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# Order Properties of the Space of Dominated Uryson Operators

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**Abstract**

The "Up-and-down" theorem which describe the structure of the Boolean algebra fragments of a positive orthogonally additive operator was recently proved in [14]. This result we apply to prove the decomposability of the lattice valued norm of the space $\mathcal{D}_U(V,W)$ of all dominated Uryson operators. We obtain that, for a lattice-normed space $V$ and a Banach-Kantorovich space $W$ the space $\mathcal{D}_U(V,W)$ is a Banach-Kantorovich space.

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## 1. Introduction

Today the theory of regular operators in vector lattices is a very large area of Functional Analysis to which many textbooks are devoted [1, 2, 4, 17]. Nonlinear maps between vector lattices in an involved subject. An interesting class of nonlinear maps called abstract Uryson operators was introduced and studied in 1990 by Mazón and de León [8, 9], and then considered to be defined on lattice-normed spaces by Kusraev and the first named author [3, 5, 6, 10, 11, 12]. Narrow operators in setting of vector lattices and lattice-normed spaces were investigated in [7, 13, 15]. The aim of this note is continue this line of investigations. We prove the space of dominated Uryson operators from decomposable lattice-normed space to Banach-Kantorovich space is Banach-Kantorovich space with respect of the dominant norm.[1]

## 2. Preliminary information

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices and lattice-normed spaces the reader can find in the books [2, 4, 17].

**Definition 2.1.** Consider a vector space $V$ and a real archimedean vector lattice $E$. A map $\lvert\!\lvert \cdot \rvert\!\rvert : V \to E$ is a *vector norm* if it satisfies the following axioms:

1) $\lvert\!\lvert v \rvert\!\rvert \geq 0$; $\lvert\!\lvert v \rvert\!\rvert = 0 \Leftrightarrow v = 0$; $(v \in V)$;
2) $\lvert\!\lvert v_1 + v_2 \rvert\!\rvert \leq \lvert\!\lvert v_1 \rvert\!\rvert + \lvert\!\lvert v_2 \rvert\!\rvert$ ; $(v_1, v_2 \in V)$;
3) $\lvert\!\lvert \lambda v \rvert\!\rvert = |\lambda| \lvert\!\lvert v \rvert\!\rvert$ ; $(\lambda \in \mathbb{R},\, v \in V)$.

A vector norm is said to be *decomposable* if

4) for all $e_1, e_2 \in E_+$ and $x \in V$ the condition $\lvert\!\lvert x \rvert\!\rvert = e_1 + e_2$ implies the existence of $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $\lvert\!\lvert x_k \rvert\!\rvert = e_k$, $(k := 1, 2)$.

In the case where condition (4) is valid only for disjoint $e_1, e_2 \in E_+$, the norm is said to be *disjointly-decomposable* or, in short, *d-decomposable*.

A triple $(V, \lvert\!\lvert \cdot \rvert\!\rvert, E)$ (in brief $(V, E)$, $(V, \lvert\!\lvert \cdot \rvert\!\rvert)$ or $V$ with default parameters omitted) is a *lattice-normed space* if $\lvert\!\lvert \cdot \rvert\!\rvert$ is an $E$-valued vector norm in the vector space $V$. If the norm $\lvert\!\lvert \cdot \rvert\!\rvert$ is decomposable then the space $V$ itself is called decomposable. We say that a net $(v_\alpha)_{\alpha \in \Delta}$ *(bo)-converges* to an element $v \in V$ and write $v = bo\text{-}\lim v_\alpha$ if there exists a decreasing net $(e_\gamma)_{\gamma \in \Gamma}$ in $E_+$ such that $\inf_{\gamma \in \Gamma}(e_\gamma) = 0$ and for every $\gamma \in \Gamma$ there is an index $\alpha(\gamma) \in \Delta$ such that $\lvert\!\lvert v - v_{\alpha(\gamma)} \rvert\!\rvert \leq e_\gamma$ for all $\alpha \geq \alpha(\gamma)$. A net $(v_\alpha)_{\alpha \in \Delta}$ is called *(bo)-fundamental* if the net $(v_\alpha - v_\beta)_{(\alpha,\beta) \in \Delta \times \Delta}$ $(bo)$-converges to zero. A lattice-normed space is called *(bo)-complete* if every $(bo)$-fundamental net $(bo)$-converges to an element of this space. Let $e$ be a positive element of a vector lattice $E$. By $[0, e]$ we denote the set $\{v \in V : \lvert\!\lvert v \rvert\!\rvert \leq e\}$. A set $M \subset V$ is called (bo)-*bounded* if there exists $e \in E_+$ such that $M \subset [0, e]$. Every decomposable $(bo)$-complete lattice-normed space is called a *Banach-Kantorovich space* (a BKS for short).

