In this paper, we introduced the concept of minimum covering color energy $E_C^c(G)$ of a graph $G$ and computed minimum covering chromatic energies of star graph, complete graph, crown graph, bipartite graph and cocktail graphs. Upper and lower bounds for $E_C^c(G)$ are also established.

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**Keywords:** Minimum covering set, Minimum covering color matrix, Minimum covering chromatic eigenvalues, Minimum covering chromatic energy of a graph
values of the eigenvalues of $G$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

For details on the mathematical aspects of the theory of graph energy see the reviews [7], papers [3, 4, 8] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [10, 12, 13], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 9].

Prof. Chandrashekara Adiga et al. [1] have defined the minimum covering energy which depends on its particular minimum cover. Again Prof. Chandrashekara Adiga et al. [2] have defined color energy $E_c(G)$ of a graph $G$. Motivated by these two papers, we introduced the concept of minimum covering color energy $E_{C}^c(G)$ of a graph $G$ and computed minimum covering chromatic energies of star graph, complete graph, crown graph, bipartite graph and cocktail graphs. Upper and lower bounds for $E_{C}^c(G)$ are also established.

2 Definitions and examples:

2.1 THE MINIMUM COVERING ENERGY

Let $G$ be a simple graph of order $n$ with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E$. A subset $C$ of $V$ is called a covering set of $G$ if every edge of $G$ is incident to at least one vertex of $V$. Any covering set with minimum cardinality is called a minimum covering set. Let $C$ be a minimum covering set of a graph $G$. The minimum covering matrix of $G$ is the $n \times n$ matrix defined by $A^C(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 
1 & \text{if } v_i v_j \in E \\
1 & \text{if } i = j \text{ and } v_i \in C \\
0 & \text{otherwise}
\end{cases}$$

The characteristic polynomial of $A^C(G)$ is denoted by $f_n(G, \lambda) = \det(\lambda I - A^C(G))$. The minimum covering eigenvalues of the graph $G$ are the eigenvalues of $A^C(G)$. Since $A^C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum covering energy[1] of $G$ is then defined as $E^C(G) = \sum_{i=1}^{n} |\lambda_i|$.

2.2 COLORING

A coloring of graph $G$ is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph $G$ is called chromatic number and is denoted by $\chi(G)$. 
2.3 COLOR ENERGY

Let $G$ be a simple graph of order $n$ with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E$. The color matrix of $G$ is the $n \times n$ matrix defined by $A_c(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j) \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non adjacent with } c(v_i) = c(v_j) \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by $f_n(G, \rho) = \det(\rho I - A_c(G))$. If the color used is minimum then the adjacency matrix is denoted by $A_\chi(G)$. The eigenvalues of the graph $G$ are the eigenvalues of $A_c(G)$. Since $A_c(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$.

The color energy \[2\] of $G$ is defined as $E_c(G) : = \sum_{i=1}^n |\rho_i|$. If the color used is minimum then the energy is called chromatic energy and it is denoted by $E_{\chi}(G)$.

2.4 THE MINIMUM COVERING COLOR ENERGY

Let $G$ be a simple graph of order $n$ with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E$. Let $C$ be a minimum covering set of a graph $G$. The minimum covering color matrix of $G$ is the $n \times n$ matrix defined by $A_c^C(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j) \text{ or if } i = j \text{ and } v_i \in C \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non adjacent with } c(v_i) = c(v_j) \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_c^C(G)$ is denoted by $f_n(G, \rho) = \det(\rho I - A_c^C(G))$. If the color used is minimum then the adjacency matrix is denoted by $A_{\chi}^C(G)$. The minimum covering color eigenvalues of the graph $G$ are the eigenvalues of $A_c^C(G)$. Since $A_c^C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The minimum covering color energy of $G$ is defined as $E_c^C(G)$ :

$$E_c^C(G) : = \sum_{i=1}^n |\rho_i|.$$ If the color used is minimum then the energy is called minimum covering chromatic energy and it is denoted by $E_{\chi}^C(G)$.

Note that the trace of $A_c^C(G) = |C|$.

In this paper, we are interested in studying mathematical aspects of the minimum covering color energy of a graph. The application of minimum covering color energy in other branches of science have to be investigated.

