Differential and Operations on Graphs

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Abstract

The differential of $D$ is defined as $\partial(D) = |B(D)| - |D|$ and the differential of a graph $G$, written $\partial(G)$, is equal to $\max\{\partial(D) : D \subseteq V\}$. The Differential in Graphs, was introduced in [13]. The research and application of the $\partial(G)$ appears mainly in Computational Mathematics. The aim of this paper is to obtain new inequalities involving the operations in graphs, the Differential $\partial(G)$ and other well known parameter in graphs.

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1 Introduction

The differential in graphs is a subject of increasing interest, both in pure and applied mathematics. In particular the study of the mathematical properties of the differential in graphs, together with a variety of other kinds of differentials of a set, started in [11, 13]. In these works, several bounds on the differential of a graph were given. This parameter has also been studied in [1, 2, 3, 14], and the differential of some products of graphs has been studied in [4, 15]. The differential of a set $D$ was also considered in [7], where it was denoted by $\eta(D)$, and the minimum differential of an independent set was considered in
The case of the $B$-differential of a graph or enclaveless number, defined as $\psi(G) = \max\{|B(D)| : D \subseteq V\}$, was investigated in [13, 17].

Throughout this paper, $G = (V, E)$ denote a simple graph of order $n = |V|$ and size $m = |E|$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a vertex $v \in V$ we denote $N(v) = \{u \in V : u \sim v\}$ and $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ will be denoted by $\delta(v) = |N(v)|$. We denote by $\delta$ and $\Delta$ the minimum and maximum degree of the graph, respectively. The subgraph induced by a set $S \subseteq V$ will be denoted by $G[S]$. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors $v$ has in $S$: $N_S(v) := \{u \in S : u \sim v\}$ and $\delta_S(v) = |N_S(v)|$. Finally, we denote $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Let $G = (V, E)$ be a graph of order $n$, for every set $D \subseteq V$ let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set $D$, and let $C(D) = V \setminus (D \cup B(D))$. The differential of $D$ is defined as $\partial(D) = |B(D)| - |D|$ and the differential of a graph $G$, written $\partial(G)$, is equal to $\max\{|\partial(D) : D \subseteq V\}$. A set $D$ satisfying $\partial(D) = \partial(G)$ is called a $\partial$-set or differential set. Note that the connectivity of $G$ is not an important restriction, since if $G$ has connected components $G_1, \ldots, G_k$, then $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. Therefore, we will only consider connected graphs.

## 2 Preliminaries

We start with the following basic results.

**Proposition 2.1.** For graphs with order $n$.

- If $G'$ is a induced subgraph of $G$, $\partial(G') \leq \partial(G)$.
- For Complete Graphs $K_n$, $\partial(K_n) = n - 2$.
- For Paths Graphs $P_n$, $n \geq 1$, $\partial(P_n) = \left\lfloor \frac{n}{3} \right\rfloor$.
- For $K_{r,t}$ is a Complete Bipartite Graphs, $\partial(K_{r,t}) = \max\{r - 1, t - 1, r + t - 4\}$.
- For Wheel Graphs $W_n$, $\partial(W_n) = n - 2$.
- For cycles Graphs $C_n$, $n \geq 3$, $\partial(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$.

Note that depending on the election of the differential set $D$ of the graph, we have different properties for the partition $\{D, B(D), C(D)\}$. 
Proposition 2.2. If $D$ is a minimum $\partial$-set, then the set $\{D, B(D), C(D)\}$ is a partition of $V$ such that:

(a) for all $v \in D$, $|\text{epn}[v, D]| \geq 2$,
(b) for all $v \in B(D)$, $\delta_{C(D)}(v) \leq 2$,
(c) for all $v \in C(D)$, $\delta_{C(D)}(v) \leq 1$.

