Restrained Locating-Domination and Restrained Differentiating-Domination in Graphs

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Abstract

In this paper, the restrained differentiating-dominating sets in the join, corona and composition of graphs are characterized. Also, the differentiating-domination numbers of these graphs are determined. Moreover, we rectify some of the results on restrained locating-dominating sets found in [5].

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph and $u \in V(G)$. The neighborhood of $u$ is the set $N_G(u) = N(u) = \{v \in V(G) : uv \in E(G)\}$. The degree of a vertex $u \in V(G)$ is equal to the cardinality of $N_G(u)$ and the maximum degree of $G$ is $\Delta(G) = \max \{\text{deg}_G(u) : u \in V(G)\}$. The distance $d_G(u, v)$ in $G$ of two vertices $u$ and $v$ is defined as the length of a shortest path between $u$ and $v$ in $G$. The diameter of $G$ is $\text{diam}(G) = \max \{d_G(u, v) : u, v \in V(G)\}$.
vertices $u$ and $v$ of $G$ is the length of the shortest $u$-$v$ path in $G$. If $X \subseteq V(G)$, then the open neighborhood of $X$ is the set $N_G(X) = N(X) = \bigcup_{v \in X} N_G(v)$. The closed neighborhood of $X$ is $N_G[X] = N[X] = X \cup N(X)$.

A connected graph $G$ of order $n \geq 3$ is point distinguishing if for any two distinct vertices $u$ and $v$ of $G$, $N_G(u) \neq N_G(v)$. It is totally point determining if for any two distinct vertices $u$ and $v$ of $G$, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in (V(G) \setminus S)$, there exists $w \in S$ such that $vw \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set of $G$.

A subset $S$ of $V(G)$ is a restrained dominating set of $G$ if $S$ is a dominating set of $G$ and for each $v \in V(G) \setminus S$, there exists $u \in (V(G) \setminus S) \cap N_G(v)$. Equivalently, a dominating subset $S$ of $V(G)$ is a restrained dominating set of $G$ if $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The minimum cardinality of a restrained dominating set of $G$, denoted by $\gamma_r(G)$, is called the restrained domination number of $G$.

Let $G$ be a point distinguishing graph. A subset $S$ of $V(G)$ is a differentiating set of $G$ if for every two distinct vertices $u, v \in V(G)$, $N_G[u] \cap S \neq N_G[v] \cap S$. It is a strictly differentiating set if it is differentiating and $N_G[u] \cap S \neq S$, for all $u \in V(G)$. The minimum cardinality of a differentiating set of $G$, denoted by $dn(G)$, is called the differentiating number of $G$. The minimum cardinality of a strictly differentiating set of $G$ denoted by $sdn(G)$, is called the strictly differentiating number of $G$.

A differentiating (resp. strictly differentiating) subset $S$ of $V(G)$ is a restrained differentiating (resp. restrained strictly differentiating) set of $G$ if either $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The minimum cardinality of a restrained differentiating (resp. restrained strictly differentiating) set of $G$, denoted by $rtn(G)$ (resp. $rstdn(G)$), is called the restrained differentiating number (resp. restrained strictly differentiating number) of $G$.

A differentiating (resp. strictly differentiating) subset $S$ of $V(G)$ which is also dominating is called a differentiating-dominating (resp. strictly differentiating-dominating) set of $G$. The minimum cardinality of a differentiating-dominating (resp. strictly differentiating-dominating) set of $G$, denoted by $\gamma_D(G)$ (resp. $\gamma_{SD}(G)$), is called the differentiating domination (resp. strictly differentiating domination) number of $G$.

A set $S \subseteq V(G)$ of a point distinguishing graph $G$ is called a restrained differentiating-dominating set of $G$ if $S$ is a differentiating-dominating set of $G$ and $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The restrained differentiating domination number of $G$, denoted by $\gamma_{rD}(G)$, is the minimum cardinality of a restrained differentiating-dominating set of $G$.

Let $G$ be a connected graph and suppose that there exist (distinct) adjacent vertices $u$ and $v$ of $V(G)$ such that $N_G[u] = N_G[v]$. Then $N_G[u] \cap S = N_G[v] \cap S$ for any subset $S$ of $V(G)$. This implies that $G$ cannot have a differentiating
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set. Also, if $\Delta(G) = n - 1$ and $v \in V(G)$ such that $\text{deg}(v) = n - 1$, then $N_G[v] \cap S = S$ for any subset $S$ of $V(G)$. Thus, $G$ cannot have a strictly differentiating set.

The concepts of differentiating sets, differentiating-dominating sets and the associated parameters are studied in [2] and [4]. On the other hand, restrained domination in graphs is defined and studied in [6] and [7].