**Definition 2.2.** Let $E$ be a vector lattice and $X$ a vector space. An orthogonally additive map $T : E \to X$ is called even if $T(x) = T(-x)$ for every $x \in E$.

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If $E, F$ are vector lattices, the set of all even abstract Uryson operators from $E$ to $F$ we denote by $\mathcal{U}^{ev}(E, F)$.

If $E, F$ are vector lattices with $F$ Dedekind complete, the space $\mathcal{U}^{ev}(E, F)$ is not empty. Indeed, for every $T \in \mathcal{U}(E, F)$ by ([8],Proposition 3.4) there exists an even operator $\widetilde{T} \in U_+^{ev}(E, F)$ which is defined by the formula,

$$\widetilde{T}f = \sup\{|T|g : |g| \leq |f|\}.$$

**Lemma 2.3.** *([16], Lemma 3.2.) Let $E, F$ be vector lattices with $F$ Dedekind complete. Then $\mathcal{U}^{ev}(E, F)$ is a Dedekind complete sublattice of $\mathcal{U}(E, F)$.*

**Definition 2.4.** Let $(V, E)$ and $(W, F)$ be lattice-normed spaces. A map $T : V \to W$ is called *orthogonally additive* if $T(u + v) = Tu + Tv$ for every $u, v \in V$, $u \perp v$. An orthogonally additive map $T : V \to W$ is called a *dominated Uryson* operator if there exists $S \in \mathcal{U}_+^{ev}(E, F)$ such that $\lvert Tv \rvert \leq S \lvert v \rvert$ for every $v \in V$. In this case we say that $S$ is a *dominant* for $T$. The set of all dominants of the operator $T$ is denoted by $\mathrm{Domin}(T)$. If there is the least element in $\mathrm{Domin}(T)$ with respect to the order induced by $\mathcal{U}_+^{ev}(E, F)$ then it is called the *least* or the *exact dominant* of $T$ and is denoted by $\lvert T \rvert$. The set of all dominated Uryson operators from $V$ to $W$ is denoted by $\mathcal{D}_U(V, W)$.

**Theorem 2.5.** *([16],Theor. 3.4.) Let $(V, E)$, $(W, F)$ be lattice-normed spaces with $V$ decomposable and $F$ Dedekind complete. Then every dominated Uryson operator $T : V \to W$ has an exact dominant $\lvert T \rvert$.*

**Definition 2.6.** A subset $D$ of a vector lattice $E$ is called *lateral ideal if the following conditions hold*

(1) *if $x \in D$ then $y \in D$ for every $y \in \mathcal{F}_x$;*
(2) *if $x, y \in D$, $x \perp y$ then $x + y \in D$.*

Consider some examples.

**Example 1.** Let $E$ be a vector lattice. Every order ideal in $E$ is a lateral set.

**Example 2.** Let $E, F$ be a vector lattices and $T \in \mathcal{U}_+(E, F)$. Then $\mathcal{N}_T := \{e \in E : T(e) = 0\}$ is a lateral ideal.

