On a Generalization of Rubel’s Equation

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Abstract

All pairs linear functionals on the space $\mathcal{H}(G)$ which satisfy generalized Rubel’s equation are described.

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1 Introduction

Let $G$ be an arbitrary domain of the complex plane. Let $\mathcal{H}(G)$ denote the space of all analytic functions in $G$ equipped with the topology of compact convergence. In [1], L.A. Rubel posed and solved the problem of finding all pairs of linear continuous functionals $L$ and $M$ on the space $\mathcal{H}(G)$ that satisfy the relation

$$L(fg) = L(f)M(g) + L(g)M(f)$$ (1)

for arbitrary functions $f$ and $g$ of $\mathcal{H}(G)$. Nandakumar [2] solved the Rubel problem in the class of linear functionals on the space $\mathcal{H}(G)$. Further investigations related to the description of pairs of linear functionals on the space $\mathcal{H}(G)$ that satisfy similar relations were carried out by Nandakumar and Kannappan in [3]-[4]. These results were systematized in [5].
In the light of the above-mentioned results there naturally arises the problem of finding of all linear functionals $L, M, N$ on $\mathcal{H}(G)$ that satisfy

$$L(fg) = L(f)M(g) + L(g)N(f)$$  \hspace{1cm} (2)

for arbitrary $f$ and $g$ of $\mathcal{H}(G)$.

The purpose of this paper is to solve this problem. Note that in the case when $M = N$ the equation (2) coincides with Rubel’s equation (1).

## 2 Main Results

Assume that linear functionals $L, M, N$ on $\mathcal{H}(G)$ satisfy (2) for any $f, g \in \mathcal{H}(G)$. Setting $f = g = 1$ in (2) we get

$$L(1)(1 - M(1) - N(1)) = 0.$$  \hspace{1cm} (3)

Let us consider the following two possible cases:

Assume $L(1) \neq 0$. Then (3) implies that $M(1) + N(1) = 1$. Setting $g = 1$ in (2) we get

$$L(f)N(1) = L(1)N(f),$$  \hspace{1cm} (4)

for any $f \in \mathcal{H}(G)$.

We first consider the case $N(1) = 0$. Then $N(f) = 0$ for any $f \in \mathcal{H}(G)$. Then (2) takes the form $L(fg) = L(f)M(g)$, where $f, g \in \mathcal{H}(G)$. Setting $f = g = 1$ in the previous relation we get $L(g) = CM(g)$, where $C = L(1) \neq 0$, $g \in \mathcal{H}(G)$. Then the relation $L(fg) = L(f)M(g)$ implies $M(fg) = M(f)M(g)$ for $f, g \in \mathcal{H}(G)$. Hence, $M$ is a multiplicative functional on $\mathcal{H}(G)$. Using the description of all multiplicative functionals on $\mathcal{H}(G)$ \[6\] we get either $M = 0$, or $M(f) = f(z_0)$, $z_0 \in G$. The first of these cases is not possible because $L \neq 0$. In the second case the relation $L(g) = CM(g)$ implies that $L(f) = Cf(z_0)$, where $C \in \mathbb{C}$, $C \neq 0$. Then (2) implies that $N = 0$. Thus, we obtain $L(f) = Cf(z_0)$, $M(f) = f(z_0)$, $N = 0$, where $z_0 \in G$, $C \in \mathbb{C}$, $C \neq 0$.

Now consider the case $N(1) \neq 0$. Then (4) implies that $L(f) = CN(f)$, where $C = \frac{L(1)}{N(1)}$, $C \neq 0$. Then (2) takes the form:

$$N(fg) = N(f)M(g) + N(g)N(f),$$  \hspace{1cm} (5)

$f, g \in \mathcal{H}(G)$. Setting $f = 1$ in (5) we get

$$N(g)(1 - N(1)) = N(1)M(g),$$  \hspace{1cm} (6)

where $g \in \mathcal{H}(G)$. Since $N(1) \neq 0$, (6) implies that $M(g) = \frac{1 - N(1)}{N(1)}N(g)$, $g \in \mathcal{H}(G)$. Then (5) implies that $N_1 = \frac{1}{N(1)}N$ is the multiplicative functional on $\mathcal{H}(G)$. Hence, either $N = 0$, or $N(f) = N(1)f(z_0)$, where $z_0 \in G$. The
first of these cases is not possible because \( N(1) \neq 0 \). In the second case we get \( L(f) = Af(z_0), \ M(f) = (1 - B)f(z_0), \ N(f) = Bf(z_0) \), where \( z_0 \in G, \ A, B \in \mathbb{C} \).

Let us consider the case \( L(1) = 0 \). Setting \( f = 1 \) in (2) we get \( L(g)(1 - N(1)) = 0 \) for any \( g \in \mathcal{H}(G) \). If \( N(1) \neq 1 \), then \( L = 0 \). Therefore, in this case we obtain \( L = 0, \ M, N \), where \( M, N \) are arbitrary linear functionals on the space \( \mathcal{H}(G) \).