Recall that a graph consisting of one central vertex $c$ and $d$ neighbors that in turn have no further neighbors other than $c$ is also known as a star $S_d = K_{1,d}$. We also denote an $S_d$ star $S$ by $S = \{c; v_1, \ldots, v_d\}$ to indicate that $c$ is its center and $v_1, \ldots, v_d$ are its ray vertices. We will call an $S_d$ star big if $d \geq 2$.

Given a graph $G = (V, E)$, a big star packing is given by a vertex-disjoint collection $S = \{X_i \mid 1 \leq i \leq k\}$ of (not necessarily induced) big stars $X_i \subseteq V$, i.e., the graph induced by $X_i$, written $G[X_i]$ for short, contains some $S_d$ with $d = |X_i| - 1 \geq 2$. We will write $S(D)$ when we want to specify that $D$ is the set of vertices which are star centers of $S$. The set $S_t(D)$ collects all $S_t$ stars from $S(D)$ for $t \geq 2$, and $S_{\geq t}(D)$ collects all $S_d$ stars from $S(D)$ such that $d \geq t$. If $S$ is a big star packing of $G$, we denote this property by $S \in SP(G)$.

In [1] it was proved that $\partial(G) = \max \{\sum_{X \in S}(|X| - 2) : S \in SP(G)\}$. For every $S \in SP(G)$ we write $\partial(S) = \sum_{X \in S}(|X| - 2)$ and call this the differential of the big star packing $S$. We call a star packing $S \in SP(G)$ a differential (star) packing if it assumes the differential of the graph, i.e., if $\partial(S) = \partial(G)$. A maximum differential (star) packing is a differential packing of maximum cardinality, i.e., with the maximum number of stars contained in it. Let max $SP(G)$ collect all maximum differential packings of $G$.

Lemma 2.3. For every big star packing $S(D) \in \text{max } SP$ it is satisfied that $\delta_{C(D)}(v) \leq 1$ for every $v \in B(D)$. Moreover, there exists a big star packing $S(D) \in \text{max } SP$ such that $B(D)$ is a dominating set in $G$.

Theorem 2.4. Let $G$ be a graph of order $n$ and minimum degree $\delta$. Then

$$\partial(G) \geq \left\lceil \frac{n(\delta - 1)}{3\delta - 1} \right\rceil.$$  

Proof. If we take the big star packing $S(D) \in \text{max } SP$ given in Lemma 2.3, we have that $\delta_{C(D)}(v) \leq 1$ for every $v \in B(D)$ and $\delta_{C(D)}(v) \leq 1$ for every $v \in C(D)$, then, if $C(D) = \{c_1, \ldots, c_t\}$, we have that $(\delta - 1)|C(D)| \leq |B(D)|$. Thus,

$$n = |D| + |B(D)| + |C(D)| \leq |D| + |B(D)| + \frac{|B(D)|}{\delta - 1}.$$
Since $|D| \leq \partial(G)$, we conclude that $n(\delta - 1) \leq \partial(G)(3\delta - 1)$. That is,

$$\partial(G) \geq \frac{n(\delta - 1)}{3\delta - 1}.$$  

A set $S \subset V$ is a dominating set if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. For more information on domination in graphs see [8, 9].

It is known that $\gamma(G) \geq n\Delta + 1$ for every graph $G$ of order $n$ and maximum degree $\Delta$. The following result shows that the graphs attaining this lower bound can be characterized by its differential (see [5]).

**Theorem 2.5.** Let $G$ be a graph with order $n$ and maximum degree $\Delta$. Then, $\gamma(G) = \frac{n}{\Delta+1}$ if and only if $\partial(G) = \frac{n(\Delta-1)}{\Delta+1}$.

A graph $G$ is said to be dominant differential if it contains a $\partial$-set which is also a dominating set. Some examples of dominant differential graphs are complete graphs, star graphs, wheel graphs, and path graphs $P_n$ and cycle graphs $C_n$ with $n = 3k$ or $n = 3k + 2$. For more information see [15].

