Secure Weakly Convex Domination
in the Join of Graphs

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Abstract

In this paper, we explore the concept of secure weakly convex
domination in the join of graphs. In particular, we characterized the
secure weakly convex dominating set of the join $K_1+G$. As a consequence,
the secure weakly convex domination number of this graph is obtained.

Mathematics Subject Classification: 05C69

Keywords: domination, secure domination, weakly convex domination,
secure weakly convex domination, join

1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any two vertices
$u$ and $v$ of $G$, the distance $d_G(u, v)$ is the length of the shortest $u$-$v$ path in
$G$. A $u$-$v$ path of length $d_G(u, v)$ is called $u$-$v$ geodesic. A set $C \subseteq V(G)$ is
a weakly convex set of $G$ if for every two vertices $u, v \in C$ there exists a $u$-$v$
geodesic whose vertices belongs to $C$, or equivalently, if for every two vertices
$u, v \in C$, $d_{\langle C \rangle}(u, v) = d_G(u, v)$. A set $C$ is a convex set of $G$ if for every two
vertices $u, v \in C$, the vertex-set of every $u$-$v$ geodesic is contained in $C$.

A set $S$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$
such that $uv \in E(G)$. The *domination number* of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ which is weakly convex (respectively, convex) is called a *weakly convex* (respectively, *convex*) dominating set. The *weakly convex* (respectively, *convex*) domination number of $G$, denoted by $\gamma_{swc}(G)$ (respectively, $\gamma_{con}(G)$), is the smallest cardinality of a weakly convex (respectively, convex) dominating set of $G$.

A set $S$ is a *secure dominating set* of $G$ if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. A set $S$ is a *secure weakly convex* (respectively, *secure convex*) dominating set of $G$ if $S$ is a weakly convex (respectively, convex) set of $G$ and if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$ is a weakly convex dominating set of $G$. The *secure* (resp. *secure weakly convex*, *secure convex*) domination number of $G$, denoted by $\gamma_s(G)$ (resp. $\gamma_{swc}(G), \gamma_{con}(G)$), is the smallest cardinality of a secure (resp. secure weakly convex, secure convex) dominating set of $G$.

The concept of weakly convex domination was introduced by Jerzy Topp and is discussed in [3] and [4]. Another domination parameter is the secure domination which was discussed in [1], [2], and [5]. A combination of these two concepts give rise to a new variant of domination called secure weakly convex domination.

**Remark 1.1** Let $G$ be a connected graph of order $n \geq 2$. Then $G = K_n$ if and only if $\gamma_{swc}(G) = 1$.

**Remark 1.2** Let $n$ be a positive integer. Then

(i) $\gamma_{swc}(P_n) = n$ for $n \geq 3$.

(ii) $\gamma_{swc}(C_n) = n$ for $n \geq 5$.

2 Results

**Theorem 2.1** Let $G$ be a non-complete graph. Then $\gamma_{swc}(K_1 + G) = 2$ if and only if one of the following is satisfied:

(i) $\gamma(G) = 1$.

(ii) $\gamma_{swc}(G) = 2$.

**Proof:** Suppose $\gamma_{swc}(K_1 + G) = 2$ and let $V(K_1) = \{v\}$. Let $S = \{x, w\}$, where $w \in V(G)$, be a secure weakly convex dominating set of $K_1 + G$. Consider the following cases:

Case 1. $x = v$.

Let $u \in V(G) \setminus S$. Clearly, $vu \in E(K_1 + G)$. Suppose $uw \notin E(G)$. Then $(S \setminus \{v\}) \cup \{u\} = \{u, w\}$. This implies that $(S \setminus \{v\}) \cup \{u\}$ is not weakly convex.
This contradicts the assumption that \( S \) is a secure weakly convex dominating set of \( K_1 + G \). Hence, \( uw \in E(G) \), which implies that \( \{w\} \) is a dominating set of \( G \). Thus, \( \gamma(G) = 1 \).

Case 2. \( x \neq v \).

Since \( S \) is a secure weakly convex dominating set of \( K_1 + G \), \( S \) must be a secure weakly convex dominating set of \( G \). Therefore, \( \gamma_{\text{swc}}(G) = 2 \).

Conversely, suppose that \( \gamma(G) = 1 \). Let \( S = \{a, b\} \), where \( a \in V(K_1) \) and \( \{b\} \) is a dominating set of \( G \). Then for every \( c \in V(G) \), \( ac, bc \in E(K_1 + G) \). Thus, \((S \setminus \{b\}) \cup \{c\} = \{a, c\} \). Hence, \( S \) is a secure weakly convex dominating set of \( K_1 + G \). This means that \( \gamma_{\text{swc}}(K_1 + G) \leq |S| = 2 \). Since \( K_1 + G \) is non-complete, \( \gamma_{\text{swc}}(K_1 + G) \geq 2 \). Therefore, \( \gamma_{\text{swc}}(K_1 + G) = 2 \).