**EXAMPLE 1 :** The possible minimum covering sets for the following graph $G$ are

i) $C_1 = \{v_1, v_2\}$ ii) $C_2 = \{v_1, v_3\}$. 
FIGURE - 1

Case 1 : If the vertices $v_1, v_2, v_3$ and $v_4$ are colored with the colors $c_1, c_2, c_3$ and $c_4$ respectively then (i) $A^c_1(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Characteristic equation is $\rho^4 - 2\rho^3 - 3\rho^2 + \rho + 1 = 0$.

Minimum covering color eigenvalues are $\rho_1 \approx -1$, $\rho_2 \approx -0.5320888862$, $\rho_3 \approx 0.6527036447$, $\rho_4 \approx 2.879385242$.

Minimum covering color energy, $E^c_1(G) \approx 5.0641777729$.

(ii) $A^c_2(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Characteristic equation is $\rho^4 - 2\rho^3 - 3\rho^2 + \rho + 1 = 0$.

Minimum covering color eigenvalues are $\rho_1 \approx -1$, $\rho_2 \approx -0.5320888862$, $\rho_3 \approx 0.6527036447$, $\rho_4 \approx 2.879385242$.

Minimum covering color energy, $E^c_2(G) \approx 5.0641777729$.

Case 2 : If the vertices $v_1, v_2, v_3$ and $v_4$ are colored with the minimum colors $c_1, c_2, c_3$ and $c_2$ respectively then (i) $A^\chi_1(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

Characteristic equation is $\rho^4 - 2\rho^3 - 4\rho^2 + 4\rho + 4 = 0$.

Minimum covering chromatic eigenvalues are $\rho_1 \approx 1.414213562$, $\rho_2 \approx -1.414213562$, $\rho_3 \approx 2.732050808$, $\rho_4 \approx -2.732050808$.

Minimum covering chromatic energy, $E^\chi_1(G) \approx 6.2925287396$
Minimum covering color energy of a graph

(ii) $A^C_χ(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

Characteristic equation is $\rho^4 - 2\rho^3 - 4\rho^2 + 5\rho + 1 = 0$.

Minimum covering chromatic eigenvalues are $\rho_1 \approx -1.745281240$, $\rho_2 \approx 2.745281240$, $\rho_3 \approx -0.1772829191$, $\rho_4 \approx 1.177282919$.

Minimum covering chromatic energy, $E^C_χ(G) \approx 5.8451283181$.

\[ \therefore \text{Minimum covering chromatic energy depends on the covering set.} \]

Thus minimum covering color energy depends on the covering set and the colors used for coloring of vertices.

3 MINIMUM COVERING CHROMATIC ENERGY OF SOME STANDARD GRAPHS

Theorem 3.1. The minimum covering chromatic energy of star graph $K_{1,n-1}$ is equal to

\[ \begin{cases} \sqrt{5}, & \text{if } n = 2 \\ (n - 2) + \sqrt{n^2 + 2n - 3}, & \text{if } n \geq 3 \end{cases} \]

Proof. For the star graph $K_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$. The minimum covering set $C = \{v_0\}$ and since $\chi(K_{1,n-1}) = 2$.

$A^C_χ(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & -1 & \cdots & -1 & -1 \\ 1 & -1 & 0 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & 0 & -1 \\ 1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}_{n \times n}$

Case i. Characteristic equation for $n = 2$ is $(\rho^2 - \rho - 1) = 0$.

Minimum covering chromatic eigenvalues for $n = 2$ are $\rho = \frac{1 \pm \sqrt{5}}{2}$.

$\therefore E^C_χ(K_{1,n-1}) = \sqrt{5}$.

Case ii. Characteristic equation for $n \geq 3$ is $(\rho - 1)^{n-2}[\rho^2 + (n - 3)\rho - (2n - 3)] = 0$.

Minimum covering chromatic eigenvalues for $n \geq 3$ are $\rho = 1[(n-2) \text{ times each}]$,

$\rho = \frac{(n - 3) \pm \sqrt{n^2 + 2n - 3}}{2} \text{ [one time each]}$.

Minimum covering chromatic energy,

$E^C_χ(K_{1,n-1}) = |1|(n - 2) + \left| \frac{(n - 3) + \sqrt{n^2 + 2n - 3}}{2} \right| + \left| \frac{(n - 3) - \sqrt{n^2 + 2n - 3}}{2} \right|$.
\[ = (n - 2) + \sqrt{n^2 + 2n - 3}. \]

\textbf{Theorem 3.2.} The minimum covering chromatic energy of cocktail party graph \( k_{n \times 2} \) is equal to \((4n - 5) + \sqrt{4n^2 + 14n - 7}\) for \( n \geq 2 \).