For further consideration we introduce the following set

$$\widetilde{E}_+ = \{e \in E_+ : e = \bigsqcup_{i=1}^{n} \lvert v_i \rvert ; v_i \in V; n \in \mathbb{N}\}.$$

**Example 3.** ([16],Lemma 3.6.) Let $(V, E)$ be lattice-normed spaces with $V$ decomposable. Then $\widetilde{E}_+$ is a lateral ideal.

**Theorem 2.7.** *([16],Theor.3.7.) Let $(V, E)$, $(W, F)$ be the same as in Theorem 2.5. Then the exact dominant of a dominated Uryson operator $T : V \to W$ can be calculated by the following formulas*

(1) $\mathbf{|}T\mathbf{|}(e) = \sup\Big\{\sum\limits_{i=1}^{n} \mathbf{|}Tu_i\mathbf{|} : \coprod_{i=1}^{n} \mathbf{|}u_i\mathbf{|} = e,\, n \in \mathbb{N}\Big\}\,(e \in \widetilde{E}_+)$;

(2) $\mathbf{|}T\mathbf{|}(e) = \sup\Big\{\mathbf{|}T\mathbf{|}(e_0) : e_0 \in \widetilde{E}_+,\, e_0 \sqsubseteq e\Big\}; (e \in E_+)$

(3) $\mathbf{|}T\mathbf{|}(e) = \mathbf{|}T\mathbf{|}(e_+) + \mathbf{|}T\mathbf{|}(e_-), e \in E$.

Let $E, F$ be vector lattices with $F$ Dedekind complete and $T \in \mathcal{U}_+(E, F)$. For further considerations is very important to describe the fragments of $T$. That is

$$\mathcal{F}_T = \{S \in \mathcal{U}_+(E, F) : S \wedge (T - S) = 0\}.$$

Like in the linear case we consider elementary fragments. For a subset $\mathcal{A}$ of a vector lattice $W$ we employ the following notation:

$$\mathcal{A}^{\uparrow} = \{x \in W : \exists \text{ a net } (x_\alpha) \subset \mathcal{A} \text{ with } x_\alpha \uparrow x\}.$$

The meanings of $\mathcal{A}^{\downarrow}$ is analogous. As usual, we also write

$$\mathcal{A}^{\downarrow\uparrow} = (\mathcal{A}^{\downarrow})^{\uparrow}; \mathcal{A}^{\uparrow\downarrow\uparrow} = ((\mathcal{A}^{\uparrow})^{\downarrow})^{\uparrow}.$$

It is clear that $\mathcal{A}^{\downarrow\downarrow} = \mathcal{A}^{\downarrow}$, $\mathcal{A}^{\uparrow\uparrow} = \mathcal{A}^{\uparrow}$. Consider a positive abstract Uryson operator $T : E \to F$, where $F$ is Dedekind complete. Since $\mathcal{F}_T$ is a Boolean algebra, it is closed under finite suprema and infima. In particular, all "ups and downs" of $\mathcal{F}_T$ are likewise closed under finite suprema and infima, and therefore they are also directed upward and, respectively, downward.

Let $T \in \mathcal{U}_+(E, F)$ and $D \subset E$ be a lateral ideal. Then for every $x \in E$, is defined a map $\pi^D T : E \to F_+$ by the following formula

$$\pi^D T(x) = \sup\{Ty : y \in \mathcal{F}_x \cap D\}. \tag{2.1}$$

**Lemma 2.8.** *([3],Lemma 3.6). Let $E, F$ be vector lattices with $F$ Dedekind complete, $\rho \in \mathfrak{B}(F)$, $T \in \mathcal{U}_+(E, F)$ and $D$ be a lateral ideal. Then $\pi^D T$ is a positive abstract Uryson operator and $\rho\pi^D T \in \mathcal{F}_T$.*

