Some Properties of Quasi-Convex Functions
on Abstract Convex Spaces

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Abstract

In this paper, we introduce $F$-convex sets which are modified $E$-convex sets and show that $F$-convex sets are abstract convex spaces. Using generalized KKM maps on abstract convex spaces, we obtain a fixed point theorem and an equilibrium result for generalized quasiconcave functions on $F$-convex sets. We discuss properties of quasi-convex functions on abstract convex spaces and apply them to $F$-convex sets. We also give new results of semi-convex programming.

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Keywords: $E$-convex set, $F$-convex set, $G$-convex spaces, abstract convex spaces, generalized KKM, generalized quasiconcavity, quasi-convex programming, quasi-semi-$E$-convex, quasi-$F$-convex, strictly quasi-semi-$E$-convex

1 Introduction

The concept of convexity and its generalizations are very important in the studies of nonlinear analysis and convex analysis. Recently, S. Park [8, 9, 10] introduced the new concept of abstract convex spaces as a far-reaching generalization of convex spaces, $H$-spaces, generalized convex (or $G$-convex) spaces and other abstract convex structures. Another generalization of convexity is an $E$-convex set which was introduced by Youness [15] and generalized by
Kim([6], [7]) as follows; Let $X$ be a nonempty subset of a vector space $Y$. A set $X$ is said to be $E$-convex if there is a function $E : Y \to Y$ such that $(1 - \lambda)E(x) + \lambda E(y) \in X$ for each $x, y \in X$ and $0 \leq \lambda \leq 1$.

Using the $E$-convexity, Kim introduced the $E$-KKM maps [6] as follows; Let $X$ be a nonempty subset of a vector space $Y$. A multimap $T : X \rightrightarrows Y$ is called an $E$-KKM map on $X$ if for any finite $\{x_1, \cdots, x_n\} \subset X$, $\text{co}(\{E(x_1), \cdots, E(x_n)\}) \subset \bigcup_{i=1}^{n} T(x_i)$,

where $\text{co}$ denotes the convex hull operation on $Y$.

Kim proved $E$-KKM theorems and fixed point theorems in $E$-convex sets [6]. In order to prove the fixed point theorem in an $E$-convex set $X$ using the $E$-KKM map $T : X \rightrightarrows X$, $X$ must contain $\text{co}(\{E(x_1), \cdots, E(x_n)\})$ for each $\{x_1, \cdots, x_n\} \subset X$. But $E$-convexity of $X$ doesn’t guarantee it; Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$E(x, y) = \begin{cases} (0, 0), & \text{if } y = 0; \\ (1, 0), & \text{if } x = 0, y \neq 0; \\ (0, 1), & \text{otherwise.} \end{cases}$$

Then the set $X = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0, 0 \leq y \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : x + y = 1, x > 0, y > 0\}$ is $E$-convex, but $\frac{1}{3}E(0, 0) + \frac{1}{3}E(0, 1) + \frac{1}{3}E(\frac{1}{2}, \frac{1}{2}) \notin X$.

So we modify the $E$-convexity as follows: Let $X$ and $D$ be nonempty subsets of a topological vector space $Y$. $(X, D)$ is said to be $F$-convex in $Y$ iff there is a function $F : Y \to Y$ such that $\text{co}F(N) \subset X$ for each $N \in \langle D \rangle$. In case $X = D$, let $X = (X, X)$.

For any nonempty $D' \subset D$, the $F$-convex hull of $D'$ is denoted and defined by $\text{co}_F D' := \bigcup\{\text{co}F(A) \mid A \in \langle D' \rangle\} \subset X$.

A subset $X'$ of $X$ is called an $F$-convex subset of $(X, D)$ relative to $D'$ if for any $N \in \langle D' \rangle$, we have $\text{co}F(N) \subset X'$. Note that $Y$ is $F$-convex.

In this paper, we show that $F$-convex spaces are $G$-convex spaces. Using generalized KKM maps on abstract convex spaces, we obtain a fixed point theorem and an equilibrium result for generalized quasiconcave functions on $F$-convex spaces. We discuss properties of quasi-convex functions on abstract convex spaces and apply them to $F$-convex sets. We also give new results of semi-convex programming.