Henceforth, suppose that \( L \neq 0 \). Then \( N(1) = 1 \). Since \( L \neq 0 \), setting \( g = 1 \) in (2) we have \( M(1) = 1 \). Therefore, if \( L(1) = 0 \), then \( M(1) = N(1) = 1 \).

We now show that there exists a polynomial of the form \( p(z) = z^2 + bz + c \) which is a zero of each of functionals \( L \) and \( M \). This is equivalent to the fact that the system

\[
\begin{cases}
L(z^2) + bL(z) + cL(1) = 0; \\
M(z^2) + bM(z) + cM(1) = 0,
\end{cases}
\tag{7}
\]

has at least one solution \((b, c)\) in the set of complex numbers. Since \( L(1) = 0 \) and \( L(z^2) = L(z)(M(z) + N(z)) \), the first equation of (7) takes the form \( L(z)(M(z) + N(z) + b) = 0 \). If \( L(z) = 0 \), then \( b \) we can take any complex number. If \( L(z) \neq 0 \), then \( b = -(M(z) + N(z)) \). Hence, we uniquely obtain \( c \) from the second equation of (7).

Since \( L, M, N \) satisfy (2) and \( L \neq 0 \), the constructed polynomial \( p(z) \) has the following properties:

\( \alpha) \ N(p) = 0; \)
\( \beta) \ L(pf) = M(pf) = N(pf) = 0 \) for any \( f \in \mathcal{H}(G) \).

Indeed, setting \( g = p \) in (2) we get \( L(pf) = 0 \) for any \( f \in \mathcal{H}(G) \). Since \( L \neq 0 \), there exists \( g_0 \in \mathcal{H}(G) \) such that \( L(g_0) \neq 0 \). Setting \( f = p, \ g = g_0 \) in (2) we get \( N(p) = 0 \). Hence, the property \( \alpha \) and the first of properties \( \beta \) are proved. Replacing in (2) \( pf \) instead of \( f \) and setting \( g = g_0 \) we get \( N(pf) = 0 \) for any \( f \in \mathcal{H}(G) \). Similarly, we see that \( M(pf) = 0 \) for any \( f \in \mathcal{H}(G) \). Thus, the property \( \beta \) is proved.

Let us consider the following two possible cases.

1) Assume that the polynomial \( p(z) \) has two different real roots \( z_1, z_2 \) i.e., \( p(z) = (z - z_1)(z - z_2) \). We consider different cases of location of \( z_1, z_2 \):

a) suppose that \( z_1 \notin G, \ z_2 \notin G \). The function \( h(z) = \frac{f(z)}{(z-z_1)(z-z_2)} \) belongs to the space \( \mathcal{H}(G) \) for any \( f(z) \in \mathcal{H}(G) \). Using \( \beta \), we obtain that \( L(f) = L(ph) = 0 \) for an arbitrary function \( f \) from \( \mathcal{H}(G) \). We get the contradiction because \( L \neq 0 \).

b) suppose that \( z_1 \in G, \ z_2 \notin G \). Then an arbitrary function \( f \in \mathcal{H}(G) \) we can present in the form \( f(z) = (z - z_1)g_1(z) + f(z_1) \), where \( g_1(z) \in \mathcal{H}(G) \). Then using \( \beta \) and \( L(1) = 0 \) we get \( L(f) = L \left( p(z)\frac{g_1(z)}{z-z_2} + f(z_1) \right) = 0 \) for any \( f \in \mathcal{H}(G) \). We obtain a contradiction, since \( L \neq 0 \).
c) let \( z_2 \in G, \ z_1 \notin G \). Similarly, as in the previous case we get a contradiction.

d) suppose that \( z_1 \in G, \ z_2 \in G \). Then an arbitrary function \( f \in \mathcal{H}(G) \) we can present in the form
\[
 f(z) = (z - z_1)(z - z_2)g_2(z) + \frac{f(z_1) - f(z_2)}{z_1 - z_2}z + \frac{z_1f(z_2) - z_2f(z_1)}{z_1 - z_2}
\]
where \( g_2(z) \) is some function of \( \mathcal{H}(G) \). Using the property \( \beta \), we obtain that for an arbitrary \( f \in \mathcal{H}(G) \) the following equalities hold:
\[
 L(f) = C(f(z_1) - f(z_2)) 
\]
\[
 M(f) = A_1f(z_1) + A_2f(z_2), 
\]
\[
 N(f) = B_1f(z_1) + B_2f(z_2), 
\]
where \( A_1, A_2, B_1, B_2, C \in \mathbb{C}, \ C \neq 0 \).

