Strong Convergence Theorem for Fixed Points of Nearly Uniformly $L-$Lipschitzian Asymptotically Generalized $\Phi$-Hemicontractive Mappings

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Abstract

Let $C$ be a nonempty convex subset of a real Banach space $E$ in which the normalized duality map is norm-to-norm uniformly continuous on bounded subsets of $E$. Let $T$ be a nearly uniformly $L-$Lipschitzian asymptotically generalized $\Phi$-hemicontractive map in the intermediate sense. An iterative process of the Mann-type is proved to converge strongly to the unique fixed point of $T$. Under this setting, our theorem is a significant improvement on the results of Kim et al. (Nonlinear Analysis 71(2009), 2833-2838), Okeke et al. (International Journal of Mathematical Analysis, Vol. 7, 2013, No. 40, 1991-2003), and a host of other results.

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1. Introduction

Let $E$ be an arbitrary real normed linear space with dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ into $2^{E^*}$ defined by

$$J(x) := \{f^* \in E^*: \langle x, f^* \rangle = \| x \|^2 = \| f^* \|^2\},$$

(1.1)
where \( \langle ., . \rangle \) denotes the duality pairing.

We give the following definitions which will be needed in the sequel.

**Definition 1.1.**

Let \( C \) be a nonempty subset of a real normed linear space \( E \). Let \( T : C \to E \) be a mapping.

- \( T \) is called a **generalized \( \Phi \)-pseudocontractive mapping** if there exists a strictly increasing continuous function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to \infty} \Phi(t) = \infty \), \( \Phi(0) = 0 \), such that for all \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) satisfying the following inequality:
  \[
  \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|). \tag{1.2}
  \]

- The mapping \( T \) is called **generalized \( \Phi \)-hemicontractive** if \( F(T) := \{ x \in C : Tx = x \} \neq \emptyset \), and there exists a strictly increasing continuous function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to \infty} \Phi(t) = \infty \), \( \Phi(0) = 0 \), such that for all \( x \in C \), \( x^* \in F(T) \), there exists \( j(x - x^*) \in J(x - x^*) \) such that the following inequality holds:
  \[
  \langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|). \tag{1.3}
  \]

Clearly, the class of generalized \( \Phi \)-hemicontractive mappings includes the class of generalized \( \Phi \)-pseudocontractive mappings in which the fixed points set \( F(T) \) is not empty.

- The mapping \( T \) is called **generalized strongly successively \( \Phi \)-pseudocontractive** if there exists a strictly increasing continuous function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to \infty} \Phi(t) = \infty \), \( \Phi(0) = 0 \) such that for each \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) such that the following inequality holds:
  \[
  \langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|) \quad \forall n \geq 1. \tag{1.4}
  \]

Observe that if \( T^n = T \) for all \( n \in \mathbb{N} \) in (1.4), we obtain (1.2).

- \( T \) is **asymptotically generalized \( \Phi \)-pseudocontractive** with sequence \( \{k_n\} \), \( k_n \geq 1 \) and \( \lim_{n \to \infty} k_n = 1 \) if there exists a strictly increasing continuous function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to \infty} \Phi(t) = \infty \), \( \Phi(0) = 0 \) such that for each \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) satisfying the following inequality:
  \[
  \langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 - \Phi(\|x - y\|). \tag{1.5}
  \]

The class of asymptotically generalized \( \Phi \)-pseudocontractive maps was introduced by Kim et al. [25] in 2009 as a generalization of the class of
generalized strongly successively $\Phi$-pseudocontractive mappings. Observe that if $k_n = 1$ for all $n \in \mathbb{N}$ in (1.5), we obtain (1.4).

$T$ is called asymptotically generalized $\Phi$-hemicontractive with sequence $\{k_n\}$, $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$ if there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ with

$$\lim_{t \to \infty} \Phi(t) = \infty, \Phi(0) = 0,$$

such that for each $x \in C$, $p \in F(T)$, there exists $j(x - p) \in J(x - p)$ such that the following inequality holds:

$$\langle T^nx - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \Phi(\|x - p\|).$$ (1.6)

The class of asymptotically generalized $\Phi$-pseudocontractive mappings is a generalization of the class of strongly pseudocontractive maps, (i.e., mappings $T$ satisfying the following condition:

