Periodic Travelling Waves for a Generalized Dispersive Equation

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Abstract

In this paper we establish the existence of periodic travelling waves for a certain generalized hyperelastic dispersive equation. We follow a variational approach by characterizing periodic travelling waves as critical points of an action functional. The existence result follows as a consequence of the Arzela-Ascoli Theorem and the fact that the action functional is (sequentially) weakly lower semi-continuous.

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1 Introduction

In this work we consider the following generalized two-dimensional nonlinear dispersive equation

\[
\partial_x \left( I - \partial^2_x + \mu \partial^4_x \right) \partial_t u + \partial^2_x \left( \frac{p+2}{p+1} u^{p+1} - \gamma \left[ u \partial_x (\partial_x u)^p + \frac{p}{p+1} (\partial_x u)^{p+1} \right] \right) - \alpha \partial_y^2 u + \beta \partial_x^2 \partial_y^2 u = 0. \tag{1}
\]
In the case \( p = 1 \), the equation (1) was derived by R. M. Chen in [5] as a model for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state. In this model \( u \) represents vertical displacement of the plate relative to a uniformly pre-stressed state, while \( x \) and \( y \) are rescaled longitudinal and lateral coordinates in the horizontal plane. To reduce the full three-dimensional field equation to an approximate two-dimensional plate equation, an assumption has been made that the thickness of the plate is small in comparison to the other dimensions. It is also assumed that the small perturbations superimposed on the pre-stressed state only appear in the vertical direction (the \( z \)-direction) and in one horizontal direction (the \( x \)-direction). Hence the variation of waves in the transverse direction (the \( y \)-direction) is small. Equation (1) is obtained under the additional assumption that the wavelength in the \( x \)-direction is short. On the other hand, if the wavelength is large, we obtain the Kadomtsev-Petviashvili (KP) equation.

The parameters in equation (1) are all material constants. The scalar \( \mu \) describes the stiffness of the plate which is nonnegative. The coefficients \( \alpha \) and \( \beta \) are material constants that measure weak transverse effects. The material constant \( \gamma \) occurs as a consequence of the balance between the nonlinear and dispersive effects. Note that there is no dissipation in this model.

Equation (1) generalizes several well-known equations including the BBM equation [1] when \( \mu = \alpha = \beta = \gamma = 0 \), the regularized long-wave Kadomtsev-Petviashvili (KP) equation [3] (also referred as KP-BBM equation, see [7]) when \( \mu = \beta = \gamma = 0 \), and the Camassa-Holm (CH) equation [4] when \( \delta = \alpha = \beta = 0, \gamma = 1 \). In contrast to the derivation in [5] of nonlinear dispersive waves in a hyperelastic plate, these particular equations are usually derived as models of water waves. In equation (1), the two spatial dimensions make the analysis very different from the CH equation. The \( \mu \)-terms include a nonlinear term of fourth order, which makes equation (1) very different from the KP-BBM equation.

For equations that model the evolution of nonlinear waves, it is very important to determine the existence and uniqueness of solution for the associated initial value problem, and the existence of special solutions as the travelling waves. For instance, travelling wave solutions are important in the study of dynamics of wave propagation in many applied models such as fluid dynamics, acoustic, oceanography, and weather forecasting. An important application is the use of solitons (travelling waves of finite energy) as an efficient means of long-distance communication.

For \( \gamma \in \mathbb{R}, \mu, \alpha, \beta > 0 \) and \( p = 1 \), R. M. Chen (see [6]) showed, in the space Sobolev type \( W(\mathbb{R}^2) \) equipped with the norm

\[
\|u\|^2_w = \int_{\mathbb{R}} [u^2 + u_x^2 + u_{xx}^2 + (\partial_x^{-1}u_y)^2 + u_y^2] dxdy,
\]
the existence of two-dimensional travelling wave solutions (2D-solitons) which propagate with speed wave \( c > 0 \), i.e. solutions of the form

\[ u(x, y, t) = v(x - ct, y). \]

For this, R. M. Chen used the Concentration-Compactness Theorem. Formally \( \partial_x^{-1}u_y \) is defined via the Fourier transform as

\[ \widehat{\partial_x^{-1}u_y} = \frac{\eta}{\xi} \widehat{u}(\xi, \eta). \]

