The Upper Bound for Sugeno Integral
of Nonnegative Convex Functions

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Abstract

Since the Sugeno integral first introduced by Sugeno in [8], many authors have studied its properties and applications. Recently, Caballero [1] proposed an upper bound for Sugeno integral of a nonnegative convex function. But the difference between the upper bound and the value of Sugeno integral is considerable when the convex function is not monotone (See Example 3.4). We introduce new type (that is, the triangular type) of an upper bound function for a nonnegative convex function, which has less value of Sugeno integral than that of Caballero’s. We give some examples to illustrate it.

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1 Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [8] as a tool for modeling nondeterministic problems. The properties and applications of the Sugeno integral have been studied by many authors. Ralescu and Adams [3] generalized the range of fuzzy measures from $[0, 1]$ to $[0, \infty]$ and gave an equivalent definition of fuzzy integral. Román-Flores et al. [4,5] studied the level-continuity of fuzzy integrals and H-continuity of fuzzy measures and geometric inequalities for fuzzy measures and integral. Wang and Klir [9] provided a general overview on fuzzy measurement and fuzzy integration theory. The authors in [2,6,7] presented some fuzzy integral inequalities for monotone functions with applications for solving fuzzy integrals. Recently, Caballero [1] proposed Hermite-Hadamard type inequality for convex functions and gave the upper bound for Sugeno integral of convex functions. He gave some examples for monotone convex functions. But the difference between the upper bound and the value of Sugeno integral is considerable when the convex function is not monotone. The purpose of this paper is to introduce new type (that is, the triangular type) of an upper bound function for a nonnegative convex functions which has less value of Sugeno integral than that of Caballero’s. We give some examples to illustrate it.

2 Preliminaries

In this section, we present some definitions and basic properties of the Sugeno integral.

**Definition 2.1** Let $\Sigma$ be a $\sigma$-algebra of subsets of $R$ and let $\mu : \Sigma \to [0, \infty]$ be a nonnegative, extended real-valued set function. Then we say that $\mu$ is a fuzzy measure if and only if the following properties are satisfied:

(a) $\mu(\emptyset) = 0$.

(b) (monotonicity) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$.

(c) (continuity from below) $\{E_p\} \subseteq \Sigma$, $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$.

(d) (continuity from above) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, $\mu(E_1) < \infty$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$.

We denote by $L_\alpha f = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$ for $\alpha > 0$ the $\alpha$-level of $f$ and $L_0 f = \{x \in X \mid f(x) > 0\} = \text{supp}(f)$ the support of $f$. 
We note that \( \alpha \leq \beta \) means \( \{ f \geq \beta \} \subseteq \{ f \geq \alpha \} \).

For a fuzzy measure \( \mu \) on \( A \subset R \), let \( F^\mu(A) = \{ f : A \to [0, \infty) \mid f \text{ is } \mu\text{-measurable} \} \) be the set of \( \mu\)-measurable functions.

**Definition 2.2** Let \( \mu \) be a fuzzy measure on \( (R, \Sigma) \). If \( f \in F^\mu(R) \) and \( A \in \Sigma \), then the Sugeno integral or fuzzy integral of \( f \) on \( A \) with respect to the fuzzy measure \( \mu \) is defined by

\[
(S) \int_A f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(A \cap F_\alpha)].
\]

If \( A \) is the set \( R \) of real numbers, then

\[
(S) \int_R f d\mu = (S) \int f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(F_\alpha)].
\]

**Remark 2.3** Let \( \text{Dom} f \) be the domain of the function \( f \). Then

\[
(S) \int_A f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu(\text{Dom} f \cap A \cap F_\alpha)].
\]

For example,

\[
(S) \int_0^1 \frac{1}{x} dx = (S) \int_{[0,1]} \frac{1}{x} dx = \sup_{\alpha \in [0, \infty)} [\alpha \land \mu((0,1] \cap F_\alpha)].
\]

The following properties of the Sugeno integral are well-known.

**Proposition 2.4** If \( \mu \) is a fuzzy measure on \( R \) and \( f, g \in F^\mu(X) \), then

1. \( (S) \int_A f d\mu \leq \mu(A) \);
2. If \( f \leq g \) on \( A \), then \( (S) \int_A f d\mu \leq (S) \int_A g d\mu \);
3. \( (S) \int_A k d\mu = k \land \mu(A) \), \( k \) is a nonnegative constant;
4. \( \mu(A \cap \{ f \geq \alpha \}) \geq \alpha \Rightarrow (S) \int_A f d\mu \geq \alpha \);
5. \( \mu(A \cap \{ f \geq \alpha \}) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha \);
6. \( (S) \int_A f d\mu < \alpha \Leftrightarrow \) there exists a \( \gamma < \alpha \) such that \( \mu(A \cap \{ f \geq \gamma \}) < \alpha \);
7. \( (S) \int_A f d\mu > \alpha \Leftrightarrow \) there exists a \( \gamma > \alpha \) such that \( \mu(A \cap \{ f \geq \gamma \}) > \alpha \);
8. If \( \mu(A) < \infty \), then \( (S) \int_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{ f \geq \alpha \}) \geq \alpha \).