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2 \forall x, y \in C, k \in (0, 1),$$

and the class of $\phi$-strongly pseudocontractive maps, (i.e., mappings $T$ satisfying the following condition:

$$\langle Tx - Ty, j(x - y) \rangle \leq \phi(\|x - y\|)\|x - y\| \forall x, y \in C),$$

where $\phi$ has the same properties as $\Phi$. Both classes have been studied extensively by numerous authors under various continuity conditions on $T$: for early results on these topics, see e.g. Chidume [5], [6], Chidume et al. [9], Chidume and Zegeye [13], Goebel and Reich [22], Matoushova and Reich [28], Reich [33], [34], [35]; for Lipschitz $\Phi-$pseudocontractive-type mappings, see e.g., Chidume [7], Chidume and Chidume [8], Chidume and Mutangadura [10], Chidume and Zegeye [12], [13], Deng [19], Ding [21], Sahu [36], Ofoedu [29], Haiyun et al., Lim [26]; for $\Phi-$pseudocontractive maps with bounded range, see e.g., Hirano and Huang [23], Browder and Petryshyn [2]; for uniformly continuous $\Phi-$hemicontractive-type mappings, see e.g., Chidume and Chidume [18], Hirano and Huang [23], and Chang et al. [3]; for recent theorems on $\Phi-$hemicontractive maps associated with Hammerstein integral equations with various continuity assumptions, see e.g., Chidume [11], Chidume and Djitte [14]; Chidume and Shehu [15]; for more general results, see e.g., Chidume et al. [16], Zegeye et al. [39], Kim et al. [25], Shazad and Zegeye [37], [38], Chang et al. [3], Liu and Kang [27]; see also the following recent monographs: Berinde [1], Chidume [4], and the references contained in them.

**Definition 1.2.** Let $C$ be a nonempty subset of a real normed linear space $E$ and $\{a_n\}$ in $[0, \infty)$ be a fixed sequence with $a_n \to 0$.

$T$ is a mapping $T : C \to C$ is said to be nearly asymptotically nonexpansive with respect to the sequence $\{a_n\}$ if there exists $k_n \geq 1, \lim_{n \to \infty} k_n = 1$ such that

$$\|T^nx - T^ny\| \leq k_n(\|x - y\| + a_n) \forall x, y \in C.$$ (1.7)

The infimum of $k_n$ in inequality (1.7) is called nearly asymptotically nonexpansive constant and is denoted by $\eta(T^n)$.
• A nearly asymptotically nonexpansive map \( T \) with sequence \( \{(a_n, \eta(T^n))\} \) is said to be nearly uniformly \( L \)-Lipschitz if \( k_n = L \) for all \( n \in \mathbb{N} \), i.e., if
\[
\|T^n x - T^n y\| \leq L(\|x - y\| + a_n) \quad \forall \, x, y \in C.
\] (1.8)

**Definition 1.3.** Let \( C \) be a nonempty subset of a real normed linear space \( E \). A mapping \( T : C \to C \) is said to be asymptotically generalized \( \Phi \)-hemicontractive in the intermediate sense with sequence \( \{k_n\} \), \( k_n \geq 1 \), \( \lim_{n \to \infty} k_n = 1 \), and \( F(T) \neq \emptyset \), if there exists a strictly increasing continuous function \( \Phi : [0, \infty) \to [0, \infty) \), with \( \Phi(0) = 0 \), \( \Phi(t) \to \infty \) as \( t \to \infty \), such that for each \( x \in C \) and \( p \in F(T) \), there exists \( j(x - p) \in J(x - p) \) satisfying the following inequality:
\[
\limsup_{n \to \infty} \sup_{(x, p) \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \leq 0.
\] (1.9)

Set
\[
\tau_n := \max \left\{ 0, \sup_{(x, p) \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \right\}.
\]

It follows that \( \tau_n \geq 0 \), \( \tau_n \to 0 \) as \( n \to \infty \). Hence, (1.9) yields the following inequality:
\[
\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 + \tau_n - \Phi(\|x - p\|). \quad (1.10)
\]

Clearly, the class of asymptotically generalized \( \Phi \)-hemicontractive mappings in the intermediate sense is more general than that of asymptotically generalized \( \Phi \)-hemicontractive mappings. In [25], the authors proved the following theorem which is their main theorem.

**Theorem 1.4.** ([25]) Let \( C \) be a nonempty convex subset of a real Banach space \( E \) and \( T : C \to C \) be a nearly uniformly \( L \)-Lipschitzian mapping with sequence \( \{a_n\} \) and asymptotically generalized \( \Phi \)-hemicontractive mapping with sequence \( \{k_n\} \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \) be a sequence in \([0, 1]\) satisfying the conditions:

(i) \( \left\{ \frac{a_n}{\alpha_n} \right\} \) is bounded, (ii) \( \sum_{n=1}^{\infty} \alpha_n = \infty \), (iii) \( \sum_{n=1}^{\infty} \alpha_n^2 < \infty \),

(iv) \( \sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty \).