A. Montes and R. Córdoba in work in revision, using the Mountain Pass Lemma, for \( \gamma \in \mathbb{R}, \mu, \alpha, \beta > 0 \) and \( p \in \mathbb{Z}^+ \) proved the existence of one-dimensional travelling wave solutions in the energy space (1D-solitons) for the equation (1). This is, solutions of the form

\[ u(x, y, t) = v(x + y - ct), \quad (2) \]

in the Sobolev space \( H^2(\mathbb{R}) \) equipped with the norm

\[ \| u \|_{H^2}^2 = \int_{\mathbb{R}} \left[ u^2 + (u')^2 + (u'')^2 \right] dx, \]

which propagate with speed wave \( c > 0 \).

In this paper, when \( \gamma \in \mathbb{R}, \mu, \alpha, \beta > 0 \) and \( p \in \mathbb{Z}^+ \) we establish the existence of one-dimensional periodic travelling waves of period \( T \) and wave velocity \( c > 0 \). I. e. we show the existence of solutions of the form (2) in the Sobolev space of periodic functions with the mean zero property \( H^2_{per}[0, T] \) equipped with the norm

\[ \| u \|_{H^2_{per}}^2 = \int_0^T \left[ u^2 + (u')^2 + (u'')^2 \right] dx. \]

We will get the result by using a variational approach for which periodic travelling wave solutions corresponding to critical points of a suitable action functional, for which the existence of critical points follows as a consequence of the Arzela-Ascoli Theorem and the fact that the action functional is weakly lower semi-continuous in an appropriate subset and is coercive. The paper is organized as follows. In section 2 we define the appropriate spaces to look for periodic travelling waves and state our main result. In section 3 we present some preliminaries result and prove the main result of this work. Throughout this paper \( C \) denotes a generic constant whose value may change from instance to instance.
2 Main result

In order to state the main result, we prepare several notations. If $X$ is a Hilbert space $\| \cdot \|_X$ denotes the norm, $\langle \cdot, \cdot \rangle_X$ its inner product and $X'$ represents the dual space. If $\Omega \subset \mathbb{R}, L^q(\Omega), 1 \leq q \leq \infty$, denote the usual Lebesgue space. Given $T > 0$, $L^q_{\text{per}} = L^q_{\text{per}}[0, T]$ denotes the space of periodic real functions $v$ with period $T$ such that $v \in L^p[0, T]$ and have the mean zero property. This is,

$$L^q_{\text{per}}[0, T] = \{ v : \mathbb{R} \to \mathbb{R} : v \in L^q[0, T], \int_0^T v(x) \, dx = 0 \text{ and } T\text{-periodic} \}.$$

The Sobolev space $W^k_{\text{per}} = W^k_{\text{per}}[0, T]$ of periodic functions with period $T$ is defined in the following way. If $C^\infty_c([0, T])$ denotes the space of smooth compactly, we define

$$Y_{\text{per}} = \{ v : \mathbb{R} \to \mathbb{R} : v|_{[0,T]} \in C^\infty_c([0, T]) \text{ and } T\text{-periodic} \}.$$

Then we define the Sobolev space $W^k_{\text{per}}[0, T]$ as the closure of $Y_{\text{per}}$ with respect to the norm given by

$$\| v \|^2_{W^k_{\text{per}}} = \sum_{i=0}^k \int_0^T |v^{(i)}(x)|^2 \, dx.$$

Then we have that the space $W^k_{\text{per}}[0, T]$ is a Hilbert space with the inner product defined as

$$\langle v, w \rangle_{W^k_{\text{per}}} = \sum_{i=0}^k \int_0^T v(x)w(x) \, dx.$$

Now, we define the Sobolev space of periodic functions with the mean zero property, $H^k_{\text{per}} = H^k_{\text{per}}[0, T]$, as the closed subspace of $W^k_{\text{per}}[0, T]$ given by

$$H^k_{\text{per}}[0, T] = \{ v \in W^k_{\text{per}}[0, T] : \int_0^L v^{(i)} \, dx = 0, \, i = 0, \ldots, k - 1 \}.$$

In this work we show the existence of periodic travelling waves in the space $H^2_{\text{per}}[0, T]$. The following theorem is our main result.