**Remark 2.5** Let \( F \) be the distribution function associated with \( f \) on \( A \), that is, \( F(\alpha) = \mu(A \cap \{ f \geq \alpha \}) \). Then, by the properties (4) and (5) of Proposition 2.4, \( F(\alpha) = \alpha \) implies \( (S) \int_A f d\mu = \alpha \). Thus, from a numerical point of view, the fuzzy integral can be calculated by solving the equation \( F(\alpha) = \alpha \).
3 Main Results

In [1], Caballero proposed Hermite-Hadamard type inequality for convex functions and they gave estimates for Sugeno integral.

**Theorem 3.1** [1] Let \( f : [0, 1] \to [0, \infty) \) be a convex function such that \( f(0) < f(1) \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
(S) \int_0^1 f \, d\mu \leq \min \left\{ \frac{f(1)}{1 + f(1) - f(0)}, 1 \right\}.
\]

**Theorem 3.2** [1] Let \( f : [0, 1] \to [0, \infty) \) be a convex function such that \( f(0) > f(1) \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
(S) \int_0^1 f \, d\mu \leq \min \left\{ \frac{f(0)}{1 + f(0) - f(1)}, 1 \right\}.
\]

**Remark 3.3** If \( f(0) = f(1) \) in the above Theorem 3.1 and 3.2, then

\[
(S) \int_0^1 f \, d\mu \leq \min \{f(0), 1\}.
\]

In fact the upper bound of Sugeno integral for the convex function in Theorem 3.1, 3.2 and Remark 3.3 is the value of Sugeno integral of the line that connects \((0, f(0))\) and \((1, f(1))\). But since \( f \) is convex, if \( f \) is not monotone, then the difference between the upper bound and the values of Sugeno integral of \( f \) is considerable. We now consider more general situation for a convex function. Hence we take one more point on the graph of \( f \) so that three points on the graph are connected with line segments. Then we get another upper bound that is smaller than Caballero’s. We give the next example to demonstrate this.

**Example 3.4** Consider \( f(x) = (x - \frac{1}{3})^2 + \frac{1}{4} \) on \([0, 1]\). Then \( f \) is nonnegative, nondecreasing and convex on the interval \([0, 1]\). Let

\[
g(x) = \begin{cases} 
2(f(\frac{1}{2}) - f(0))x + f(0) & \text{if } 0 \leq x \leq \frac{1}{2} \\
2(f(1) - f(\frac{1}{2}))(x - 1) + f(1) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

and \( h(x) = (f(1) - f(0))x + f(0) \). It is easy to calculate the values for Sugeno integral as follows;

\[
(S) \int_0^1 f \, d\mu = 3 - \sqrt{7} \approx 0.354248689
\]

\[
(S) \int_0^1 g \, d\mu = \frac{15}{41} \approx 0.3658536585
\]

\[
(S) \int_0^1 h \, d\mu = \frac{25}{48} \approx 0.5208333333
\]
In Theorem 3.1, 3.2 and Remark 3.3, h is used as the upper bound of f and we noticed that the value of Sugeno integral of g is less than that of h. Thus we will show that g is an another upper bound of f.

We now give the definitions of a triangular type and a line type of upper bound function for a convex function.

**Definition 3.5** Let \( f : [0, 1] \to [0, \infty) \) be a convex function. Then for a convex function \( f \) at any \( t \) in \([0,1]\), the triangular type of an upper bound function \( T_f \) is defined as a mapping from \([0,1]\) to \([0, \infty)\) given by

\[
T_f(x) = \begin{cases} 
   t(0) - f(0) & \text{if } 0 \leq x \leq t \\
   f(1) - f(t) & \text{if } t \leq x \leq 1
\end{cases}
\]

Similarly, line type of an upper bound function \( L_f \) is defined as a mapping from \([0,1]\) to \([0, \infty)\) given by

\[
L_f = f(1) - f(0) x + f(0), \quad \text{for } x \in [0,1]
\]

**Note.** In Example 3.4, we see that \( g = T_f \) and \( h = L_f \) by Definition 3.5.

**Theorem 3.6** Let \( f : [0, 1] \to [0, \infty) \) be a convex function. Then

\[
(S) \int_0^1 f d\mu \leq (S) \int_0^1 T_f d\mu \leq (S) \int_0^1 L_f d\mu \quad \text{for all } t \in (0, 1).
\]
Proof. It is clear by the convexity of \( f \).

Theorem 3.6 shows that when we evaluate the Sugeno integral, the value for the triangular type of an upper bound function is smaller than the value for the line type. We now give the definition of \( t \)-estimate for a convex function by using Sugeno integral of \( T_{f_t} \).