## 2 Preliminaries

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set $D$. 
A generalized convex space or a G-convex space \((X, D; \Gamma)\) consists of a topological space \(X\) and a nonempty set \(D\) such that for each \(A \in \langle D \rangle\) with the cardinality \(|A| = n + 1\), there exist a subset \(\Gamma(A)\) of \(X\) and a continuous function \(\phi_A : \Delta_n \to \Gamma(A)\) such that \(J \in \langle A \rangle\) implies \(\phi_A(\Delta_J) \subset \Gamma(J)\). Here, \(\Delta_n\) is the standard \(n\)-simplex with vertices \(\{e_i\}_{i=0}^n\), and \(\Delta_J\) is the face of \(\Delta_n\) corresponding to \(J \in \langle A \rangle\); that is, if \(A = \{a_0, a_1, \ldots, a_n\}\) and \(J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A\), then \(\Delta_J \cap \Gamma(A) = \co\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}\). For details on G-convex spaces, see [5, 12, 13, 14].

**Theorem 2.1.** An \(F\)-convex subset \((X, D)\) in a topological vector space \(Y\) is a G-convex space.

**Proof.** For each \(A = \{y_0, \ldots, y_n\} \in \langle D \rangle\), put \(\Gamma(A) = \co F(A)\) and define a function \(\phi_A : \Delta_n \to \Gamma(A)\) by \(\phi_A(\Sigma_{i=0}^n \lambda_i e_i) := \Sigma_{i=0}^n \lambda_i F(y_i)\). Since \(\phi_A\) is continuous, \((X, D; \Gamma)\) is a G-convex space.

A multimap (or simply, a map) \(T : X \to Y\) is a function from a set \(X\) into the power set of \(Y\); that is, a function with the values \(T(x) \subset Y\) for \(x \in X\) and the fibers \(T^-(y) := \{x \in X \mid y \in T(x)\}\) for \(y \in Y\). For \(A \subset X\), let \(T(A) := \bigcup\{T(x) \mid x \in A\}\).

Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context. The closure operation and graph of \(T\) are denoted by \(\cor T\) and \(\Gr T\), respectively.

The following is due to Park [8]. An abstract convex space \((X, D; \Gamma)\) consists of a topological space \(X\), a non-empty set \(D\), and a multimap \(\Gamma : \langle D \rangle \to X\) with nonempty values \(\Gamma_A := \Gamma(A)\) for \(A \in \langle D \rangle\).

For any nonempty \(D' \subset D\), the \(\Gamma\)-convex hull of \(D'\) is denoted and defined by

\[
\cor_D D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset X.
\]

A subset \(X'\) of \(X\) is called a \(\Gamma\)-convex subset of \((X, D; \Gamma)\) relative to \(D'\) if for any \(N \in \langle D' \rangle\), we have \(\Gamma_N \subset X'\), that is, \(\cor_D D' \subset X'\).

When \(D \subset X\) in \((X, D; \Gamma)\), a subset \(X'\) of \(X\) is said to be \(\Gamma\)-convex if \(\cor_D (X' \cap D) \subset X'\); in other words, \(X'\) is \(\Gamma\)-convex relative to \(D' := X' \cap D\).

When \(X = D\), let \((X; \Gamma) := (X, X; \Gamma)\).

## 3 Generalized quasiconcavity and Generalized KKM maps

Let \((X, D; \Gamma)\) be an abstract convex space. If a map \(T : D \to X\) satisfies \(\Gamma_A \subset T(A)\) for all \(A \in \langle D \rangle\), then \(T\) is called a KKM map.
The \textit{partial KKM principle} for an abstract convex space \((X, D; \Gamma)\) is the statement that, for any closed-valued KKM map \(T : D \to X\), the family \(\{T(z)\}_{z \in D}\) has the finite intersection property.

Known examples of abstract convex spaces satisfying the partial KKM principle are given in Park [8, 9, 10] and the references therein. Note that \(G\)-convex spaces are abstract convex spaces satisfying the partial KKM principle.