**Theorem 2.1.** Suppose $\gamma \in \mathbb{R}, \mu, \alpha, \beta > 0$, $p \in \mathbb{Z}^+$ and $c > 0$. Then the generalized equation (1) admits nontrivial one-dimensional travelling wave solutions, $u(x, y, t) = \phi(x + y - ct)$, in the space $H^2_{\text{per}}[0, T]$.

We will prove the existence of periodic travelling wave solutions by using the Arzela-Ascoli Theorem and the following result (see Theorem 1.2 in [8]).
Theorem 2.2. Let $X$ be a Hilbert space and let $M \subset X$ be a weakly closed subset of $X$. Suppose that $E : M \rightarrow \mathbb{R} \cup \{+\infty\}$ is coercive and that is (sequentially) weakly lower semi-continuous on $M$ with respect to $X$, that is, suppose the following conditions are fulfilled:

1. $E(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$, with $u \in M$.

2. For any $u \in M$, any sequence $(u_n)_n$ in $M$ such that $u_n \rightharpoonup u$ (weakly) in $X$ there holds:

$$E(u) \leq \liminf_{n \to \infty} E(u_n).$$

Then $E$ is bounded below on $M$ and attains its minimum in $M$.

3 Existence of periodic travelling waves

By a one-dimensional travelling wave solution we shall mean a solution $\phi$ for the equation (1) of the form

$$u(x, y, t) = \phi(x + y - ct),$$

where $c$ denotes the velocity of wave. Then, one sees that $\phi$ must satisfy

$$\left[ (c + \alpha)\phi - (c + \beta)\phi' + c\mu\phi'''' - \frac{p + 2}{p + 1}\phi^{p+1} + \gamma \left( \phi'[\phi']^p + \frac{p}{p + 1}(\phi')^{p+1} \right) \right]' = 0.$$

By using integration and the mean zero property, we obtain the travelling wave equation

$$(c+\alpha)\phi - (c+\beta)\phi'' + c\mu\phi'''' - \frac{p + 2}{p + 1}\phi^{p+1} + \gamma \left( \phi'[\phi']^p + \frac{p}{p + 1}(\phi')^{p+1} \right) = 0.$$  (3)

Now, we can see that solutions $\phi$ of the equation (3) are critical points of the functional $J_c$ given by

$$J_c(\phi) = I_c(\phi) + G(\phi),$$

where the functionals $I_c$ and $G$ are defined on the space $H^2_{per}[0, T]$ by

$$I_c(\phi) = \frac{1}{2} \int_0^T \left[ (c + \alpha)\phi^2 + (c + \beta)(\phi')^2 + c\mu(\phi'')^2 \right] dx,$$

$$G(\phi) = \frac{-1}{p + 1} \int_0^T \left[ \phi^{p+2} + \gamma \phi(\phi')^{p+1} \right] dx.$$
First we have that $I_c, G, J_c \in C^1(H^2_{\text{per}}[0,T], \mathbb{R})$ and its derivatives in $\phi$ in the direction of $w$ are given by

$$I'_c(\phi)(w) = \int_0^T [(c + \alpha)\phi w + (c + \beta)\phi' w' + c\mu\phi'' w''] dx,$$

$$G'(\phi)(w) = \frac{-1}{p+1} \int_0^T [(p + 2)\phi^{p+1} w + \gamma((\phi')^{p+1} w + (p + 1)\phi(\phi')^p w')] dx.$$

As a consequence of this, after integration by parts, we conclude that

$$J'_c(\phi) = (c + \alpha)\phi - (c + \beta)\phi'' + c\mu\phi''' - \frac{p+2}{p+1}(\phi')^p + \frac{p}{p+1}(\phi')^{p+1},$$

meaning that a critical point $\phi$ of the functional $J_c$ satisfies the travelling wave equation (3). Hereafter, we will say that weak solutions for (3) are critical points of the functional $J_c$. In particular, we have that

$$\langle J'_c(\phi), \phi \rangle = 2I_c(\phi) + (p + 2)G(\phi) = 2J_c(\phi) + pG(\phi). \quad (4)$$

Next, we show the following results on properties of $I_c$ and $G$, assuming that $\gamma \in \mathbb{R}$, $\mu, \alpha, \beta > 0$ and $p \in \mathbb{Z}^+$.