Let \( \{x_n\} \) be the sequence in \( E \) generated from an arbitrary \( x_1 \in C \) by
\[
x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n u_n, \quad n \in \mathbb{N},
\] (1.11)
where \( \{u_n\} \) is a bounded sequence in \( C \). Then, the sequence \( \{x_n\} \) in \( C \) defined by (1.11) converges strongly to the unique fixed point of \( T \).
In [31], the authors stated the following theorem as a generalization of theorem 1.4.

**Theorem 1.5.** ([31]) Let $C$ be a nonempty convex subset of a real Banach space $E$ and $T : C \to C$ a nearly uniformly $L$-Lipschitzian mapping with sequence $\{a_n\}$ and asymptotically generalized $\Phi$-hemiconttractive mapping in the intermediate sense with sequences $\{\tau_n\}$ and $\{k_n\}$ as defined in (1.10), and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the conditions:

- $\frac{1}{\alpha_n + a_n L + \beta_n}$ is bounded,
- $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$,
- $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, and $\sum_{n=1}^{\infty} \tau_n < \infty$,
- $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

Let $\{x_n\}$ be the sequence in $C$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n u_n, \quad n \in \mathbb{N}.$$  \hspace{1cm} (1.12)

where $\{u_n\}$ is a bounded sequence in $C$. Then, the sequence $\{x_n\}$ in $C$ defined by (1.12) converges to a unique fixed point of $T$.

**Remark 1.**

(a). Condition (i) in theorem 1.4 restricts the class of nearly uniformly $L$-Lipschitzian mappings with sequence $\{a_n\}$ to which the theorem is applicable. The sequence $\{a_n\}$ comes with the definition of the operator. So, we do not choose it. For example if $T$ satisfies the following equation:

$$\|T^n x - T^n y\| = L(\|x - y\| + \frac{1}{\log n}),$$  \hspace{1cm} (1.13)

then $a_n = \frac{1}{\log n} \in [0, \infty)$, $\lim_{n \to \infty} a_n = 0$, and if we make the canonical choice $\alpha_n = \frac{1}{n}$, we have that $\frac{\alpha_n}{a_n}$ is unbounded so that condition (i) is not satisfied. So, theorem 1.4, in this case, does not guarantee convergence of $\{x_n\}$ to the fixed point of $T$.

(b). The authors of theorems 1.4 and 1.5 did not place any condition on the sequence $\{\beta_n\}$ in theorems 1.4 and 1.5. For the sequence to be well defined, $\beta_n$ must satisfy the following conditions:

$$\beta_n \in (0, 1), \quad \alpha_n + \beta_n < 1 \forall n \geq 1.$$

In [31], the authors defined $A_n$ by

$$A_n := 2\alpha_n (k_n - 1) + 2\alpha_n L(\alpha_n + \alpha_n L + \beta_n)$$

and claimed that the conditions $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ imply that $\sum_{n=1}^{\infty} A_n < \infty$. This is not correct. One needs to also
assume, that $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$. This would imply that $\liminf_{n \to \infty} \beta_n = 0$. If this is not the case, since $\beta_n \in (0,1)$, one would then have that $\beta_n \geq a > 0 \ \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$ and some $a \in (0,1)$, and this would imply that $\forall n \geq n_0$,

$$\infty > \sum_{n=1}^{\infty} \alpha_n \beta_n \geq a \sum_{n=1}^{\infty} \alpha_n,$$

contradicting condition (ii). Hence, $\liminf \beta_n = 0$. This then implies that condition (i) of the theorem is impossible to be satisfied since $\liminf (\alpha_n + \alpha_n L + \beta_n = 0)$. Since condition (i) is used in a nontrivial manner in [31], the proof of theorem 1.5 given in [31] in not correct.

(c). The condition $\lim_{n \to \infty} \alpha_n = 0$ which is part of condition (ii) in theorem 1.5 is redundant because of condition (iii).

(d). Condition (iv) in both theorems 1.4 and 1.5 imposes restriction on the class of asymptotically generalized $\Phi$-hemicontractive mappings to which theorems 1.4 and 1.5 are applicable. For example, if $k_n$ is of the form $k_n = 1 + \frac{1}{\log n}$ and $\alpha_n = \frac{1}{n}$, then

$$\sum_{n=1}^{\infty} \alpha_n (k_n - 1) = \sum_{n=1}^{\infty} \frac{1}{n \log n},$$

which does not satisfy condition (iv).