A subset $S$ of $V(G)$ is a locating set in a connected graph $G$ if every two vertices $u$ and $v$ of $V(G)\setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. It is a strictly locating set if it is locating and $N_G(u) \cap S \neq S$ for all $u \in V(G)\setminus S$. The minimum cardinality of a locating set of $G$, denoted by $ln(G)$, is called the locating number of $G$. The minimum cardinality of a strictly locating set of $G$, denoted by $sln(G)$, is called the strictly locating number of $G$.

A locating subset $S$ of $V(G)$ is a restrained locating set of a connected graph $G$ if $S = V(G)$ or $(V(G)\setminus S)$ has no isolated vertex. The restrained locating number of $G$, denoted by $rln(G)$, is the smallest cardinality of a restrained locating set of $G$.

A locating (resp. strictly locating) subset $S$ of $V(G)$ which is dominating is called a locating-dominating (resp. strictly locating-dominating) set or simply $L$-dominating (resp. $SL$-dominating) set of a graph $G$. The minimum cardinality of a locating-dominating (resp. strictly locating-dominating) set of $G$, denoted by $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$), is called the $L$-domination (resp. $SL$-domination) number of $G$.

A subset $S$ of a connected graph $G$ is a restrained locating-dominating set of $G$ if $S$ is a locating-dominating set of $G$ and either $S = V(G)$ or $(V(G)\setminus S)$ has no isolated vertex. The restrained $L$-domination number of $G$, denoted by $\gamma_{rL}(G)$, is the smallest cardinality of a restrained locating-dominating set of $G$.

2 Results on Restrained Locating-Dominating Sets in Graphs

**Theorem 2.1** [1] Let $G$ be a connected graph of order $n \geq 2$. If $ln(G) < sln(G)$, then $1 + ln(G) = sln(G)$.

**Theorem 2.2** [5] Let $G$ and $H$ be connected non-trivial graphs. A set $S \subseteq V(G + H)$ is a restrained locating-dominating set of $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets of $G$ and $H$, respectively, where $S_1$ or $S_2$ is a strictly locating set and one of the following holds:

(i) $S_1 = V(G)$ and $S_2$ is a restrained locating set of $H$;

(ii) $S_2 = V(H)$ and $S_1$ is a restrained locating set of $G$;
Suppose that $\gamma$.

Case 1.

Corollary 2.3 Let $G$ and $H$ be connected non-trivial graphs of order $m$ and $n$, respectively. Then

$$\gamma_{rL}(G + H) = \begin{cases} m + n, & \text{if } sln(G) = m \text{ and } sln(H) = n \\ \min \{sln(G) + ln(H), sln(H) + ln(G)\}, & \text{otherwise.} \end{cases}$$

Proof: Suppose that $sln(G) = m$ and $sln(H) = n$. Then $V(G)$ and $V(H)$ are the only strictly locating sets of $G$ and $H$, respectively. Since $ln(G) \leq m - 1$, it follows that by Theorem 2.1, $1 + ln(G) = sln(G)$. Thus, $ln(G) = m - 1$. Since $ln(G) \leq rln(G)$ and $rln(G)$ cannot be equal to $m - 1$, it follows that $rln(G) = m$. Similarly, $rln(H) = n$. Thus, $V(G)$ and $V(H)$ are the only restrained locating sets of $G$ and $H$, respectively. Hence, by Theorem 2.2, if $S$ is a minimum restrained locating-dominating set of $G + H$, then $S = V(G) \cup V(H)$. Therefore, $\gamma_{rL}(G + H) = m + n$.

Suppose that $sln(G) \neq m$ or $sln(H) \neq n$. Consider the following cases:

Case 1. Suppose that $sln(G) = m$ and $sln(H) \neq n$ or $sln(G) \neq m$ and $sln(H) = n$.

Suppose first that $sln(G) = m$ and $sln(H) \neq n$. Let $S_1$ and $S_2$ be minimum locating set and minimum strictly locating set of $G$ and $H$, respectively. Then by Theorem 2.2, $S = S_1 \cup S_2$ is a restrained locating-dominating set of $G + H$. Thus, $\gamma_{rL}(G + H) \leq |S| = sln(H) + ln(G)$.

Now, suppose that $S'$ is a minimum restrained locating-dominating set of $G + H$. Then by Theorem 2.2, $S' = S'_1 \cup S'_2$, where $S'_1 \neq V(G)$ is a locating set of $G$ and $S'_2 \neq V(H)$ is a strictly locating set of $H$. Hence, $\gamma_{rL}(G + H) = |S'| \geq ln(G) + sln(H)$. Therefore, $\gamma_{rL}(G + H) = ln(G) + sln(H)$. Similarly, if $sln(G) \neq m$ and $sln(H) = n$, then $\gamma_{rL}(G + H) = sln(G) + ln(H)$.

Case 2. Suppose that $sln(G) \neq m$ and $sln(H) \neq n$.