\textbf{Proof.} Let \( k_{n \times 2} \) be the cocktail party graph with vertex set \( V = \bigcup_{i=1}^{n} \{u_i, v_i\} \).

The minimum covering set \( C = \bigcup_{i=1}^{n-1} \{u_i, v_i\} \). Since \( \chi(k_{n \times 2}) = n \), we chromatic the vertices in such a way that \( c(v_i) = c(u_i) \), for \( 1 \leq i \leq n \).

Then,

\[
A_{\chi}^C(K_{n \times 2}) = \begin{pmatrix}
1 & -1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & -1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & -1 & 0
\end{pmatrix}_{2n \times 2n}
\]

\textbf{Case i.} Characteristic equation for \( n = 2 \) is \((\rho - 1) (\rho - 2)(\rho^2 + \rho - 4) = 0\).

Minimum covering chromatic eigenvalues for \( n = 2 \) are

\[ \rho = 1, \quad \rho = 2, \quad \rho = \frac{1 \pm \sqrt{17}}{2}. \]

\[
E_{\chi}^C(K_{n \times 2}) = 3 + \sqrt{17}.
\]

\textbf{Case ii.} Characteristic equation for \( n \geq 3 \) is \((\rho - 1) (\rho - 2)^{n-1} (\rho + 2)^{n-2}[\rho^2 - (2n - 5)\rho - (6n - 8)] = 0\).

Minimum covering chromatic eigenvalues for \( n \geq 3 \) are

\[ \rho = 1, \quad \rho = 2[(n - 1) \text{ times each}], \quad \rho = -2[(n - 2) \text{ times each}] \]

\[ \rho = \frac{(2n - 5) \pm \sqrt{4n^2 + 14n - 7}}{2} \]

Minimum covering chromatic energy for \( n \geq 3 \) are,

\[ E_{\chi}^C(K_{n \times 2}) = |1| + |2|(n - 1) + \frac{(2n - 5) + \sqrt{4n^2 + 14n - 7}}{2} + \frac{(2n - 5) - \sqrt{4n^2 + 14n - 7}}{2} = (4n - 5) + \sqrt{4n^2 + 14n - 7}. \]

\textbf{Theorem 3.3.} The minimum covering chromatic energy of crown graph \( S_{2n}^0 \) is equal to \((3n - 3) + \sqrt{4n^2 - 10n + 13}\).
Proof. For the crown graph with vertex set \( V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \).

The minimum covering set is \( C = \{u_1, u_2, \ldots, u_n\} \).

Since \( \chi(S^0_{2n}) = 2 \).

Then,

\[
A^C_{\chi}(S^0_{2n}) = \begin{pmatrix}
1 & -1 & \cdots & -1 & -1 & 0 & 1 & \cdots & 1 & 1 \\
-1 & 1 & \cdots & -1 & -1 & 1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & 1 & -1 & 1 & 1 & \cdots & 0 & 1 \\
-1 & -1 & \cdots & -1 & 1 & 1 & 1 & \cdots & 1 & 0 \\
0 & 1 & \cdots & 1 & 1 & 0 & -1 & \cdots & -1 & -1 \\
1 & 0 & \cdots & 1 & 1 & -1 & 0 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & 1 & -1 & -1 & \cdots & 0 & -1 \\
1 & 1 & \cdots & 1 & 0 & -1 & -1 & \cdots & -1 & 0
\end{pmatrix}_{2n \times 2n}
\]

Characteristic equation for \( n \geq 2 \) is

\[
(\rho^2 - 3\rho + 1)^{n-1} [\rho^2 + (2n - 3)\rho - (n - 1)] = 0.
\]

Minimum covering chromatic eigenvalues for \( n \geq 2 \) are

\[
\rho = \frac{3 \pm \sqrt{5}}{2} \text{[}(n - 1) \text{ times]}, \quad \rho = \frac{(3 - 2n) \pm \sqrt{4n^2 - 10n + 13}}{2}.
\]

Minimum covering chromatic energy for \( n \geq 2 \) are,

\[
E^C_{\chi}(S^0_{2n}) = \left| \frac{3 + \sqrt{5}}{2} \right|(n - 1) + \left| \frac{3 - \sqrt{5}}{2} \right|(n - 1) + \left| \frac{(3 - 2n) + \sqrt{4n^2 - 10n + 13}}{2} \right| + \left| \frac{(3 - 2n) - \sqrt{4n^2 - 10n + 13}}{2} \right| = (3n - 3) + \sqrt{4n^2 - 10n + 13}.
\]

\[\square\]

\textbf{Theorem 3.4.} The minimum covering chromatic energy of complete graph \( K_n, n \geq 2 \) is equal to \( \sqrt{n^2 + 2n - 3} \).