(e). The addition of bounded error term $u_n$ in the recursion formula (1.11) or (1.12) leads to no generalization (see e.g., [4] for further details; see also theorem 3.4 below). Consequently, there is no loss of generality if $\beta_n$ is set equal to 0 for all $n \geq 1$ in the recursion formula (1.11) or (1.12).

For recent results on the approximation of fixed points of mappings which are generalized asymptotically $\Phi$-hemicontractive in the intermediate sense, see e.g., Chidume et al. [17], Okeke et al. [31], Olaleru and Okeke [30], Kaczor et al. [24], Qin et al. [32], and the references contained in them.

It is our purpose in this paper to consider the recurrence formula (1.12) with $\beta_n \equiv 0 \ \forall n \geq 1$, and prove a strong convergence theorem for the unique fixed point of any nearly uniformly $L$-Lipschitzian asymptotically generalized $\Phi$-hemicontractive map in the intermediate sense in real Banach spaces in which the normalized duality mapping is norm-to-norm uniformly continuous on bounded sets. These spaces include all uniformly smooth spaces. In particular, they include the Sobolev spaces $W^{m,p}(\Omega)$, $L_p$ spaces, $1 < p < \infty$.

2. Preliminaries

In the sequel, we shall need the following results and definitions.
Lemma 2.1. (see e.g., [4]). Let \( E \) be a real normed linear space. Then, for each \( x, y \in E \), there exists \( j(x + y) \in J(x + y) \) such that
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.
\]

Lemma 2.2. (see e.g., [4]). Let \( \{\lambda_n\} \) and \( \{\gamma_n\} \) be sequences of nonnegative real numbers and \( \{\alpha_n\} \) be a sequence of positive real numbers satisfying the conditions:
\[
\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \frac{\gamma_n}{\alpha_n} \to 0, \quad \text{as} \quad n \to \infty.
\]
Let the recursive inequality
\[
\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, \quad n = 1, 2, 3, \ldots,
\]
be given where \( \psi : [0, \infty) \to [0, \infty) \) is a strictly increasing continuous function such that it is positive on \((0, \infty)\) and \( \psi(0) = 0 \). Then, \( \lambda_n \to 0 \) as \( n \to \infty \).

Definition 2.3. Let \( E \) be a real normed space and \( S := \{x \in E : \|x\| = 1\} \). \( E \) is said to be uniformly smooth if
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
exists uniformly for \( (x, y) \in S \times S \).

3. Main Results

We prove the following theorem.

Theorem 3.1. Let \( C \) be a nonempty convex subset of a real Banach space \( E \) in which the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of \( E \). Let \( T \) be a nearly uniformly \( L \)-Lipschitzian asymptotically generalized \( \Phi \)-hemicontractive map in the intermediate sense with sequences \( \{\tau_n\} \) and \( \{k_n\} \) as defined in inequality (1.10). Define a sequence \( \{x_n\} \) in \( C \) by:
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \quad (3.1)
\]
where \( \alpha_n \in (0, 1) \) is such that (i) \( \sum_{i=1}^{\infty} \alpha_n = \infty \), (ii) \( \lim_{n \to \infty} \alpha_n = 0 \). Then, there exists \( \gamma_0 > 0 \) such that if \( \alpha_n \leq \gamma_0 \forall \ n \geq n_0 \), for some \( n_0 \in \mathbb{N} \), the sequence \( \{x_n\} \) defined by (3.1) converges strongly to the unique fixed point of \( T \).

Proof. Let \( x^* \in F(T) \). It is clear that \( x^* \) is unique. For, if \( x^{**} \) is another fixed point of \( T \), then \( T^n x^* = x^* \) and \( T^n x^{**} = x^{**} \), \( \forall n \geq 1 \). Thus,
\[
\langle T^n x^* - T^n x^{**}, j(x^* - x^{**}) \rangle \leq k_n \|x^* - x^{**}\|^2 - \Phi(\|x^* - x^{**}\|), \quad \forall n \geq 1.
\]
That is, \( \|x^* - x^{**}\|^2 \leq k_n \|x^* - x^{**}\|^2 - \Phi(\|x^* - x^{**}\|), \quad \forall n \geq 1 \). Taking limits on both sides as \( n \to \infty \), we have:
\[
\|x^* - x^{**}\|^2 \leq \|x^* - x^{**}\|^2 - \Phi(\|x^* - x^{**}\|) < \|x^* - x^{**}\|^2,
\]
a contradiction. Hence, \( x^* \) is unique.
Since $T$ is asymptotically generalized $\Phi$-hemicontractive in the intermediate sense and nearly uniformly $L$–Lipschitzian, we compute as follows:

$$
\Phi(||x - x^*||) \leq k_n ||x - x^*||^2 + \tau_n - \langle T^n x - x^*, j(x - x^*) \rangle
\leq k_n ||x - x^*||^2 + L(||x - x^*|| + a_n) ||x - x^*|| + \tau_n.
$$