If $D = \mathcal{F}_x$ then the operator $\pi^D T$ is denoted by $\pi^x T$. Let $F$ be a vector lattice. Recall that a family of mutually disjoint order projections $(\rho_\xi)_{\xi\in\Xi}$ on $F$ is said to be *partitions of unity* if $\bigvee\limits_{\xi\in\Xi}(\rho_\xi)_{\xi\in\Xi} = Id_F$. Any fragment of the form $\sum\limits_{i=1}^{n} \rho_i \pi^{x_i} T$, $n \in \mathbb{N}$, where $\rho_1, \dots, \rho_n$ is a finite family of mutually disjoint order projections in $F$, like in the linear case is called an *elementary* fragment of $T$. The set of all elementary fragments of $T$ we denote by $\mathcal{A}_T$.

The following theorem is important for further considerations.

**Theorem 2.9.** *([14],Theorem 3.14). Let $E, F$ be vector lattices, $F$ Dedekind complete, $T \in \mathcal{U}_+(E, F)$ and $S \in \mathcal{F}_T$. Then $S \in \mathcal{A}_T^{\uparrow\downarrow\uparrow}$.*

## 3. Decomposability of the space of dominated Uryson operator

In this section we give an application of the Theorem 2.9 and establish that the set of all dominated Uryson operators is a Banach-Kantorovich space with respect to the the dominant norm.

The following theorem is a main result of the article.

**Theorem 3.1.** *Let $(V, E)$, $(W, F)$ be lattice-normed spaces with $V$ decomposable and $W$ a Banach-Kantorovich space. Then the space of all dominated Uryson operators $\mathcal{D}_U(V, W)$ is a Banach-Kantorovich space.*

First we need the following lemma.

**Lemma 3.2.** *Let $(V, E)$, $(W, F)$ be lattice-normed spaces with $V$ decomposable and $F$ Dedekind complete. Then the set of all dominated Uryson operator $\mathcal{D}(V, W)$ with the map $|\!|\cdot|\!| : \mathcal{D}(V, W) \to \mathcal{U}^{ev}_{+}(E, F)$ is a lattice-normed space.*

*Proof.* As we saw in ([16], Theorem 3.4), in the case of a decomposable lattice-normed space $V$ and a Dedekind complete vector lattice $F$, the dominant norm $|\!|T|\!| \in \mathcal{U}^{ev}_{+}(E, F)$ is well defined for each dominated Uryson operator $T \in \mathcal{D}(V, W)$. So, the set $\mathcal{D}(V, W)$ with the map $|\!|\cdot|\!| : \mathcal{D}(V, W) \to \mathcal{U}^{ev}_{+}(E, F)$ is a lattice-normed space too. The first and second axioms of the Definition 2.1 are obvious. Now, for every $T_1, T_2 \in \mathcal{D}(V, W)$ and $v \in V$ we may write

$$|\!|(T_1 + T_2)v|\!| \leq |\!|T_1 v|\!| + |\!|T_2 v|\!| \leq S_1\, |\!|v|\!| + S_2\, |\!|v|\!| =$$

$$= (S_1 + S_2)\, |\!|v|\!|\,,$$

where $S_1, S_2$ are dominants for $T_1$ and $T_2$ respectively. Therefore Domin$(T_1 + T_2) \supset$ Domin$(T_1)+$Domin$(T_2)$ and the third axiom of the Definition 2.1 is also valid. □

**Lemma 3.3.** *Let $(V, E)$, $(W, F)$ be lattice normed spaces with $V$ decomposable and $W$ $(bo)$-complete. Then the lattice-normed space $\mathcal{D}(V, W)$ is $(bo)$-complete.*