Motivated by Kassay and Kolumbán [3], generalized KKM maps on abstract convex spaces is defined in [4] as follows: Let \((X, D; \Gamma)\) be an abstract convex space and \(I\) be a nonempty set. A map \(T : I \to X\) is called a \textit{generalized KKM map} provided that for each \(N \in \langle I \rangle\), there exists a function \(\sigma : N \to D\) such that \(\Gamma_{\sigma(M)} \subset T(M)\) for each \(M \in \langle N \rangle\). If \(\sigma\) is an identity function on \(D\), then \(T\) is a KKM map.

The following generalized KKM theorem is in [4];

\textbf{Theorem 3.1.} Let \(I\) be a nonempty set, \((X, D; \Gamma)\) be an abstract convex space satisfying the partial KKM principle, and \(T : I \to X\) be a multimap such that \(T\) is a generalized KKM map. Then \(\{T(z)\}_{z \in I}\) has the finite intersection property.

Further, if there exists a nonempty compact subset \(K\) of \(X\) such that \(\bigcap_{z \in M} T(z) \subset K\) for some \(M \in \langle I \rangle\), then \(K \cap \bigcap \{T(z) \mid z \in I\} \neq \emptyset\).

From Theorem 3.1, we obtain the following theorem on an \(F\)-convex space;

\textbf{Theorem 3.2.} Let \(I\) be a nonempty set, \((X, D)\) be nonempty \(F\)-convex in a topological vector space \(Y\), and \(T : I \to X\) be a multimap satisfying (3.2.1) \(T\) is a generalized KKM map. Then \(\{T(z)\}_{z \in I}\) has the finite intersection property.

Further, if (3.2.2) there exists a nonempty compact subset \(K\) of \(X\) such that \(\bigcap_{z \in M} T(z) \subset K\) for some \(M \in \langle I \rangle\).

Then \(K \cap \bigcap \{T(z) \mid z \in I\} \neq \emptyset\).

Theorem 3.2 generalizes Theorem 3.1 in [7].

Consider the following related three conditions for \(T : I \to X\);

(a) \(\bigcap_{z \in I} T(z) = \bigcap_{z \in I} T(z)\) (\(T\) is intersectionally closed-valued [2]).

(b) \(\bigcap_{z \in I} T(z) = \bigcap_{z \in I} T(z)\) (\(T\) is transfer closed-valued).

(c) \(T\) is closed-valued.

Luc et al. [2] noted that (c) \(\implies\) (b) \(\implies\) (a).

From Theorem 3.2, we obtain the following KKM type theorem;

\textbf{Theorem 3.3.} Let \(I\) be a nonempty set, \((X, D)\) be nonempty \(F\)-convex in a topological vector space \(Y\), and \(T : I \to X\) a map satisfying conditions (3.2.1) and (3.2.2). Then
(α) If $T$ is transfer closed-valued, then $K \cap \bigcap_{z \in I} T(z) \neq \emptyset$.

(β) If $T$ is intersectionally closed-valued, then $\bigcap_{z \in I} T(z) \neq \emptyset$.

Proof. Since $T$ is a generalized KKM map with closed values, by Theorem 3.2, we have $K \cap \bigcap_{z \in I} T(z) \neq \emptyset$.

(α) $T$ is transfer closed-valued, so we have $K \cap \bigcap_{z \in I} T(z) = K \cap \bigcap_{z \in I} \overline{T(z)} \neq \emptyset$.

(β) Since $T$ is intersectionally closed-valued, we have $\bigcap_{z \in I} T(z) = \bigcap_{z \in I} \overline{T(z)} \neq \emptyset$. □

Theorem 3.3 generalizes Theorem 3.1 in [7].

Let $I$ be a nonempty set, $(X, D; \Gamma)$ be an abstract convex space, and $f : I \times X \to \mathbb{R}$ be a function. Let $\gamma \in \mathbb{R}$. We say that $f$ is generalized $\gamma$-quasiconcave in the first variable $z \in I$ if for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\emptyset \neq M \subset N$ implies $\gamma \geq \min_{z \in M} f(z, x)$ for all $x \in \Gamma_{\sigma(M)}$.