Taking limits on both sides as $n \to \infty$, we obtain that:

$$
\Phi(||x - x^*||) \leq (1 + L)||x - x^*||^2.
$$

If $||x_n - x^*|| = 0$ for all $n \in \mathbb{N}$, then we are done. So, we assume $x_{n_0} \neq x^*$ for some $n_0 \in \mathbb{N}$. Since $k_n - 1 \to 0$, $a_n \to 0$, $\alpha_n \to 0$, $\tau_n \to 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, the following conditions hold:

(i) $x_{n_0} \neq x^*$; (ii) $\Phi(||x_{n_0} - x^*||) \leq (1 + L)||x_{n_0} - x^*||^2 =: a_0 > 0$;

(iii) $k_n - 1 \leq \frac{a_0}{(2 + 3L)\Phi^{-1}(a_0)}$; (iv) $a_n \leq \Phi^{-1}(a_0)$; (v) $\tau_n \leq \frac{1}{4}a_0$.

Furthermore, since $j$ is uniformly continuous on bounded subsets of $E$, given

$$
\epsilon_0 := \frac{a_0}{4[(2 + 3L)\Phi^{-1}(a_0)]}; \exists \delta > 0
$$

such that if $x, y \in B(x^*, r_0)$, where $r_0 := (2 + 3L)\Phi^{-1}(a_0)$, $s = \sup_{n \geq 1} a_n$ then, $||j(x) - j(y)|| < \epsilon_0$.

Now, define $\gamma_0 \in \mathbb{R}^+$ by

$$
\gamma_0 := \frac{1}{4} \min \left\{1, \frac{a_0}{8\Phi^{-1}(a_0)}, \frac{\delta}{(2 + 3L)\Phi^{-1}(a_0)}, \frac{1}{(2 + 3L)} \right\}.
$$

We first prove that $\{x_n\}$ is bounded.

Claim: $||x_n - x^*|| \leq 2\Phi^{-1}(a_0) \forall n \geq n_0$.

Clearly, the claim holds for $n = n_0$. Assume it holds for some $n \geq n_0$. We show that it holds for $(n + 1)$. Suppose this is not the case. Then,

$$
||x_{n+1} - x^*|| > 2\Phi^{-1}(a_0).
$$
Now, from the recurrence formula (3.1) and Lemma 2.1, we compute as follows:

\[
2\Phi^{-1}(a_0)^2 < \| x_{n+1} - x^* \|^2
\]

\[
\leq \| x_n - x^* \|^2 - 2\alpha_n \langle (I - T^n)x_n, j(x_{n+1} - x^*) \rangle
\]

\[
= \| x_n - x^* \|^2 - 2\alpha_n \langle (I - T^n)x_n - (I - T^n)x^*, j(x_n - x^*) \rangle
\]

\[
- 2\alpha_n \langle (I - T^n)x_n - (I - T^n)x^*, j(x_{n+1} - x^*) - j(x_n - x^*) \rangle
\]

\[
\leq \| x_n - x^* \|^2 - 2\alpha_n \| x_n - x^* \|^2 + 2\alpha_n k_n \| x_n - x^* \|^2
\]

\[
- 2\alpha_n \Phi(\| x_n - x^* \|) + 2\alpha_n \tau_n
\]

\[
+ 2\alpha_n \| (I - T^n)x_n - (I - T^n)x^* \| \cdot \| j(x_{n+1} - x^*) - j(x_n - x^*) \|
\]

\[
\leq \| x_n - x^* \|^2 + 2\alpha_n (k_n - 1) \| x_n - x^* \|^2 - 2\alpha_n \Phi(\| x_n - x^* \|) + 2\alpha_n \tau_n
\]

\[
+ 2\alpha_n \| (I - T^n)x_n \| \cdot \| j(x_{n+1} - x^*) - j(x_n - x^*) \|.
\]

Furthermore, we establish the following estimates:

\[(vi) \quad \Phi(\| x_n - x^* \|) \geq a_0 \quad \text{and} \quad (vii) \quad \| x_{n+1} - x^* \| \leq 3\Phi^{-1}(a_0).
\]