*Proof.* Let $(T_\alpha)$ be $bo$-fundamental net in $\mathcal{D}(V, W)$. It means that for $\alpha, \beta \geq \gamma$ we have $|\!|T_\alpha - T_\beta|\!| \leq S_\gamma$, where the net is decreasing and converges to zero in $\mathcal{U}^{ev}(E, F)$. Then we may write

$$|\!|T_\alpha v - T_\beta v|\!| \leq S_\gamma(|\!|v|\!|).\ (\star)$$

Hence the net $(T_\alpha v)$ is also $bo$-fundamental in $(W, F)$ for every $v \in V$. Since $W$ is $bo$-complete, there exists an orthogonally additive operator $T : V \to W$ defined by the formula $Tv = bo\text{-}\lim_\alpha T v_\alpha$. Passage to the limit over $\alpha$ in the $(\star)$ gives $|\!|Tv - T_\beta v|\!| \leq S_\gamma(|\!|v|\!|)$. Thus, we have

$$|\!|Tv|\!| \leq |\!|Tv - T_\beta v|\!| + |\!|T_\beta v|\!| \leq S_\gamma(|\!|v|\!|) + |\!|T_\beta|\!|\, |\!|v|\!|\,,$$

and the operator $T$ is dominated. Let us show that $T = bo\text{-}\lim_\alpha T\alpha$. Fix $e \in E_+$ and take $v_1, \ldots, v_n \in V$ so that $(|\!|v_1|\!| + \cdots + |\!|v_n|\!|) \sqsubseteq e$. Then we may write

$$\sum_{i=1}^{n} \left| T_\alpha v_i - T_\beta v_i \right| \leq \sum_{i=1}^{n} S_\gamma(\left| v_i \right|) \leq S_\gamma(e).$$

Passing to the order limit over $\alpha$ and taking the supremum over all finite families $(v_1, \dots, v_v)$ we have $\left| T - T_\beta \right| \leq S_\gamma$ for all $\alpha \geq \gamma$ and therefore finishing the proof. □

**Lemma 3.4.** *([4],Proposition 2.1.2.3). Let $(V, E)$ be a lattice-normed space with $V$ decomposable. Then for a pair of disjoint elements $e_1, e_2 \in E_+$ the decomposition $v = v_1 + v_2$, where $\left| v_1 \right| = e_1$, $\left| v_2 \right| = e_2$ is unique.*

**Lemma 3.5.** *Let $E, F$ be vector lattices with $F$ Dedekind complete, and let $D \subset E$ be a lateral ideal. Then for every $S \in \mathcal{U}_+(E, F)$, $e \in D$ the following equality holds*

$$(\pi^D)^\perp Se = 0.$$

*Proof.* By definition, $(\pi^D)Se = \sup\{Se_0 : e_0 \sqsubseteq e, e_0 \in D\}$. Then for an arbitrary $e \in D$ we have $Se = (\pi^D)Se$. Therefore,

$$(\pi^D)^\perp Se = (\pi^D)^\perp \circ (\pi^D)Se = 0.$$

□

**Lemma 3.6.** *Let $(V, E)$, $(W, F)$ be lattice normed spaces with $V$ decomposable and $W$ $(bo)$-complete. Suppose $D$ is an arbitrary lateral ideal in $E$. For every $T \in \mathcal{D}_U(V, W)$ there exists a unique operator $\pi^D T \in \mathcal{D}_U(V, W)$ such that $\left| \pi^D T \right| = \pi^D \left| T \right|$ and $\left| T - \pi^D T \right| = \left| T \right| - \pi^D \left| T \right|$.*

*Proof.* Denote $\Psi := \left| T \right|$. Take an element $v \in V$. We are going to construct a net $(v_\alpha)_{\alpha \in \Lambda}$ (which depends on the $D$) for $v$ with the following properties: $o\text{-}\lim_\alpha \pi^D \Psi \left| v - v_\alpha \right| = 0$ and $o\text{-}\lim_\alpha (\pi^D)^\perp \Psi \left| v_\alpha \right| = 0$. This net can be constructed by the following procedure. Assign $e_\alpha := \alpha \sqsubseteq \left| v \right|$, $\alpha \in D$ and $f_\alpha = \left| v \right| - e_\alpha$. Observe that $(e_\alpha)_{\alpha \in \Delta}$ is the set of all fragments of $\left| v \right|$, which ordered by inclusion, $e_\alpha \perp f_\alpha$ for every $\alpha$ and $e_\alpha \sqsubseteq e_\beta$, $\alpha, \beta \in \Lambda$, $\beta \geq \alpha$. By the decomposability of $V$, there exists a net $(v_\alpha)_{\alpha \in \Lambda} \subset V$, such that