### 3 Main Results

The Corona Product of graphs was introduced in [6] as a new and important operation on two graphs. Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively, the corona product $G \circ H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i$th copy of $G$ with the $i$th-vertex of $H$. We will denote by $V = \{v_1, v_2, \ldots, v_{n_1}\}$ the set of vertices of $G$ and by $H_i = (V_i, E_i)$, $V_i = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_{n_2}^{(i)}\}$, the copy of $H$ such that $v_k^{(i)} \sim v_i$ for every $i \leq k \leq n_2$. For more information see [10].

**Theorem 3.1.** Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively.

(a) If $n_2 \geq 2$, then $\delta(G \circ H) = n_1(n_2 - 1)$.

(b) If $n_2 = 1$, then $\delta(G \circ H) = n_1 - \gamma(G)$.

**Proof.** Firstly, let us see that, if $D$ is a $\delta$-set in $G \circ H$, we can assume that $D \subseteq V$. We suppose that

$$\{v_1^{(i)}, v_2^{(i)}, \ldots, v_{n_2}^{(i)}\} = D \cap H_i,$$
with $1 \leq k \leq n_2$. If $v_i \in D$, then
\[
\delta(D \setminus \{v_{j_1}^{(i)}\}, v_{j_2}^{(i)}, \ldots, v_{j_k}^{(i)}) = \delta(D) + 2k > \delta(D)
\]
which is a contradiction.

If $v_i \notin D$, then $\delta((D \setminus \{v_{j_1}^{(i)}\}, v_{j_2}^{(i)}, \ldots, v_{j_k}^{(i)}) \cup \{v_i\}) \geq \delta(D) + 2k - 2 \geq \delta(D)$, so we can take
\[
D' = (D \setminus \{v_{j_1}^{(i)}\}, v_{j_2}^{(i)}, \ldots, v_{j_k}^{(i)}) \cup \{v_i\}
\]
instead of $D$.

(a) If $n_2 \geq 2$ and $v_i \notin D$, then $\delta(D \cup \{v_i\}) = |B(d \cup \{v_i\})| - |D \cup \{v_i\}| \geq |B(D)| \geq \delta(D)$, in consequence, we can take $D = V$ and, therefore,
\[
\delta(G \circ H) = \delta(V) = n_1n_2 - n_1 = n_1(n_2 - 1).
\]

(b) For every $D \subseteq V$ we have $\delta(D) = \delta_G(D) + |D| = |B_G(D)|$. Thus, since $\max_{D \subseteq V} |B_D(D)| = n_1 - \gamma(G)$ (see [17]), we conclude
\[
\delta(G \circ H) = \max_{D \subseteq V} |B_D(D)| = n_1 - \gamma(G).
\]

The Join of two graphs $G$ and $H$, denoted by $H + G$, is defined as the graph obtained from disjoint graphs $G$ and $H$ by taking one copy of $G$ and one copy of $H$ and joining by an edge each vertex of $G$ with each vertex of $H$. In this section we will give explicit formulas for the differential of a join graph. The following result was proved in [13], it relates the domination number and the differential of a graph.

**Theorem 3.2.** For any connected graph $G$ of order $n$,
\[
n - 2\gamma(G) \leq \delta(G) \leq n - \gamma(G) - 1.
\]

Moreover, it is easy to check the following statement.

**Proposition 3.3.** For any graph $G + H$,
\[
1 \leq \gamma(G + H) \leq 2.
\]

(a) $\gamma(G + H) = 1$ if and only if $\gamma(G) = 1$ or $\gamma(H) = 1$.

(b) $\gamma(G + H) = 2$ if and only if $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. 


From propositions 3.2 and 3.3 we have following proposition.

**Proposition 3.4.** For any graphs $G$ and $H$ of order $n_1$ and $n_2$, respectively,

$$n_1 + n_2 - 4 \leq \delta(G + H) \leq n_1 + n_2 - 2.$$ 

Next proposition was proved in [3].

