [4].

(b) Theorem 3.1 generalizes the results of [4] from a single asymptotically nonexpansive nonself mapping to two asymptotically nonexpansive nonself mappings

**References**


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