$$\left| v_\alpha \right| = e_\alpha; \ \left| v - v_\alpha \right| = f_\alpha, \ \alpha \in \Lambda.$$

Moreover, using the fact that $e_\alpha \sqsubseteq e_\beta$ we may write

$$e_\beta = e_\alpha + (e_\beta - e_\alpha); \ e_\alpha \perp (e_\beta - e_\alpha);$$

$$\left| v_\beta - v_\alpha \right| = (e_\beta - e_\alpha) = \left| v_\beta \right| - \left| v_\alpha \right| \in D.$$

The net $(v_\alpha)_{\alpha \in \Lambda}$ is said to be *cut* for $v \in V$ (with respect to the $D$). For such a net the limit $bo\text{-}\lim_\alpha Tv_\alpha$ exists. Indeed, for all $\beta \geq \alpha$ we have

$$\begin{aligned} \left| Tv_\beta - Tv_\alpha \right| &= \left| T(v_\beta - v_\alpha) + Tv_\alpha - Tv_\alpha \right| \leq \Psi \left| v_\beta - v_\alpha \right| \\ &= \pi^D \Psi(\left| v_\beta - v_\alpha \right|) + (\pi^D)^\perp \Psi(\left| v_\beta - v_\alpha \right|) = \\ &= \pi^D \Psi(\left| v_\beta \right| - \left| v_\alpha \right|) + (\pi^D)^\perp \Psi(\left| v_\beta \right| - \left| v_\alpha \right|) \leq \end{aligned}$$

$$\leq \pi^D\Psi(|v| - |v_\alpha|) + (\pi^D)^\perp\Psi(|v_\beta| - |v_\alpha|) =$$
$$= \pi^D\Psi(|v| - |v_\alpha|) \downarrow 0.$$

Observe that by Lemma 3.5 $(\pi^D)^\perp\Psi(|v_\beta| - |v_\alpha|) = 0$ for every $\alpha, \beta \in \Lambda$, $\beta \geq \alpha$. Thus, the net $(Tv_\alpha)$ is $(bo)$-fundamental and $(bo)$-limit exists by $(bo)$-completeness of $W$. By Lemma 3.4, the net $(v_\alpha)$ is unique. Hence, the operator $\pi^D T : V \to W$ is defined by the formula $\pi^D Tv = bo\text{-}\lim_\alpha Tv_\alpha$, where $v_\alpha$ is a cut net for $v$. It is clear, that if $v_1, v_2 \in V$, $v_1 \perp v_2$ and $(v^1_\alpha)$, $(v^2_\alpha)$ are cut nets for $v_1, v_2$ then $(v^1_\alpha + v^2_\alpha)$ is a cut net for $v_1 + v_2$. Consequently, taking into account the definition of the operator $\pi^D T$, we have

$$\pi^D T(v_1 + v_2) = \pi^D Tv_1 + \pi^D Tv_2;\ v_1, v_2 \in V;\ v_1 \perp v_2.$$

Moreover, the operator $\pi^D T$ is a dominated Uryson operator by the following inequalities

$$|\pi^D Tv| = o\text{-}\lim_\alpha |Tv_\alpha| \leq o\text{-}\lim_\alpha \Psi(|v_\alpha|) = \pi^D\Psi(|v|).$$