For (vi), we compute as follows:

\[
\alpha_n \| x_n - T^n x_n \| = \alpha_n \| x_n - x^* - (T^n x_n - x^*) \|
\]

\[
\leq \alpha_n \| x_n - x^* \| + \alpha_n \| T^n x_n - x^* \|
\]

\[
\leq \alpha_n \| x_n - x^* \| + \alpha_n L (\| x_n - x^* \| + a_n)
\]

\[
= \alpha_n \| x_n - x^* \| + L\alpha_n \| x_n - x^* \| + L\alpha_n a_n.
\]

\[
\leq \alpha_n (L + 1)2\Phi^{-1}(a_0) + L\alpha_n \Phi^{-1}(a_0) = (2 + 3L)\alpha_n \Phi^{-1}(a_0)
\]

\[
\leq (2 + 3L)\gamma_0 \Phi^{-1}(a_0) \leq \Phi^{-1}(a_0) \quad \forall \ n \geq n_0.
\]

So,

\[
\| x_n - x^* \| \geq 2\Phi^{-1}(a_0) - \Phi^{-1}(a_0) = \Phi^{-1}(a_0),
\]

establishing (vi). For (vii), we have:

\[
\| x_{n+1} - x^* \| \leq 2\Phi^{-1}(a_0) + \Phi^{-1}(a_0) = 3\Phi^{-1}(a_0),
\]

establishing (vii). Using estimates (iii), (iv), (v), (vi), (vii), the induction hypothesis, the fact that j is norm-to-norm uniformly continuous on bounded subsets of E, the definition of \( \gamma_0 \), and the estimate \( \| x_n - T^n x_n \| \leq (2 + \)
3L)Φ−1(a0), we obtain that:
\[
\left[2\Phi^{-1}(a_0)\right]^2 < \|x_{n+1} - x^*\|^2 \leq \left[2\Phi^{-1}(a_0)\right]^2 - 2\alpha_n a_0 + \frac{\alpha_n}{2} a_0 + \frac{\alpha_n}{2} a_0
\]
\[
= \left[2\Phi^{-1}(a_0)\right]^2 - \frac{1}{2} \alpha_n a_0 < \left[2\Phi^{-1}(a_0)\right]^2,
\]
which is a contradiction. Hence, \(\|x_{n+1} - x^*\| \leq 2\Phi^{-1}(a_0), \forall n \geq n_0\), and therefore, \(\{x_n\}\) is bounded. We now establish the convergence of the sequence \(\{x_n\}\) to the unique fixed point of \(T\). Set \(M_1 := 2\sup_{n \geq 1} \|x_n - x^*\|^2\) and \(M_2 := 2\sup_{n \geq 1} \|T^n x_n - x_n\|\). Then, we have that:
\[
\|x_{n+1} - x^\ast\|^2 = \|x_n - x^\ast - \alpha_n (I - T^n)x_n\|^2
\]
\[
\leq \|x_n - x^\ast\|^2 - 2\alpha_n \|x_n - x^\ast\|^2 + 2\alpha_n (k_n \|x_n - x^\ast\|^2 - \Phi(\|x_n - x^\ast\|))
\]
\[
+ 2\alpha_n ((I - T^n)(x_n - x^\ast)) \cdot (j(x_{n+1} - x^\ast) - j(x_n - x^\ast)) + 2\alpha_n \tau_n,
\]
\[
\leq \|x_n - x^\ast\|^2 - 2\alpha_n \Phi(\|x_n - x^\ast\|) + M_1 \alpha_n (k_n - 1)
\]
\[
+ M_2 \alpha_n \|j(x_{n+1} - x^\ast) - j(x_n - x^\ast)\| + 2\alpha_n \tau_n.
\]
Applying Lemma 2.2 with \(\psi\) defined by \(\psi(t) = \Phi(\sqrt{t})\),
\[
\gamma_n := M_1 \alpha_n (k_n - 1) + M_2 \alpha_n \|j(x_{n+1} - x^\ast) - j(x_n - x^\ast)\| + 2\alpha_n \tau_n,
\]
we have that
\[
\|x_n - x^\ast\| \to 0 \text{ as } n \to \infty.
\]
Hence, \(x_n \to x^\ast\) as \(n \to \infty\). □

**Remark 2.** It is known (see e.g., [4]) that if \(E\) is a uniformly smooth real Banach space, the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of \(E\). In particular, the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of \(L_p\), \(1 < p < \infty\). Consequently, we have the following corollaries.