**Proposition 3.5.** Let $G$ be a graph of order $n$ and maximum degree $\Delta$,

a) $\delta(G) = n - 2$ if and only if $\Delta = n - 1$.

b) $\delta(G) = n - 3$ if and only if $\Delta = n - 2$.

In order to proof Theorem 3.6 note that for two graphs $G$ and $H$ of order $n_1$ and $n_2$ and maximum degrees $\Delta_1$ and $\Delta_2$, respectively, the maximum degree of the join of $G$ and $H$ is

$$\Delta(G + H) = \max\{\Delta_1 + n_2, \Delta_2 + n_1\},$$

as a direct consequence of the previous propositions, we can give the exact value for the differential of a join of two graphs depending on the maximum degree of these graphs.

**Theorem 3.6.** Let $G$ and $H$ two graphs of order $n_1$ and $n_2$ and maximum degrees $\Delta_1$ and $\Delta_2$, respectively. Then

(a) $\delta(G + H) = n_1 + n_2 - 2$ if and only if $\Delta_1 = n_1 - 1$ or $\Delta_2 = n_2 - 1$.

(b) $\delta(G + H) = n_1 + n_2 - 3$ if and only if $\Delta_1 = n_1 - 2$ and $\Delta_2 \leq n_2 - 2$ or $\Delta_1 = n_1 - 2$ and $\Delta_2 = n_2 - 2$.

(c) $\delta(G + H) = n_1 + n_2 - 4$ if and only if $\Delta_1 \leq n_1 - 3$ and $\Delta_2 = n_2 - 3$.

The Cartesian product $G \times H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and any two vertices $(a, c)$ and $(b, d)$ are adjacent in $G \times H$ if and only if either

- $a = b$ and $c$ is adjacent $d$ in $G$, or
- $c = d$ and $a$ is adjacent $b$ in $H$.

Some applications of this type of product can be seen in [16]. We will need the following results.
Lemma 3.7. For any graph, $\partial(\Gamma) = 1$ if and only if $\Gamma = C_3, C_4, C_5, P_3, P_4 \circ P_5$.

Lemma 3.8. For any graph $\Gamma$ with maximum degree $\Delta$, the following statement hold: $\partial(\Gamma) = 2$ if and only if $\Gamma$ is a graph with either:

(a) $\Gamma$ is isomorphic to $C_6, C_7, C_8, P_6, P_7 \circ P_8$. or

(b) $\Delta = 3$, and, for every vertex $v \in V$ such that $\delta(v) = 3$, the subgraph induced by $V \setminus N[v]$ has no subgraph isomorphic to $P_3$, and $\Gamma$, has no 3 independent subgraphs isomorphic to $P_3$.

Proposition 3.9. Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively. If $n_1 + n_2 \geq 11$, then

$$\partial(G \times H) \geq 3.$$ 

Proof. By contradiction, suppose that $\partial(G \times H) \leq 2$. Then, by previous lemmas, we need only consider the case $\partial(G \times H) = 2$ with $\Delta_{G \times H} = 3$. Taking a vertex $u$ such that $N(u) = \{u_1, u_2, u_3\}$. The grater number of adjacent vertices to $N[u]$ could be 6, in this case, $\delta(u_1) = \delta(u_2) = \delta(u_3) = 3$. But, taking $S = \{u_1, u_2, u_3\}$ we obtain $\partial(S) = 4$. If the number of adjacent vertices to $N[u]$ is 5, it is, $\delta(u_1) = 2$ and $\delta(u_2) = \delta(u_3) = 3$, taking $S = \{u_1, u_2, u_3\}$ we obtain $\partial(S) = 3$. Therefore, the number of adjacent vertices to $N[u]$ is 4, for example, $\delta(u_1) = \delta(u_2) = 2$ and $\delta(u_3) = 3$. Denote by $N(u_1) = \{u, v_1\}$, $N(u_2) = \{u, v_2\}$ and $N(u_3) = \{u, v_3, v_4\}$. Denote by $A = \{u, u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$. If exist a vertex $z \notin A$ adjacent to $u_1$ or $u_2$, we could form a path $P_3$ in the induced subgraph by $V \setminus N(u_3)$, it is a contradiction to Lemma 3.8. If $v_3$ has two adjacent vertices outside of $A$, taking $S = \{u, v_3\}$, we obtain $\partial(S) = 4$. The same applies to $v_4$. Therefore, the maximum number of adjacent vertices to $v_3$ or $v_4$ outside of $A$ is one. Is clear that adding any other vertex we get a contradiction with Lemma 3.7. Therefore, the maximum number of adjacent vertices is 10; which is a contradiction. \hfill $\square$