Proof. For the complete graph \( K_n \) with the vertex set \( V = \{v_1, v_2, \ldots, v_n\} \).

The minimum covering set is \( C = \{v_1, v_2, \ldots, v_{n-1}\} \).

Since \( \chi(K_n) = n \).

\[
A^C_{\chi}(K_n) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{pmatrix}_{n \times m}
\]

Characteristic equation for \( n \geq 2 \) is
\[
\rho^{n-2}[\rho^2 - (n - 1)\rho - (n - 1)] = 0.
\]
Minimum covering chromatic eigenvalues for \(n \geq 2\) are
\[
\rho = 0[(n - 2)\text{times}], \quad \rho = \frac{(n - 1) \pm \sqrt{n^2 + 2n - 3}}{2}.
\]
Minimum covering chromatic energy for \(n \geq 2\),
\[
E_C^\chi(K_n) = \sqrt{n^2 + 2n - 3}.
\]

\[\square\]

**Theorem 3.5.** The minimum covering chromatic energy of complete bipartite graph \(K_{m,n}\) is equal to \((2m + n - 3) + \sqrt{m^2 + n^2 + 2mn - 2n - 2m + 1}\).

**Proof.** For the complete bipartite graph \(K_{m,n}\) \((m \leq n)\) with vertex set \(V = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}\).

The minimum covering set is \(C = \{u_1, u_2, \ldots, u_m\}\).

Since \(\chi(K_{m,n}) = 2\).

Then,
\[
A_C^\chi(K_{m,n}) = \begin{pmatrix}
1 & -1 & \cdots & -1 & -1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & \cdots & -1 & -1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & 1 & -1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & -1 & \cdots & -1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 0 & -1 & \cdots & -1 & -1 \\
1 & 1 & \cdots & 1 & 1 & -1 & 0 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ddots & 1 & 1 & -1 & -1 & \cdots & 0 & -1 \\
1 & 1 & \cdots & 1 & 1 & -1 & -1 & \cdots & -1 & 0
\end{pmatrix}_{m+n \times m+n}
\]

Characteristic equation for \(K_{m,n}\) is
\[
(\rho - 1)^{n-1}(\rho - 2)^{m-1}[\rho^2 + (m + n - 3)\rho - (2n + m - 2)] = 0.
\]
Minimum covering chromatic eigenvalues for \(K_{m,n}\) are
\[
\rho = 1[(n - 1)\text{times}], \quad \rho = 2[(m - 1)\text{times}],
\]
\[
\rho = \frac{(3 - m - n) \pm \sqrt{m^2 + n^2 + 2mn + 2n - 2m + 1}}{2}.
\]
Minimum covering chromatic energy for \(K_{m,n}\) are
\[
E_C^\chi(K_{m,n}) = \frac{|1|(n - 1) + |2|(m - 1) + \left|\frac{(3 - m - n) \pm \sqrt{m^2 + n^2 + 2mn + 2n - 2m + 1}}{2}\right|}{2} + \frac{|3 - m - n| - \sqrt{m^2 + n^2 + 2mn + 2n - 2m + 1}}{2}
\]
\[
= (n - 1) + (2m - 2) + \sqrt{m^2 + n^2 + 2mn + 2n - 2m + 1}
\]
\[
= (2m + n - 3) + \sqrt{m^2 + n^2 + 2mn + 2n - 2m + 1}.
\]
4 Properties of minimum covering color energy of a graph

Theorem 4.1. If $\lambda_1, \lambda_2, ..., \lambda_n$ represents minimum covering color eigenvalues of a matrix $A_c(G)$ of a graph $G$, then (i) $\sum_{i=1}^{n} \lambda_i = |C|$ (ii) $\sum_{i=1}^{n} \lambda_i^2 = 2[m + m'_c] + |C|$, where $m'_c$ is the number of pairs of non-adjacent vertices receiving the same color in $G$.