So, $\pi^D T \in \mathcal{D}_U(E, F)$ and $|\pi^D T| \leq \pi^D\Psi$. For the operator $T - \pi^D T$ and arbitrary $v \in V$ we may write

$$|(T - \pi^D T)v| = o\text{-}\lim_\alpha |Tv - Tv_\alpha| \leq o\text{-}\lim_\alpha \Psi(|v - v_\alpha|) = (\pi^D)^\perp\Psi\,|v|\,.$$

Therefore $|T - \pi^D T| \leq (\pi^D)^\perp\Psi$. Next, we obtain

$$|T| \leq |\pi^D T| + |T - \pi^D T| \leq \pi^D\Psi + (\pi^D)^\perp\Psi = \Psi = |T|\,;$$

and therefore $|\pi^D T| = \pi^D\Psi$ and $|T - \pi^D T| = (\pi^D)^\perp\Psi$. Let us prove the uniqueness of the operator $\pi^D T$. Assume that there is an operator $\overline{T}$ with the same properties as $\pi^D T$:

$$|\overline{T}| = \pi^D\,|T|\,;\ |T - \overline{T}| = |T| - \pi^D\,|T|\,.$$

Then we may write

$$\pi^D\,|\overline{T} - \pi^D T| \leq \pi^D\,|T - \pi^D T| +$$
$$+\pi^D\,|T - \overline{T}| = \pi^D(2(\pi^D)^\perp\,|T|) = 0;$$
$$(\pi^D)^\perp\,|\overline{T} - \pi^D T| \leq (\pi^D)^\perp(|\pi^D T| + |\overline{T}|) =$$
$$= (\pi^D)^\perp(2\pi^D\,|T|) = 0.$$

Finally we have $|\overline{T} - \pi^D T| = 0$ or $\overline{T} = \pi^D T$. □

**Corollary 3.7.** *Let $T$, $D$ be the same as in Lemma 3.3. Then the map $\pi^D$ : $T \mapsto \pi^D T$ is a linear projection in $\mathcal{D}_U(V, W)$.*

*Proof.* It is proven that $\pi^D$ is a band projection in $\mathcal{U}(E, F)$. Therefore we may write

$$|T - \pi^D T| = |T| - \pi^D\,|T|\,,$$

and replacing $T$ with $\pi^D T$ we have

$$|\pi^D T - (\pi^D)^2 T| = |\pi^D T| - (\pi^D)^2\,|T| = 0.$$

□

**Lemma 3.8.** *Let $(V,E)$, $(W,F)$ be the same as in Lemma 3.6. Suppose $(T_\alpha)_{\alpha\in\Lambda}$ is a net of dominated Uryson operators, so that for some $R\in\mathcal{D}_U(V,W)$ the equality $|R-T_\alpha|\wedge|T_\alpha|=0$ is valid for all $\alpha\in\Lambda$ and there exists $S:=o\text{-}\lim_\alpha|T_\alpha|$. Then $S\in\mathcal{F}_{|R|}$ and the equality $T:=bo\text{-}\lim_\alpha T_\alpha$ well defines a dominated Uryson operator $T:V\to W$ with $|T|=S$.*

*Proof.* By $T_\alpha\perp(R-T_\alpha)$ in view ([4], 2.1.2) we deduce

$$|R|=|R-T_\alpha+T_\alpha|=|R-T_\alpha|+|T_\alpha|$$

and $|R-T_\alpha|=|R|-|T_\alpha|$ for every $\alpha\in\Lambda$. Therefore $|T_\alpha|\in\mathcal{F}_{|R|}$ and $S\in\mathcal{F}_{|R|}$. Denote $\Psi:=|R|$ for short. Since $|T_\alpha-T_\beta|\leq 2\Psi$, and $2S$ is a fragment of $2\Psi$, we may write

$$\begin{aligned}|T_\alpha-T_\beta|&=(2\Psi-2S+2S)\wedge|T_\alpha-T_\beta|\\&\leq(2\Psi-2S)\wedge|T_\alpha-T_\beta|+2S\wedge|T_\alpha-T_\beta|\leq\\&\leq(2\Psi-2S)\wedge(|T_\alpha|+|T_\beta|)+2S\wedge(|R-T_\alpha|+|R-T_\beta|)\\&=(2\Psi-2S)\wedge(|T_\alpha|+|T_\beta|)+2S\wedge(2|R|-|T_\alpha|-|T_\beta|).\end{aligned}$$