Using (8), (9), (10) we obtain that (2) is equivalent to
\[
 f(z_1)g(z_1) - f(z_2)g(z_2) =
\]
\[
 = (f(z_1) - f(z_2))(A_1g(z_1) + A_2g(z_2)) + (g(z_1) - g(z_2))(B_1f(z_1) + B_2f(z_2)) 
\]
for any \( f, g \in \mathcal{H}(G) \). Setting \( g(z) = 1 \), \( f(z) = z \) in (11) we get \( A_1 + A_2 = 1 \). Substituting \( f(z) = 1 \), \( g(z) = z \) in (11) we get \( B_1 + B_2 = 1 \). Setting \( f(z) = g(z) = \frac{z - z_2}{z_1 - z_2} \) in (11) we have \( A_1 + B_1 = 1 \). Setting \( f(z) = g(z) = \frac{z - z_1}{z_1 - z_2} \) in (11) we get \( A_2 + B_2 = 1 \). Using the previous equalities we get \( A_1 = B_2, \ A_2 = B_1 \).

Let \( A_1 = B_2 = A \) and \( A_2 = B_1 = B \). Since \( A_1 + A_2 = 1 \), we have \( B = 1 - A \). Thus, there exist \( z_1, z_2 \in G \) and \( A, C \in \mathbb{C} \) such that \( L(f) = C(f(z_1) - f(z_2)) \), \( M(f) = Af(z_1) + (1 - A)f(z_2) \), \( N(f) = (1 - A)f(z_1) + Af(z_2) \) for any \( f \in \mathcal{H}(G) \).

2) Assume now that the polynomial \( p(z) \) is the form \( p(z) = (z - z_0)^2 \), where \( z_0 \in \mathbb{C} \). We show that \( z_0 \) belongs to \( G \). Suppose the contrary, i.e., \( z_0 \notin G \). Then for any \( f \in \mathcal{H}(G) \) the function \( \frac{f(z)}{(z - z_0)^2} \) also belongs to \( \mathcal{H}(G) \). Hence using \( \beta \) we get \( L = 0 \), a contradiction. Thus, \( z_0 \in G \).

Let choose an arbitrary function \( f \in \mathcal{H}(G) \). Then we can present \( f \) in the following form:
\[
 f(z) = (z - z_0)^2g_3(z) + f'(z_0)z + f(z_0) - z_0f'(z_0),
\]
where \( g_3(z) \) is some function of \( \mathcal{H}(G) \). Using \( \beta \) we get
\[
 L(f) = Cf'(z_0), \ M(f) = Af'(z_0) + f(z_0), \ N(f) = Bf'(z_0) + f(z_0) \]
for any \( f \in \mathcal{H}(G) \), where \( A, B, C \in \mathbb{C}, \ C \neq 0 \). Substituting the obtained \( L, M, N \) in (2) we get
\[
 (A + B)f'(z_0)g'(z_0) = 0.
\]
for any \( f, g \in \mathcal{H}(G) \). Setting \( f = g = z \) in (12) we get \( A + B = 0 \).

Thus, in this case we establish that \( L(f) = Cf'(z_0), \ M(f) = Af'(z_0) + f(z_0) \)
\[
 N(f) = -Af'(z_0) + f(z_0),
\]
where \( z_0 \in G, \ A, C, \in \mathbb{C} \).

Summarizing all the above cases, we have proved the necessity part of the following theorem.
Theorem 2.1. Let $G$ be an arbitrary domain of the complex plane. In order that linear functionals $L$, $M$, $N$ on $\mathcal{H}(G)$ satisfy equality (2) it is necessary and sufficient that these functionals belong to one of the following classes:

1° $L = 0$, $M$, $N$ are arbitrary linear functionals on $\mathcal{H}(G)$;
2° $L(f) = Af(z_0)$, $M(f) = (1 - B)f(z_0)$, $N(f) = Bf(z_0)$, $z_0 \in G$, $A, B \in \mathbb{C}$;
3° $L(f) = Cf'(z_0)$, $M(f) = Af'(z_0) + f(z_0)$, $N(f) = -Af'(z_0) + f(z_0)$, $z_0 \in G$, $A, C \in \mathbb{C}$;
4° $L(f) = C(f(z_1) - f(z_2))$, $M(f) = Af(z_1) + (1 - A)f(z_2)$, $N(f) = (1 - A)f(z_1) + Af(z_2)$, where $z_1, z_2 \in G$, $A, C \in \mathbb{C}$.

By a direct calculation we can obtain the sufficiency part of Theorem 2.1.

In the light of the above-proved theorem, there naturally arises an interesting problem of the description of all linear operators $A$, $B$, $C$ on the space $\mathcal{H}(G)$ such that

$$
(Afg)(z) = (Af)(z)(Bg)(z) + (Ag)(z)(Cf)(z)
$$

for any $f, g \in \mathcal{H}(G)$, $z \in G$. Notice that in case $C = B$ all solutions of corresponding equation (13) in the class of linear continuous operators that act in spaces of analytic functions in arbitrary simply connected domains were described in [7]. In [8] Rubel’s operator equation was solved in the class of linear operators that act in spaces of analytic functions in domains.

References


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