The following equivalency of certain concavity of extended real functions and the related generalized KKM maps for abstract convex spaces is in [4];

**Proposition 3.4.** Let $I$ be a nonempty set, $(X, D; \Gamma)$ be an abstract convex space, $f : I \times X \to \mathbb{R}$ be a function, and $\gamma \in \mathbb{R}$. Then the followings are equivalent:

1. The multimap $T : I \to X$, defined by $T(z) = \{x \in X : f(z, x) \leq \gamma\}$ for all $z \in I$, is a generalized KKM map.
2. $f$ is generalized $\gamma$-quasiconcave in the first variable $z$.

From Proposition 3.4, we obtain the following equilibrium result;

**Theorem 3.5.** Let $I$ be a nonempty set, $(X, D)$ be nonempty $F$-convex in a topological vector space $Y$, $f : I \times X \to \mathbb{R}$, and $\gamma \in \mathbb{R}$. Suppose that there exists a nonempty compact subset $K$ of $X$ such that

1. for each $z \in I$, $\{x \in X : f(z, x) \leq \gamma\}$ is intersectionally closed;
2. $f$ is generalized $\gamma$-quasiconcave in the first variable $z$; that is, for each $N := \{y_0, y_1, \ldots, y_n\} \subset I$, there exists a function $\sigma : N \to D$ such that $\gamma \geq \min_{0 \leq j \leq k} f(y_i, \sum_{j=0}^{k} \lambda_j F(\sigma(y_i)))$ for any $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset N$; and
(3) there exists a set $M \in \langle I \rangle$ such that $\bigcap_{z \in M} \{ x \in X \mid f(z, x) \leq \gamma \} \subset K$.

Then there exists an $x_0 \in X$ such that $f(z, x_0) \leq \gamma$ for all $z \in I$.

Proof. Let us define a map $T : I \to X$ by $T(z) = \{ x \in X \mid f(z, x) \leq \gamma \}$ for $z \in I$. Then, by (1), $T$ is intersectionally closed-valued. By Proposition 3.4, (2) implies that $T$ is a generalized KKM map and so is $T$. Therefore, by Theorem 3.3, $\bigcap_{z \in I} T(z) \neq \emptyset$. Hence there exists an $x_0 \in X$ such that $x_0 \in T(z)$ or $f(z, x_0) \leq \gamma$ for all $z \in I$. This completes our proof.

The following is a fixed point theorem for an abstract convex space satisfying the partial KKM principle in Park [11, Theorem 5.4];

**Theorem 3.6.** Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $S : D \to X$, $T : X \to X$ be maps. Suppose that

(1) for each $z \in D$, $S(z)$ is unionly open in $X$;

(2) for each $y \in X$, $\text{co}_T S^-(y) \subset T^-(y)$;

(3) $X = S(D)$; and

(4) there exists a nonempty compact subset $K$ of $X$ such that either

(i) $\bigcap_{z \in N} X \setminus S(z) \subset K$ for some $N \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exists a compact $\Gamma$-convex subset $L_N$ of $X$ relative to some $D' \subset D$ such that $N \subset D'$, $(L_N, D'; \Gamma')$ satisfies the partial KKM principle and $L_N \cap \bigcap_{z \in D'} X \setminus S(z) \subset K$.

Then there exists a $\bar{z} \in X$ such that $\bar{z} \in T(\bar{z})$.

From Theorem 3.6, we obtain the following theorem which generalizes Theorem 3.2 in [6];

**Theorem 3.7.** Let $I$ be a nonempty set, $(X, D)$ be nonempty $F$-convex in a topological vector space $Y$, and $S : D \to X$, $T : X \to X$ be maps. Suppose that

(1) for each $z \in D$, $S(z)$ is unionly open in $X$;

(2) for each $y \in X$, $\text{co}_F S^-(y) \subset T^-(y)$;

(3) $X = S(D)$; and

(4) there exists a nonempty compact subset $K$ of $X$ such that either

(i) $\bigcap_{z \in N} X \setminus S(z) \subset K$ for some $N \in \langle D \rangle$; or
From Theorem 4.1, it follows that 

\[ 0 \text{ is quasi-convex on } Y. \]

Then there exists a \( z \in X \) such that \( z \in T(\bar{z}) \).