**Lemma 3.1.** The functional $I_c$ is well-defined on $H^2_{\text{per}}[0,T]$. In addition, for $c > 0$, we have that $I_c(\phi) \geq 0$. Moreover, there is a positive constant $C_1 = C_1(c, \alpha, \beta, \mu)$ such that

$$C_1^{-1} \|\phi\|_{H^2_{\text{per}}}^2 \leq I_c(\phi) \leq C_1 \|\phi\|_{H^2_{\text{per}}}^2. \quad (5)$$

**Lemma 3.2.** The functional $G$ is well-defined on $H^2_{\text{per}}[0,T]$. Moreover, there is a positive constant $C = C(\gamma, p)$ such that

$$|G(\phi)| \leq C\|\phi\|_{H^2_{\text{per}}}^{p+2}. \quad (6)$$

**Proof.** Note that if $\phi \in H^2_{\text{per}}[0,T]$ then, using the H"older inequality and the fact that the embedding $H^1_{\text{per}}[0,T] \hookrightarrow L^q_{\text{per}}[0,T]$ is continuous for $q \geq 2$, we see that there is a constant $C > 0$ such that

$$\int_0^T \phi^{p+2} dx \leq C\|\phi\|_{H^2_{\text{per}}}^{p+2} \leq C\|\phi\|_{H^2_{\text{per}}}^{p+2},$$

and also that

$$\int_0^T \phi(\phi')^{p+2} dx \leq \|\phi\|_{L^2_{\text{per}}} \|\phi\|_{L^{2(p+1)}_{\text{per}}}^{p+1} \leq C\|\phi\|_{L^2_{\text{per}}} \|\phi'\|_{H^1_{\text{per}}}^{p+1} \leq C\|\phi\|_{H^2_{\text{per}}}^{p+2}.$$
Lemma 3.3. Assume that the sequence \((\phi_n)_n \subset H^2_{per}[0,T]\) converges weakly in \(H^2_{per}[0,T]\) to \(\phi_0 \in H^2_{per}[0,T]\). If \((\phi_n)_n\) converges uniformly to \(\phi_0\) on \([0,T]\) and \((\phi'_n)_n\) converges uniformly to \(\phi'_0\) on \([0,T]\), then we have that

\[
\liminf_{n \to \infty} I_c(\phi_n) \geq I_c(\phi_0).
\]

Proof. Recall that \(I_c = I_c + G\). Now, from (5) we have that \(I_c\) is like a norm in \(H^2_{per}[0,T]\), so is convex. More exactly, for \(\lambda \in (0,1)\) we have that

\[
I_c(\phi_n) \geq I_c(\lambda \phi_0) + I'_c(\lambda \phi_0) (\phi_n - \lambda \phi_0).
\]

Using the formula of \(I'_c\) we have that

\[
I'_c(\lambda \phi_0)(\phi_n - \lambda \phi_0) = \lambda \int_0^T \left[(c + \alpha)\phi_0(\phi_n - \lambda \phi_0) + (c + \beta)\phi'_0(\phi'_n - \lambda \phi'_0) + c\mu \phi''_0(\phi''_n - \lambda \phi''_0)\right]dx.
\]

Since the sequence \((\phi_n)_n\) converges weakly to \(\phi_0\) in \(H^2_{per}[0,T]\) we conclude that

\[
\lim_{n \to \infty} I'_c(\lambda \phi_0)(\phi_n - \lambda \phi_0) = 2\lambda(1 - \lambda)I_c(\phi_0).
\]

In other words, we have that

\[
\liminf_{n \to \infty} I_c(\phi_n) \geq I_c(\lambda \phi_0) + 2\lambda(1 - \lambda)I_c(\phi_0) = \lambda(2 - \lambda)I_c(\phi_0),
\]

which implies after taking \(\lambda \to 1^-\) that

\[
\liminf_{n \to \infty} I_c(\phi_n) \geq I_c(\phi_0).
\]

Now, we need to observe that

\[
\int_0^T \phi_n(\phi'_n)^{p+1}dx = \int_0^T \phi_n((\phi'_n)^{p+1} - (\phi'_0)^{p+1})dx + \int_0^T \phi_n(\phi'_0)^{p+1}dx.
\]