**Definition 3.7** Let \( f : [0, 1] \to [0, \infty) \) be a convex function. Then \( t \)-estimate \( T_f \) of \( f \) is defined by

\[
T_f = \inf_{t \in (0, 1)} (S) \int_0^1 T_{f_t} d\mu.
\]

**Theorem 3.8** Let \( f : [0, 1] \to [0, 1] \) be a convex function with \( f(0) = f(1) \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
(S) \int_0^1 T_{f_t} d\mu = \min \left\{ \frac{f(0)}{1 - f(t) + f(0)}, 1 \right\}.
\]

Moreover, if \( f \) assumes a minimum at \( x = c \), then

\[
T_f = (S) \int_0^1 T_{f_t} d\mu.
\]

**Proof.** If \( f \) has a minimum at \( x = c \) and it holds (1), then it is trivial that \( f \) holds (2). Hence we just need to show that \( f \) holds (1). By Definition 3.7,

\[
T_{f_t}(x) = \begin{cases} 
\frac{f(0) - f(0)}{f(1) - f(t)} x + f(0) & \text{if } 0 \leq x \leq t \\
\frac{f(0) - f(0)}{1 - t} (x - 1) + f(1) & \text{if } t \leq x \leq 1.
\end{cases}
\]

To evaluate the Sugeno integral of \( T_{f_t} \), we consider the distribution function \( F \) given by

\[
F(\alpha) = \mu ([0, 1] \cap \{T_{f_t} \geq \alpha\}).
\]

Since \( f(t) < f(0) \), \( F(\alpha) \) can be written as

\[
F(\alpha) = 1 - \mu \left( \frac{(\alpha - f(0)) t (\alpha - f(1))(1 - t)}{f(t) - f(0)} \right).
\]

In particular, if \( f(0) = f(1) \), then

\[
F(\alpha) = \frac{\alpha - f(0)}{f(t) - f(0)}.
\]

Thus the solution of the equation \( F(\alpha) = \alpha \) is \( \alpha = \frac{f(0)}{1 - f(t) + f(0)} \), and then

\[
(S) \int_0^1 T_{f_t} d\mu = \min \left\{ \frac{f(0)}{1 - f(t) + f(0)}, 1 \right\}.
\]
As a result of Theorem 3.8, we may conjecture that if \( f \) has a minimum at \( x = c \), then

\[
T_f = (S) \int_0^1 T_f(x) \, d\mu.
\] (3)

But the next example shows that it does not hold.

**Example 3.9** Consider the function \( f(x) = (x - \frac{1}{3})^2 + \frac{1}{4} \) on \([0, 1] \) for which we mentioned in Example 3.4. By Definition 3.5,

\[
T_{f_t}(x) = \begin{cases} 
\frac{f(t) - f(0)}{t} x + f(0) = (t - \frac{2}{3}) x + \frac{13}{36} & \text{if } 0 \leq x \leq t \\
\frac{f(1) - f(t)}{t} (x - 1) + f(1) = (t + \frac{1}{3}) x - t + \frac{13}{36} & \text{if } t \leq x \leq 1.
\end{cases}
\]

To evaluate the Sugeno integral for \( T_{f_t} \), we consider the distribution function \( F \) given by the following two cases:

(a) When \( t \in [0, \frac{2}{3}] \), let

\[
F(\alpha) = \mu([0, 1] \cap \{h \geq \alpha\}) = \mu\left(\left[-13 + 36\alpha, \frac{36t - 13 + 36\alpha}{12(3t - 2)}\right]\right) = \frac{3(4t - 7 + 12\alpha)}{4(3t - 2)(3t + 1)}.
\]

Then the solution of the equation \( F(\alpha) = \alpha \) is

\[
\alpha = \frac{3(4t - 7)}{4(9t^2 - 3t - 11)}.
\]

(b) When \( t \in [\frac{2}{3}, 1] \), let

\[
F(\alpha) = \mu([0, 1] \cap \{h \geq \alpha\}) = \mu\left(\left[-13 + 36\alpha, \frac{36t - 11 - 36\alpha}{12(3t - 2)}\right]\right) = \frac{36t - 11 - 36\alpha}{12(3t - 2)}.
\]

Then the solution of the equation \( F(\alpha) = \alpha \) is

\[
\alpha = \frac{36t - 11}{12(3t + 1)}.
\]
Hence we get

\[
(S) \int_0^1 T_f(t) \, d\mu = \begin{cases} 
\min \left\{ \frac{3(4t - 7)}{4(9t^2 - 3t - 11)}, 1 \right\} & \text{if } 0 \leq t < \frac{2}{3} \\
\min \left\{ \frac{36t - 11}{12(3t + 1)}, 1 \right\} & \text{if } \frac{2}{3} \leq t \leq 1.
\end{cases}
\]

and then

\[T_f = \inf_{t \in (0,1)} (S) \int_0^1 T_f(t) \, d\mu \approx 0.3606\]

where \( c = \frac{7}{4} - \frac{\sqrt{185}}{12} \).

But since \( f \) assumes a minimum at \( x = \frac{1}{3} \), it does not hold (3).

\[
(S) \int_0^1 T_f(t) \, d\mu
\]

Figure 2: The graphs of \((S) \int_0^1 T_f(t) \, d\mu\) in Example 3.9

Note. For a convex function \( f \), it is quite difficult to find the point at which \((S) \int_0^1 T_f(t) \, d\mu\) has a minimum as we see in Example 3.9.

References


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