4 Some results of quasi-convex function programming

For an abstract convex space \((Y; \Gamma)\), a function \( f : Y \to \mathbb{R} \) is said to be quasi-convex on \( Y \) if \( f(z) \leq \max_{y \in N}\{f(y)\} \) for each \( N \in (Y) \) and \( z \in \Gamma_N \).

For a topological vector space \( Y \), a function \( f : Y \to \mathbb{R} \) is said to be quasi-\( F \)-convex on an \( F \)-convex subset \( X \) of \( Y \) if there is a map \( F : Y \to Y \) such that \( f(\Sigma_{i=0}^n \lambda_i F(x_i)) \leq \max_{i=0, \ldots, n}\{f(x_i)\} \) for each \( \{x_0, \ldots, x_n\} \subseteq X \) and \( \Sigma_{i=0}^n \lambda_i = 1 \).

**Theorem 4.1.** For an abstract convex space \((Y; \Gamma)\), a function \( f : Y \to \mathbb{R} \) is quasi-convex on \( Y \) if and only if the level set \( K_\alpha = \{x \in Y | f(x) \leq \alpha\} \) is \( \Gamma \)-convex for each \( \alpha \in \mathbb{R} \).

**Proof.** For each \( N \in (K_\alpha) \) and \( z \in \Gamma_N \), \( f(z) \leq \max_{x \in N}\{f(x)\} \leq \alpha \). Therefore \( \Gamma_N \subseteq K_\alpha \).

For each \( N \in (Y) \), let \( \alpha = \max_{x \in N}\{f(x)\} \), then \( N \in (K_\alpha) \). Since \( K_\alpha \) is \( \Gamma \)-convex, \( f(z) \leq \alpha \) for all \( z \in \Gamma_N \), which shows that \( f \) is quasi-convex. \( \square \)

From Theorem 4.1, we obtain the following Corollary:

**Corollary 4.2.** For a topological vector space \( Y \), a function \( f : Y \to \mathbb{R} \) is quasi-\( F \)-convex on \( Y \) if and only if the level set \( K_\alpha = \{x \in Y | f(x) \leq \alpha\} \) is \( F \)-convex for each \( \alpha \in \mathbb{R} \).

Note that the intersection of \( \Gamma \)-convex subsets of an abstract convex space \((Y; \Gamma)\) is \( \Gamma \)-convex.

**Theorem 4.3.** For an abstract convex space \((Y; \Gamma)\), a function \( g_i : Y \to \mathbb{R} \) is quasi-convex on \( Y \), \( i = 1, 2, \ldots, m \). Then the set \( X = \{x \in Y | g_i(x) \leq 0, i = 1, 2, \ldots, m\} \) is \( \Gamma \)-convex.

**Proof.** From Theorem 4.1, it follows that \( X_i = \{x \in Y | g_i(x) \leq 0\} \) is \( \Gamma \)-convex, \( i = 1, 2, \ldots, m \), which implies that the set \( \bigcap_{i=1}^m X_i \) is \( \Gamma \)-convex. \( \square \)

**Corollary 4.4.** For a topological vector space \( Y \), a function \( g_i : Y \to \mathbb{R} \) is quasi-\( F \)-convex on \( Y \), \( i = 1, 2, \ldots, m \). Then the set \( X = \{x \in Y | g_i(x) \leq 0, i = 1, 2, \ldots, m\} \) is \( F \)-convex.
For a topological vector space $Y$, a function $f : Y \to \mathbb{R}$ is said to be quasi-semi-$E$-convex on $E$-convex set $X$ of $Y$ if there is a map $E : Y \to Y$ such that $f((1 - \lambda)E(x) + \lambda E(y)) \leq \max\{f(x), f(y)\}$ for each $\{x, y\} \subset X$ and all $\lambda \in [0, 1]$. Quasi-$F$-convexity on an $F$-convex space corresponds to quasi-semi-$E$-convexity on an $E$-convex space. And Corollaries 4.2 and 4.4 are comparable to Propositions 6 and 7 in [1] respectively.