Proof. (i) We know that the sum of the eigenvalues of $A_c(G)$ is the trace of $A_c(G)$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |C|$$

(ii) We know that the sum of the squares of eigenvalues of $A_c(G)$ is the trace of $(A_c(G))^2$

i.e., $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji}$

$$= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j} a_{ij}a_{ji}$$

$$= |C| + 2\sum_{i<j} a_{ij}^2$$

$$= |C| + 2(m + m'_c), \text{ where } m'_c \text{ is the number of pairs of non-adjacent vertices receiving the same color in } G.$$

5 Bounds for minimum covering color energy of a graph

Similar to Mcclelland’s bounds [12] for energy of a graph, we have bounds for $E_c(G)$.

Theorem 5.1. Let $G$ be a graph with $n$ vertices and $m$ edges. Then $\sqrt{2(m + m'_c) + |C| + n(n-1)P^2} \leq E_c(G) \leq \sqrt{n[2(m + m'_c) + |C|]}$ where $P = |\det A_c(G)|$ and $m'_c$ is the number of pairs of non-adjacent vertices receiving the same color in $G.$
Proof.

Cauchy Schwarz inequality is \( \left( \sum a_i b_i \right)^2 \leq \left( \sum a_i^2 \right) \left( \sum b_i^2 \right) \)

If \( a_i = 1, b_i = |\lambda_i| \) then \( \left( \sum |\lambda_i| \right)^2 \leq \left( \sum 1 \right) \left( \sum |\lambda_i|^2 \right) \)

\[ [E^C_c(G)]^2 \leq n[2(m + m_c') + |C|] \]

\[ E^C_c(G) \leq \sqrt{n[2(m + m_c') + |C|]} \]

Using arithmetic mean and geometric mean inequality

\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left[ \prod_{i \neq j} |\lambda_i| |\lambda_j| \right] \frac{1}{n(n-1)} \\
= \left[ \prod_{i=1}^{n} |\lambda_i| \right]^{2(n-1)} \frac{1}{n(n-1)} \\
= \left[ \prod_{i=1}^{n} |\lambda_i| \right]^{\frac{2}{n}} \\
= \prod_{i=1}^{n} |\lambda_i|^{\frac{2}{n}} \\
= |\det A^C_c(G)|^{\frac{2}{n}} = P^{\frac{2}{n}}
\]

\[ \therefore \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq n(n-1)P^{\frac{2}{n}} \] (5.1)

Now consider, \[ [E^C_c(G)]^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \]

\[ = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i| |\lambda_j| \]

i.e., \[ E^C_c(G) \geq \sqrt{2(m + m_c') + |C| + n(n-1)P^{\frac{2}{n}}} \]
Theorem 5.2. If $\lambda_1(G)$ is the largest eigenvalue of $A_c^C(G)$, then $\lambda_1(G) \geq \frac{2(m + m'_c) + |C|}{n}$.

Proof. Let $X$ be any nonzero vector. Then $\lambda_1(A_c^C) = \max_{X \neq 0} \{X'AX\}$

$\therefore \lambda_1(A_c^C) \geq \frac{J'AJ}{J'J} = \frac{2(m + m'_c) + |C|}{n}$, where $J$ is a unit matrix. □

Similar to Koolen and Moulton’s [11] upper bound for energy of a graph, upper bound for $E_c^C(G)$ is given in the following theorem.

Theorem 5.3. If $G$ is a graph with $n$ vertices and $m$ edges and $2(m + m'_c) \geq n$ then

$E_c^C(G) \leq \frac{2[m + m'_c] + |C|}{n} + \sqrt{(n - 1)[2(m + m'_c) + |C| - \left(\frac{2(m + m'_c) + |C|}{n}\right)^2]}$.

Proof.