Since we know that \((\phi'_n)^{p+1}, (\phi'_0)^{p+1} \in L^2[0,T]\), we conclude that

\[
\lim_{n \to \infty} \int_0^T \phi_n(\phi'_n)^{p+1}dx = \int_0^T \phi_0(\phi'_0)^{p+1}dx.
\]

Moreover, using the uniform convergence of \((\phi'_n)_n\) to \(\phi'_0\) we also have that

\[
\left|\int_0^T \phi_n((\phi'_n)^{p+1} - (\phi'_0)^{p+1})dx\right|
\leq C(p) \int_0^k |\phi_n| (|\phi'_n| + |\phi'_0|)^p |\phi'_n - \phi'_0|dx
\leq C(p) \sup_{[0,T]} |\phi'_n - \phi'_0| \|\phi_n\|_{L^p} \left(\|\phi'_n\|_{L^{2p}}^p + \|\phi'_0\|_{L^{2p}}^p\right)
\leq C(p) \sup_{[0,T]} |\phi'_n - \phi'_0| \|\phi_n\|_{H^2_{per}} \left(\|\phi_n\|_{H^p_{per}}^p + \|\phi_0\|_{H^p_{per}}^p\right).
\]
which means, after recalling that the sequence \((\phi_n)_n\) is bounded, that
\[
\lim_{n \to \infty} \int_0^T \phi_n (\phi'_n)^{p+1} \, dx = \int_0^T \phi_0 (\phi'_0)^{p+1} \, dx.
\]
In a similar way we have that
\[
\lim_{n \to \infty} \int_0^T (\phi_n)^{p+2} \, dx = \int_0^T (\phi_0)^{p+2} \, dx.
\]
Therefore,
\[
\lim_{n \to \infty} G(\phi_n) = G(\phi_0).
\]
As a consequence of previous remarks, we conclude that
\[
\liminf_{n \to \infty} J_c(\phi_n) = \liminf_{n \to \infty} (I_c(\phi_n) + G(\phi_n)) \geq J_c(\phi_0).
\]
\[\square\]

We must recall that \(H^2_{\text{per}}[0,T]\) is the space of absolutely continuous functions \(\phi\) which are \(T\)-periodic and such that \(\phi, \phi', \phi'' \in L^2_{\text{loc}}(\mathbb{R})\). For \(\alpha > 0\), we consider the weakly closed subset of \(H^2_{\text{per}}[0,T]\)
\[
H^2_{\alpha,\text{per}}[0,T] = \{ \phi \in H^2_{\text{per}}[0,T] : |\phi(x)|, |\phi'(x)| \leq \alpha, \ \text{a. e. } x \in \mathbb{R} \}
\]

**Lemma 3.4.** 1. There are positive constants \(C_1\) and \(C_2\) such that for any \(\phi \in X_k\), we have that
\[
J_c(\phi) \geq C_1\|\phi\|_{H^2_{\text{per}}}^2 - C_2\|\phi\|_{H^2_{\text{per}}}^{p+2}. \tag{10}
\]
2. There exists \(\alpha_0 > 0\) such that for \(0 < \alpha < \alpha_0\) the functional \(J_c\) is coercive on \(H^2_{\alpha,\text{per}}\). More exactly, there is \(C_3 > 0\) such that for \(\phi \in H^2_{\alpha,\text{per}}\),
\[
J_c(\phi) \geq C_3\|\phi\|_{H^2_{\alpha,\text{per}}}^2. \tag{11}
\]

**Proof.** 1. From inequalities (5)-(6), there are positive constants \(C_1\) and \(C_2\) such that
\[
J_c(\phi) = I_c(\phi) + G(\phi) \geq C_1^{-1}\|\phi\|_{H^2_{\text{per}}}^2 - C_2\|\phi\|_{H^2_{\text{per}}}^{p+2}.
\]
2. Let \(\phi \in H^2_{\alpha,\text{per}}\). Then \(|\phi(x)|, |\phi'(x)| \leq \alpha\) for a.e. \(x \in \mathbb{R}\). Thus,
\[
\int_0^T |\phi| |\phi'|^{p+1} \, dx \leq \alpha^p \int_0^T |\phi| |\phi'| \, dx \leq \alpha^p \|\phi\|_{H^2_{\text{per}}}^2,
\]
and also that
\[
\int_0^T |\phi|^{p+2} \, dx \leq \alpha^p \|\phi\|_{H^2_{\text{per}}}^2.
\]
Hence, there is $C(p)$ such that