For an abstract convex space $(Y; \Gamma)$, let us consider the following programming problem;

(P) $\text{Min} f(x)$, s.t. $x \in X = \{x \in Y : g_i(x) \leq 0, i = 1, \cdots, m\},$

where $f : Y \to \mathbb{R}$ and $g_i : Y \to \mathbb{R}, i = 1, \cdots, m$ are functions on $Y$, and $X$ is $\Gamma$-convex subset of $Y$.

For an abstract convex space $(Y; \Gamma)$, a function $f : Y \to \mathbb{R}$ is said to be strictly quasi-semi-convex on $Y$ if for each $\{x, y\} \subset D$ such that $x \neq y$, $f(z) < \max\{f(x), f(y)\}$ and $z \in \Gamma_{\{x,y\}} \setminus \{x, y\}$. For a topological vector space $Y$ and an $F$-convex subset $X$ of $Y$, a function $f : Y \to \mathbb{R}$ is said to be strictly quasi-semi-$F$-convex on $X$ if for each $\{x, y\} \subset X$ such that $x \neq y$, $f((1 - \lambda)F(x) + \lambda F(y)) < \max\{f(x), f(y)\}$ for all $\lambda \in (0, 1)$.

The following Theorem shows that the global optimal solutions of problem (P) is unique;

**Theorem 4.5.** For an abstract convex space $(Y; \Gamma)$ and a $\Gamma$-convex subset $X$ of $Y$, let $f : Y \to \mathbb{R}$ be quasi-convex on $X$ and $\alpha = \min_{x \in X}\{f(x)\}$. Then the set $M = \{x \in X \mid f(x) = \alpha\}$ of optimal solutions of problem (P) is $\Gamma$-convex. If $f$ is strictly quasi-semi-convex on $X$, then the set $M$ is a singleton.

**Proof.** For any $N \in \langle M \rangle$ and $x \in N$, $f(x) = \alpha$ and since $f$ is quasi-convex on $X$, $f(z) \leq \max_{x \in N}\{f(x)\} = \alpha$ for all $z \in \Gamma_N$. Which implies that $\Gamma_N \subset M$, that is, $M$ is $\Gamma$-convex.

For the second part, assume that there are $x, y \in M$ and $x \neq y$. Since $f$ is strictly quasi-semi-convex on $X$, $f(z) < \max\{f(x), f(y)\} = \alpha$ for each $z \in \Gamma_{\{x,y\}} \setminus \{x, y\}$. This contradicts that $\alpha = \min_{x \in X}\{f(x)\}$. \qed

**Corollary 4.6.** For a topological vector space $Y$ and an $F$-convex subset $X$ of $Y$, let $f : Y \to \mathbb{R}$ be quasi-$F$-convex on $X$ and $\alpha = \min_{x \in X}\{f(x)\}$. Then the set $M = \{x \in X \mid f(x) = \alpha\}$ of optimal solutions of problem (P) is $F$-convex. If $f$ is strictly quasi-semi-$F$-convex on $X$, then the set $M$ is a singleton.

**Theorem 4.7.** For an abstract convex space $(Y; \Gamma)$, let $f : Y \to \mathbb{R}$ and $g_i : Y \to \mathbb{R}, i = 1, \cdots, m$ be quasi-convex on $Y$. Then the set of optimal solutions of problem (P) is $\Gamma$-convex.
Proof. By Theorem 4.3, the set \( X = \{ x \in Y | g_i(x) \leq 0, i = 1, \ldots, m \} \) is \( \Gamma \)-convex. Hence by Theorem 4.5, the set \( M = \{ x \in X | f(x) = \alpha \} \) of optimal solutions of problem (P) is \( \Gamma \)-convex.

Corollary 4.8. For a topological vector space \( Y \), let \( f : Y \to \mathbb{R} \) and \( g_i : Y \to \mathbb{R} \), \( i = 1, \ldots, m \) be quasi-\( F \)-convex on \( Y \). Then the set of optimal solutions of problem (P) is \( F \)-convex.

Strictly quasi-semi-\( E \)-convexity is defined in the same way as strictly quasi-semi-\( F \)-convexity. For details, see [1]. Corollaries 4.6 and 4.8 correspond to Theorems 8 and 10 in [1] respectively.

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References


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