Theorem 3.10. Let $G$ and $H$ two graphs of order $n_1$ and $n_2$ and maximum degrees $\Delta_1$ and $\Delta_2$, respectively. Then

$$n_1n_2 - 2\min\{\gamma(G)n_2, \gamma(H)n_1\} \leq \partial(G \times H) \leq \min\{\gamma(G)n_2, \gamma(H)n_1\}(\Delta_1 + \Delta_2 - 1).$$

Proof.

Let us prove the following result which will be used in the proof of the theorem. If $D$ is a $\partial$-set of a graph $G$, then $|D| \leq \gamma(G)$. Let $A$ be a minimum dominating set. If $|A| < |D|$, we have

$$\partial(A) = n - 2|A| > n - |D| - |D| \geq |B(D)| - |D| = \partial(G),$$

which is a contradiction. Therefore, $|A| = |D|$. If $|A| = |D| = n$, then $\partial(G \times H) \geq \gamma(G \times H) = n$. Let $\Gamma$ be a graph on $n$ vertices such that $\partial(\Gamma) = 1$. Then $\Gamma$ is a path $P_n$. Let $S \subseteq V(\Gamma)$ be a set of at least two vertices such that $\partial(S) = 1$. Then $\partial(S \setminus \{x\}) > 2$ for any $x \in S$. By Proposition 3.9, we obtain

$$n \geq \min\{\gamma(G)n_2, \gamma(H)n_1\}(\Delta_1 + \Delta_2 - 1) \leq n \Delta_1 \Delta_2 - 1,$$

which is a contradiction. Therefore, $|A| < n$. Then $\partial(G \times H) \geq \gamma(G \times H) = n - |A|$. Therefore, $\partial(G \times H) \geq n - |A| - 2|A| = n - 3|A| = \min\{\gamma(G)n_2, \gamma(H)n_1\}(\Delta_1 + \Delta_2 - 1)$. Therefore, $\partial(G \times H) \geq \min\{\gamma(G)n_2, \gamma(H)n_1\}(\Delta_1 + \Delta_2 - 1)$. \hfill $\square$
a contradiction. Therefore, $|D| \leq \gamma(G)$.

If $D$ is a $\partial$-set of $G \times H$, since $|B(D)| \leq \Delta_{G \times H}|D|$ we have that

$$\partial(G \times H) = |B(D)| - |D| \leq |D|\Delta_{G \times H} - 1).$$

Thus, if $G \times H$ is a graph with order $n \geq 3$ and maximum degree $\Delta_{G \times H}$, then

$$\partial(G \times H) \leq \gamma(G \times H)(\Delta_{G \times H} - 1).$$

Now, using Vizing inequality: $\gamma(G \times H) \leq \min\{\gamma(G)n_2, \gamma(H)n_1\}$; the upper bound follow.

We know that for any graph $G \times H$ of order $n$ without isolated vertices,

$$n - 2\gamma(G \times H) \leq \partial(G \times H).$$

Using Vizing inequality: $\gamma(G \times H) \leq \min\{\gamma(G)n_2, \gamma(H)n_1\}$; the lower bound follow.

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