Cauchy-Schwartz inequality for $(n - 1)$ terms is

$\left[\sum_{i=2}^{n} a_i b_i \right]^2 \leq \left(\sum_{i=2}^{n} a_i^2 \right) \left(\sum_{i=2}^{n} b_i^2 \right)$

Put $a_i = 1, b_i = |\lambda_i|$ then

$\left(\sum_{i=2}^{n} |\lambda_i| \right)^2 \leq \left(\sum_{i=2}^{n} 1 \right) \left(\sum_{i=2}^{n} \lambda_i^2 \right)$

$[E_c^C(G) - \lambda_1]^2 \leq (n - 1)[2(m + m'_c) + |C| - \lambda_1^2]$

$E_c^C(G) \leq \lambda_1 + \sqrt{(n - 1)[2(m + m'_c) + |C| - \lambda_1^2]}$

Let $f(x) = x + \sqrt{(n - 1)[2(m + m'_c) + |C| - x^2]}$

For decreasing function $f'(x) \leq 0 \Rightarrow 1 - \frac{x(n - 1)}{\sqrt{(n - 1)[2(m + m'_c) + |C| - x^2]}} \leq 0$

$\Rightarrow x \geq \sqrt{\frac{2(m + m'_c) + |C|}{n}}$

Since $2(m + m'_c) + |C| \geq n$, we have

$\sqrt{\frac{2(m + m'_c) + |C|}{n}} \leq \frac{2(m + m'_c) + |C|}{n} \leq \lambda_1$[By Theorem 5.1]

$f(\lambda_1) \leq f\left(\frac{2(m + m'_c) + |C|}{n}\right)$

$\therefore E_c^C(G) \leq \frac{2(m + m'_c) + |C|}{n} + \sqrt{(n - 1)[2(m + m'_c) + |C| - \left(\frac{2(m + m'_c) + |C|}{n}\right)^2]}$. □
Recently Milovanović et al.[13] gave a sharper lower bounds for energy of a graph. Similar bounds for minimum covering color energy of a graph are established here.

**Theorem 5.4.** Let $G$ be a graph with $n$ vertices and $m$ edges. Let $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$ be minimum covering color eigenvalues of $G$ in non-increasing order. Then

$$E_c^G(G) \geq \sqrt{n[2(m + m') + |C|]} - \alpha(n)[|\lambda_1| - |\lambda_n|]^2$$

where $\alpha(n) = n\left[\frac{n}{2}\right] \left(1 - \frac{1}{n \left[\frac{n}{2}\right]}\right)$ and $[x]$ denotes the greatest integer part of real number $x$.

**Proof.** Let $a, a_1, a_2, ... a_n, A$ and $b, b_1, b_2, ... b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B \ \forall \ i = 1, 2, ... n$ then the following inequality is valid.

$$\left| n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A - a)(B - b)$$

where $\alpha(n) = n\left[\frac{n}{2}\right] \left(1 - \frac{1}{n \left[\frac{n}{2}\right]}\right)$ and equality holds if and only if $a_1 = a_2 = ... = a_n$ and $b_1 = b_2 = ... = b_n$. If $a_i = |\lambda_i|$, $b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$, then

$$\left| n \sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

But $\sum_{i=1}^{n} |\lambda_i|^2 = 2(m + m') + |C|$ and $E_c^G(G) \leq \sqrt{n[2(m + m') + |C|]}$

$$\therefore n[2(m + m') + |C|] - (E_c^G(G))^2 \leq \alpha(n)[|\lambda_1| - |\lambda_n|]^2$$

$$\Rightarrow E_c^G(G) \geq \sqrt{n[2(m + m') + |C|]} - \alpha(n)[|\lambda_1| - |\lambda_n|]^2.$$

**Theorem 5.5.** Let $G$ be a graph with $n$ vertices and $m$ edges. Let $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$ be minimum covering color eigenvalues of $G$ in non-increasing order then $E_c^G(G) \geq \frac{2(m + m') + |C| + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}$.

**Proof:**

Let $a_i \neq 0$, $b_i$, $r$ and $R$ be real numbers satisfying $r a_i \leq b_i \leq R a_i$, then the following inequality holds.[Theorem 2, [13]]

$$\sum_{i=1}^{n} b_i^2 + r R \sum_{i=1}^{n} a_i b_i \leq (r + R) \sum_{i=1}^{n} a_i b_i$$

Put $b_i = |\lambda_i|$, $a_i = 1$, $r = |\lambda_n|$ and $R = |\lambda_1|$ then

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|.$$
Minimum covering color energy of a graph

\[ i.e., \ 2(m + m'_c) + |C| + |\lambda_1||\lambda_n|n \leq (|\lambda_1| + |\lambda_n|)E_c^c(G) \]
\[ \Rightarrow E_c^c(G) \geq \frac{2(m + m'_c) + |C| + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)} \]

\[ \square \]

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References


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