**Corollary 3.2.** Let \(E\) be a uniformly smooth real Banach space and \(C\) be a nonempty convex subset of \(E\). Let \(T\) be a nearly uniformly \(L\)-Lipschitzian asymptotically generalized \(\Phi\)-hemicontractive map in the intermediate sense with sequences \(\{\tau_n\}\) and \(\{k_n\}\) as defined in (1.10). Define a sequence \(\{x_n\}\) in \(C\) by: \(x_1 \in C\),
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,
\]
where \(\alpha_n \in (0, 1)\) is such that \(\sum_{i=1}^{\infty} \alpha_n = \infty\) and \(\lim_{n \to \infty} \alpha_n = 0\). Then, there exists \(\gamma_0 > 0\) such that if \(\alpha_n \leq \gamma_0 \forall n \geq n_0\), some \(n_0 \in \mathbb{N}\), the sequence defined by (3.2) converges strongly to the unique fixed point of \(T\).

**Corollary 3.3.** Let \(E\) be any \(L_p\) space, \(1 < p < \infty\), and \(C\) be a nonempty convex subset of \(E\). Let \(T\) be a nearly uniformly \(L\)-Lipschitzian asymptotically
generalized $\Phi$-hemicontactive map in the intermediate sense with sequences \{$\tau_n$\} and \{$k_n$\} as defined in (1.10). Define a sequence \{$x_n$\} in $C$ by: $x_1 \in C$,
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{3.3} \]
where $\alpha_n \in (0, 1)$ is such that $\sum_{i=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \alpha_n = 0$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0 \forall \ n \geq n_0$, some $n_0 \in \mathbb{N}$, the sequence \{$x_n$\} defined by (3.3) converges strongly to the unique fixed point of $T$.

**Remark 3.** Unlike as in theorems 1.4 and 1.5, theorem 3.1 and corollaries 3.2 and 3.3 are applicable for any nearly uniformly $L$-Lipschitzian asymptotically generalized $\Phi$-hemicontactive map. As we have remarked in the introduction, the addition of bounded error term $u_n$, in the recursion formula of this paper leads to no generalization. The proof of any theorem with such error term added is basically an unnecessary repetition of the computations and arguments used here. We illustrate with the following theorem (see also, e.g., [4] for further details).

**Theorem 3.4.** Let $C$ be a nonempty convex subset of a real Banach space $E$ in which the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of $E$. Let $T$ be a nearly uniformly $L$-Lipschitzian asymptotically generalized $\Phi$-hemicontactive map in the intermediate sense. Define a sequence \{$x_n$\} in $C$ by: $x_1 \in C$,
\[ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n u_n, \quad n \geq 1, \tag{3.4} \]
where $u_n$ in $C$ is bounded and $\alpha_n, \beta_n \in (0, 1)$ are such that:
(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n \to \infty} \alpha_n = 0$, (iii) $\beta_n = o(\alpha_n)$ and (iv) $\alpha_n + \beta_n < 1$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0, \beta_n \leq \gamma_0 \alpha_n \forall \ n \geq n_0$, for some $n_0 \in \mathbb{N}$, the sequence defined by (3.4) converges strongly to the unique fixed point of $T$.

**Proof.** As before, with $x_n \in B(x^*, r)$ where $r = 2\Phi^{-1}(a_0)$ and using the boundedness of \{$u_n$\}, define $M^* := \sup_{n \geq 1} \|x_n - u_n\|$, and define:
\[ \gamma_0 := \frac{1}{4} \min \left\{ 1, \frac{\delta}{[(2 + 3L)\Phi^{-1}(a_0) + M^*]}, \frac{1}{2 + 3L}, \frac{a_0}{32\Phi^{-1}(a_0)(M^* + 1)} \right\}. \]
Since $k_n - 1 \to 0$, $a_n \to 0$, $\alpha_n \to 0$, $\tau_n \to 0$, $\beta_n \to 0$, $\exists \ n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, the following conditions hold:
(i) $x_{n_0} \neq x^*$; (ii) $\Phi(||x_{n_0} - x^*||) \leq (1 + L)||x_{n_0} - x^*||^2 =: a_0 > 0$;
(iii) $k_n - 1 \leq \frac{a_0}{10\Phi^{-1}(a_0)^2}$; (iv) $a_n \leq \Phi^{-1}(a_0)$, (v) $\tau_n \leq \frac{1}{4}a_0$;
(vi) $\beta_n \leq \frac{1}{10(M^* + 1)}\Phi^{-1}(a_0)$.
Re-write the recurrence relation (3.4) as follows:

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n - \beta_n (x_n - u_n) \]

and observe that the only additional term to the recurrence relation (3.1) is the term \(-\beta_n(x_n - u_n)\). To take care of this term we have introduced the last term in \(\gamma_0\). Observing that now, \(\|x_{n+1} - x^*\| \leq 4\Phi^{-1}(a_0)\), and following the method of computation as in the proof of theorem 3.1, we obtain the following estimates:

\[
\left[ 2\Phi^{-1}(a_0) \right]^2 < \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\alpha_n \langle (I - T^n)x_n, j(x_{n+1} - x^*) \rangle
+ \frac{\alpha_n}{2} a_0 + 2\beta_n \|x^n - u_n\| \|x_{n+1} - x^*\|
\leq \left[ 2\Phi^{-1}(a_0) \right]^2 - 2\alpha_n a_0 + \frac{\alpha_n}{2} a_0 + \frac{\alpha_n}{2} a_0
+ 8\beta_n M^{*}\Phi^{-1}(a_0)
\leq \left[ 2\Phi^{-1}(a_0) \right]^2 - \alpha_n a_0 + \frac{\alpha_n}{2} a_0 + 8\gamma_0 \alpha_n M^{*}\Phi^{-1}(a_0)
\leq \left[ 2\Phi^{-1}(a_0) \right]^2 - \frac{\alpha_n}{4} a_0 < \left[ 2\Phi^{-1}(a_0) \right]^2 ,
\]

a contradiction. Hence, \(\{x_n\}\), as in the proof of theorem 3.1, is bounded.

We now establish the convergence of the iterative scheme to the unique fixed point of \(T\).

Set \(M_1 := 2\sup_{n \geq 1} \|x_n - x^*\|^2\), \(M_2 := 2\sup_{n \geq 1} \|T^n x_n - x_n\|^2\) and \(M_3 = 8M^{*}\Phi^{-1}(a_0)\). Then we have that:

\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\alpha_n \Phi(\|x_n - x^*\|) + M_1 \alpha_n (k_n - 1) + 2\alpha_n \tau_n
+ M_2 \alpha_n \|j(x_{n+1} - x^*) - j(x_n - x^*)\| + \beta_n M_3 .
\]

Applying Lemma 2.2 with \(\psi\) defined by \(\psi(t) = \Phi(\sqrt{t})\), and

\[
\gamma_n := M_1 \alpha_n (k_n - 1) + M_2 \alpha_n \|j(x_{n+1} - x^*) - j(x_n - x^*)\| + 2\alpha_n \tau_n + \beta_n M_3
\]

we have, using \(\beta_n = o(\alpha_n)\), that

\[
\|x_n - x^*\| \to 0 \text{ as } n \to \infty .
\]

Hence, \(x_n \to x^*\) as \(n \to \infty\). \(\Box\)

Remark 4. In real Banach spaces where the normalized duality map is norm-to-norm uniformly continuous on bounded sets, theorem 3.1 is a significant improvement on the theorem of Kim et al. [25], theorem 1.4; and on the theorem of Okeke et al. [31], theorem 1.5; in the following sense:
(a) Theorem 3.1 is applicable for any nearly uniformly \( L \)-Lipschitzian asymptotically generalized \( \Phi \)-hemicontractive map.

(b) Conditions (i) and (ii) in theorem 1.4 which restrict the class of nearly uniformly \( L \)-Lipschitzian asymptotically generalized \( \Phi \)-hemicontractive mappings, to which the theorem is applicable, are dispensed with in theorem 3.1. Furthermore, condition (iii): \( \sum \alpha_n^2 < \infty \), is weakened to the condition \( \alpha_n \to 0 \) as \( n \to \infty \).

(c) Condition (ii), first part of condition (iii), and condition (iv), in theorem 1.5, are dispensed with in theorem 3.1. Furthermore, the condition: \( \sum \tau_n < \infty \), is weakened to the condition that \( \tau_n \to 0 \) as \( n \to \infty \). In fact, as has been remarked above, the proof of theorem 1.5 in [31] is not correct.

Prototype. The examples of \( \alpha_n \) and \( \beta_n \) in our theorems and corollaries are:

\[
\beta_n := \frac{1}{(n+1)^2} \quad \forall \ n \geq 1, \quad \alpha_n := \frac{1}{(n+1)} \quad \forall \ n \geq 1,
\]

the always desirable canonical choice for \( \alpha_n \).

References


Strong convergence theorem


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