Next, suppose that \( \gamma_{\text{swc}}(G) = 2 \). Let \( S = \{x, y\} \) be a secure weakly convex dominating set of \( G \). Clearly, \( S \) is a secure weakly convex dominating set of \( K_1 + G \). Therefore, \( \gamma_{\text{swc}}(K_1 + G) = 2 \). \( \square \)

**Theorem 2.2** Let \( G \) be a non-complete graph and suppose \( \gamma_{\text{swc}}(K_1 + G) \neq 2 \). Then \( \gamma_{\text{swc}}(K_1 + G) = 3 \) if and only if one of the following is satisfied:

(i) \( \gamma(G) = 2 \).

(ii) \( \gamma_{\text{swc}}(G) = 3 \).

**Proof:** Suppose \( \gamma_{\text{swc}}(K_1 + G) = 3 \). Let \( V(K_1) = \{x\} \) and let \( S = \{u, v, w\} \) be a secure weakly convex dominating set of \( K_1 + G \). Consider the following cases:

Case 1. \( x \in S \), say \( x = v \).

Then \( u, w \in V(G) \). Let \( y \in V(K_1 + G) \setminus (S \setminus \{v\}) \). Then \( y \in V(G) \setminus (S \setminus \{v\}) = \{u, w\} \). Suppose that \( S \setminus \{v\} \) is not a dominating set of \( G \). Then \( uy, wy \notin E(G) \), which implies that \( uy, wy \notin E(K_1 + G) \). Since \( S \) is a secure weakly convex dominating set of \( K_1 + G \) and \( uv, vw \in E(K_1 + G) \), \((S \setminus \{v\}) \cup \{y\} = \{u, y, w\} \). Thus, \((S \setminus \{v\}) \cup \{y\} \) is not weakly convex, a contradiction. Hence, \( S \setminus \{v\} \) is a dominating set of \( G \), and so \( \gamma(G) \leq |S \setminus \{v\}| = 2 \). Since \( G \) is non-complete, \( \gamma(G) \neq 1 \). Therefore, \( \gamma(G) = 2 \).

Case 2. \( x \notin S \).

Then \( u, v, w \in V(G) \). Thus, \( S \) is a secure weakly convex dominating set of \( G \). Hence, \( \gamma_{\text{swc}}(G) = 3 \).

Conversely, suppose first that \( \gamma(G) = 2 \). Let \( S = \{x, y, z\} \), where \( V(K_1) = \{x\} \) and \( \{y, z\} \) is a dominating set of \( G \). Then \( S \) is a weakly convex dominating set of \( K_1 + G \). Let \( v \in V(K_1 + G) \setminus S \). Then \( v \in V(G) \setminus \{y, z\} \). Since \( \{y, z\} \) is a dominating set of \( G \), either \( yv \in E(G) \) or \( zv \in E(G) \). Suppose \( yv \in E(G) \). Then \((S \setminus \{y\}) \cup \{v\} = \{x, v, z\} \) is a weakly convex dominating set of \( K_1 + G \). Hence, \( S \) is a secure weakly convex dominating set of \( K_1 + G \). Thus, \( \gamma_{\text{swc}}(K_1 + G) \leq |S| = 3 \). Since \( \gamma_{\text{swc}}(K_1 + G) \neq 2 \), \( \gamma_{\text{swc}}(K_1 + G) = 3 \).

Next, suppose that \( \gamma_{\text{swc}}(G) = 3 \). Let \( S = \{a, b, c\} \) be a secure weakly convex dominating set of \( G \). Then \( S \) is a secure weakly convex dominating set of \( K_1 + G \). Hence, \( S \) is a secure weakly convex dominating set of \( G \). Thus, \( \gamma_{\text{swc}}(G) = 3 \). Therefore, \( \gamma_{\text{swc}}(K_1 + G) = 3 \). \( \square \)
$K_1 + G$. Therefore, $\gamma_{swc}(K_1 + G) = 3$. \hfill \square

Note that if $G$ is a connected graph and $S \subseteq V(G)$ such that $\text{diam}_G(\langle S \rangle) \leq 2$, then $S$ is a weakly convex set in $G$.

**Theorem 2.3** Let $G$ be a non-complete graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a secure weakly convex dominating set of $K_1 + G$ if and only if it satisfies one of the following:

(i) $S$ is a secure weakly convex dominating set of $G$.
(ii) $v \in S$ and $S \setminus \{v\}$ is a dominating set of $G$.