$$|G(\phi)| \leq \alpha^p C(p)\|\phi\|^2_{H^2_{\text{per}}}.$$ 

So, using inequality (5) and previous one, we have that

$$J_{c}(\phi) \geq \frac{1}{C_1}\|\phi\|^2_{H^2_{\text{per}}} - \alpha^p C(p)\|\phi\|^2_{H^2_{\text{per}}} = \left( \frac{1}{C_1} - \alpha^p C(p) \right)\|\phi\|^2_{H^2_{\text{per}}},$$

as desired.

Our goal now is to show the existence of a non trivial critical point for $J_c$. The result will be a direct consequence of the coerciveness of $J_c$ and that $J_c$ is (sequentially) weakly lower semi-continuous on $H^2_{\alpha,\text{per}}[0,T]$ for $0 < \alpha < \alpha_0$. We will use the Arzela-Ascoli Theorem and the Theorem 2.2.

**Theorem 3.5.** For $0 < \alpha < \alpha_0$, $J_c$ has a minimum over $H^2_{\alpha,\text{per}}[0,T]$.

**Proof.** We will verify that $J_c$ satisfies the hypotheses in Theorem 3.3. Now, it is straightforward to check that $H^2_{\alpha,\text{per}}$ is weakly closed subset of $H^2_{\text{per}}$. In fact, let $(\phi_n)_n \subset H^2_{\alpha,\text{per}}$ be a sequence that converges weakly to $\phi_0$. Then we have that the sequence $(\phi_n)_n$ is bounded in $H^2_{\alpha,\text{per}}$. Now, since $\phi_n \in H^2_{\text{per}}$ has mean zero on $[0, T]$, we have that

$$|\phi_n(x) - \phi_n(y)| \leq \int_y^x |\phi_n'(r)|dr \leq |x - y|^{\frac{1}{2}}\|\phi_n\|_{H^2_{\text{per}}} \leq M|x - y|^{\frac{1}{2}}$$

and also that

$$|\phi_n'(x) - \phi_n'(y)| \leq M|x - y|^{\frac{1}{2}}.$$ 

In other words $(\phi_n)_n$ and $(\phi_n')_n$ are equicontinuous, then by using the Arzela-Ascoli Theorem we have for some subsequence (which we denote by the same symbol) that $(\phi_n)_n$ converges uniformly to $\phi_0$ on $[0, T]$ and that $(\phi_n')_n$ converges uniformly to $\phi_0'$ on $[0, T]$, since we have that $|\phi_n(x)|, |\phi_n'(x)| \leq \alpha$ for a.e. $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. From this fact and the uniform convergence of $(\phi_n)_n$ and $(\phi_n')_n$, we conclude that $|\phi_0(x)|, |\phi_0'(x)| \leq \alpha$ for a.e. $x \in \mathbb{R}$. Then $\phi_0 \in H^2_{\alpha,\text{per}}$, meaning that $H^2_{\alpha,\text{per}}$ is weakly closed subset of $H^2_{\alpha}$. Now note that the coerciveness property of $J_c$ and condition (1) in Theorem 2.2 are obtained using the inequality (11) in previous lemma. We need now to verify condition (2). Let $\phi_0 \in H^2_{\alpha,\text{per}}$ and let $(\phi_n)_n \subset H^2_{\text{per}}$ such that $\phi_n \rightharpoonup \phi_0$ (weakly) in $H^2_{\text{per}}$. This sequence $(\phi_n)_n$ is bounded $H^2_{\text{per}}$ and the same type of arguments show that $(\phi_n)_n$ converges uniformly to $\phi_0$ on $[0, T]$ and $(\phi_n')_n$ converges uniformly to $\phi_0'$ on $[0, T]$ (up to a subsequence), so by Lemma 3.3 we conclude that

$$\liminf_{n \to \infty} J_c(\phi_n) \geq J_c(\phi_0).$$

Then, from Theorem 2.2 we conclude that $J_c$ attains a minimum over $H^2_{\alpha,\text{per}}$.

□
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