Thus, we have that the net $(T_\alpha)_{\alpha\in\Lambda}$ is $(bo)$-fundamental. Then there exists an orthogonally additive operator $T=bo\text{-}\lim_\alpha T_\alpha$. Moreover,we obtain

$$|Tv|=o\text{-}\lim|T_\alpha v|\leq o\text{-}\lim_\alpha|T_\alpha|(|v|)\leq S(|v|).$$

Therefore $|T|\leq S$ and $|R-T|\leq\Psi-S$. Finally, we obtain $|T|=S$ and $|R-T|=\Psi-S$. □

**Lemma 3.9.** *Let $(V,E)$, $(W,F)$ be the same as in Lemma 3.6. Then the dominant norm $|\cdot|:\mathcal{D}_U(W,W)\to\mathcal{U}^{ev}(E,F)$ is disjointly decomposable.*

*Proof of Lemma 3.9.* Take pairwise disjoint projections $\sigma_1,\dots,\sigma_n\in\mathfrak{B}(F)$ and elements $e_1,\dots,e_n\in E$. Assign

$$\rho T:=\bigvee_{i=1}^{n}\sigma_i\pi^{e_i}T;\ \rho T=\sigma_1\circ(\pi_1 T)+\dots+\sigma_n\circ(\pi_n T),$$

where $\pi_i=\pi^{e_i}$, $1\leq i\leq n$. Then by Lemma 3.6 we may write

$$|\rho T|=\sigma_1|\pi_1 T|+\dots+\sigma|\pi_n T|=\sigma_1\pi_1|T|+\dots+\sigma\pi_n|T|=\rho|T|;$$
$$|\rho^\perp T|=\sigma_1|\pi_1^\perp T|+\dots+\sigma|\pi_n^\perp T|=\sigma_1\pi_1^\perp|T|+\dots+\sigma\pi_n^\perp|T|=\rho^\perp|T|.$$

Then take $\rho\in\mathcal{A}(|T|)^\uparrow$. Then there exists a decreasing net $\rho_\alpha$ of elementary fragments of $|T|$ in $\mathcal{A}(|T|)$ such that $\rho=\sup_\alpha\rho_\alpha$. For each $\alpha$, the operator $\rho_\alpha T\in\mathcal{D}_U(V,W)$ is well defined, moreover $|\rho_\alpha T|=\rho_\alpha|T|$ and $|\rho_\alpha^\perp T|=\rho_\alpha^\perp|T|$. By Lemma 3.8, there exists a dominated Uryson operator $\rho T=bo\text{-}\lim_\alpha\rho_\alpha T$, and $|\rho T|=\rho|T|$, $|\rho^\perp T|=\rho^\perp|T|$. Using the same arguments, we may establish the latter equality for the case where $\rho\in\mathcal{A}(|T|)^{\uparrow\downarrow\uparrow}$. Thus, for arbitrary fragments $\Psi_1=\rho|T|$ and $\Psi_2=\rho^\perp|T|$

of the dominants norm, we have $T = T_1 + T_2$ and $|T_i| = \Psi_i$, $i \in \{1, 2\}$, whenever $T_1 = \rho T$ and $T_2 = \rho^{\perp} T$. □

*Proof of Theorem 3.1.* Using the fact that every $(bo)$-complete and d-decomposable lattice-normed space is decomposable (see [4], Theorem 4.2.6) and applying lemmas 3.2 - 3.9 we complete the proof. □

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