**Proof:** Suppose $S \subseteq V(K_1 + G)$ is a secure weakly convex dominating set of $K_1 + G$. If $S \subseteq V(G)$, then $S$ is a secure weakly convex dominating set of $G$. Suppose that $v \in S$. Suppose further that $S \setminus \{v\}$ is not a dominating set of $G$. Then there exists $w \in V(G) \setminus S \setminus \{v\}$ such that $uw \notin E(G)$ for all $u \in S \setminus \{v\}$. Hence, $(S \setminus \{v\}) \cup \{w\}$ is not a weakly convex set in $G$. This contradicts the hypothesis. Therefore, $S \setminus \{v\}$ is a dominating set of $G$.

Conversely, suppose first that $S$ is a secure weakly convex dominating set of $G$. Then $S$ is a weakly convex dominating set of $K_1 + G$. Let $v \in V(G) \setminus S$. Then there exists $u \in S$ such that $uv \in E(G)$ and $(S \setminus \{u\}) \cup \{v\}$ is a weakly convex dominating set of $G$. Since $v \in v(K_1 + G)$, $(S \setminus \{u\}) \cup \{v\}$ is a weakly convex dominating set of $K_1 + G$. Hence, $S$ is a secure weakly convex dominating set of $K_1 + G$. Next, suppose that $v \in S$ and $S \setminus \{v\}$ is a dominating set of $G$. Then $S$ is a dominating set of $K_1 + G$. Since $\text{diam}(K_1 + G) = 2$ and $v \in S$, $\text{diam}_{K_1 + G}(\langle S \rangle) \leq 2$. This shows that $S$ is a weakly convex set in $K_1 + G$. Now, let $w \in V(K_1 + G) \setminus S$. Then $w \in V(G) \setminus (S \setminus \{v\})$. Since $S \setminus \{v\}$ is a dominating set of $G$, there exists $u \in S \setminus \{v\}$ such that $uw \in E(G)$. Moreover, $\text{diam}_{K_1 + G}(\langle (S \setminus \{u\}) \cup \{w\} \rangle) \leq 2$, which implies that $(S \setminus \{u\}) \cup \{w\}$ is a weakly convex set in $K_1 + G$. Also, $(S \setminus \{u\}) \cup \{w\}$ is a dominating set of $K_1 + G$. Therefore, $S$ is a secure weakly convex dominating set of $K_1 + G$. \hfill \square

**Corollary 2.4** Let $G$ be a non-complete graph. Then $\gamma_{swc}(K_1 + G) = \min\{\gamma_{swc}(G), \gamma(G) + 1\}$.

**Proof:** Assume that $\gamma(G) + 1 \leq \gamma_{swc}(G)$. Let $S$ be a minimum secure weakly convex dominating set of $K_1 + G$. Suppose $S \setminus \{v\}$ is not a minimum dominating set of $G$. Then there exists a dominating set $S' \setminus \{v\}$ in $G$ such that $|S' \setminus \{v\}| < |S \setminus \{v\}|$. By Theorem 2.3, $S'$ is secure weakly convex dominating set of $K_1 + G$. This implies that $|S'| < |S|$. This contradicts the assumption that $S$ is a minimum secure weakly convex dominating set of $K_1 + G$. Hence, $S \setminus \{v\}$ is a minimum dominating set of $G$. Therefore,

$$\gamma_{swc}(K_1 + G) = |S| = |S \setminus \{v\}| + 1 = \gamma(G) + 1.$$
Accordingly, $\gamma_{swc}(K_1 + G) = \min\{\gamma_{swc}(G), \gamma(G) + 1\}$. □

**Corollary 2.5** Let $n \geq 3$ be a positive integer. Then

(i) $\gamma_{swc}(F_{n+1}) = \lceil \frac{n}{3} \rceil + 1$.

(ii) $\gamma_{swc}(W_{n+1}) = \lceil \frac{n}{3} \rceil + 1$.

*Proof:* (i) Applying Corollary 2.4, we get

$$\gamma_{swc}(F_{n+1}) = \gamma_{swc}(K_1 + P_n) = \min\{\gamma_{swc}(P_n), \gamma(P_n) + 1\}$$

$$= \min\{n, \lceil n/3 \rceil + 1\}$$

$$= \lceil n/3 \rceil + 1.$$ 

(ii) For $n = 3$, $\gamma_{swc}(W_{n+1}) = 1$ since $W_4 \cong K_4$. For $n = 4$, $\gamma_{swc}(W_{n+1}) = 3$. For $n \geq 5$, apply Corollary 2.4

$$\gamma_{swc}(W_{n+1}) = \gamma_{swc}(K_1 + C_n) = \min\{\gamma_{swc}(C_n), \gamma(C_n) + 1\}$$

$$= \min\{n, \lceil n/3 \rceil + 1\}$$

$$= \lceil n/3 \rceil + 1.$$ 

**References**


Received: December 16, 2014; Published: January 25, 2015