Let $S$ be a minimum restrained locating-dominating set of $G + H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. Then by Theorem 2.2, $S_1$ and $S_2$ are locating sets of $G$ and $H$, respectively, where $S_1$ or $S_2$ is a strictly locating set. If $S_1 = V(G)$, then by Theorem 2.2(i), $S_2$ is a restrained locating set of $H$. Thus, $\gamma_{rL}(G + H) = m + rln(H) \geq sln(G) + ln(H)$. Similarly, by Theorem 2.2(ii), if $S_2 = V(H)$, then $\gamma_{rL}(G + H) = n + rln(G) \geq sln(H) + ln(G)$. Suppose that $S_1 \neq V(G)$ and $S_2 \neq V(H)$. Assume first that $S_1$ is a strictly locating set of $G$. Then $sln(G) + ln(H) \leq |S_1| + |S_2| = |S| = \gamma_{rL}(G + H)$. If $S_2$ is a strictly locating set of $H$, then $sln(H) + ln(G) \leq |S_2| + |S_1| = |S| = \gamma_{rL}(G + H)$. Thus, $\gamma_{rL}(G + H) \geq \min \{sln(G) + ln(H), sln(H) + ln(G)\}$. 

Now, suppose that $sln(G) + ln(H) \leq sln(H) + ln(G)$. Let $S_1$ be a minimum strictly locating set of $G$ and $S_2$ be a minimum locating set of $H$. Then $S = S_1 \cup S_2$ is a restrained locating-dominating set of $G + H$ by Theorem 2.2.
Thus, $\gamma_{rL}(G + H) \leq |S| = |S_1| + |S_2| = sln(G) + ln(H)$.

Therefore, $\gamma_{rL}(G + H) = \min \{sln(G) + ln(H), sln(H) + ln(G)\}$. □

**Corollary 2.4** Let $G$ be connected non-trivial graph of order $m$ and let $K_n$ be a complete graph of order $n$. Then

$$\gamma_{rL}(G + K_n) = \begin{cases} sln(G) + n - 1, & \text{if } sln(G) \neq m \\ m + n, & \text{otherwise.} \end{cases}$$

**Proof:** Suppose that $sln(G) \neq m$. Since $ln(K_n) = n - 1$ and $sln(K_n) = n$, it follows that by Corollary 2.3, $\gamma_{rL}(G + K_n) = \min \{sln(G) + n - 1, ln(G) + n\}$. By Theorem 2.1, $sln(G) - 1 \leq ln(G)$. Thus, $sln(G) \leq ln(G) + 1$. Therefore, $\gamma_{rL}(G + K_n) = \min \{sln(G) + n - 1, ln(G) + n\} = sln(G) + n - 1$. Now, suppose that $sln(G) = m$. Since $sln(K_n) = n$, it follows that by Corollary 2.3, $\gamma_{rL}(G + K_n) = m + n$. □

**Theorem 2.5** [5] Let $G$ be a connected non-trivial graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(G + K_1)$ is a restrained locating-dominating set of $G + K_1$ if and only if either $S = S_1 \cup \{v\}$, where $S_1$ is a restrained locating set of $G$ or $v \notin S$ and $S$ is a strictly locating-dominating set of $G$ with $V(G) \neq S$.

**Corollary 2.6** Let $G$ be a connected non-trivial graph of order $m$. Then

$$\gamma_{rL}(G + K_1) = \begin{cases} m + 1, & \text{if } \gamma_{SL}(G) = m \text{ and } rln(G) = m \\ \min \{\gamma_{SL}(G), rln(G) + 1\}, & \text{otherwise.} \end{cases}$$

### 3 Results on Restrained Differentiating-Dominating Sets in Graphs

**Remark 3.1** If $G$ is a point distinguishing graph of order $n \geq 3$, then $\gamma_{rD}(G) \in \{3, 4, \ldots, n - 2, n\}$.

**Remark 3.2** [2] Let $G$ be a point distinguishing graph of order $n \geq 3$. Then $2 \leq \gamma_{D}(G) \leq n - 1$.

**Remark 3.3** If $G$ is a point distinguishing graph of order $n \geq 3$ with $\Delta(G) \leq n - 2$, then $dn(G) \leq \gamma_{D}(G) \leq \gamma_{rD}(G) \leq \gamma_{rSD}(G)$ and $dn(G) \leq sln(G) \leq \gamma_{rSD}(G)$.

Let $A$ and $B$ be sets which are not necessarily disjoint. The disjoint union of $A$ and $B$, denoted by $A \dot{\cup} B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. 


Theorem 3.4 [2] Let $G$ and $H$ be point distinguishing graphs of orders $m \geq 2$ and $n \geq 2$, respectively, with $\Delta(G) \leq m - 2$ and $\Delta(H) \leq n - 2$. Then $S \subseteq V(G + H)$ is a differentiating-dominating set of $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets of $G$ and $H$, respectively, and either $S_G$ or $S_H$ is a strictly differentiating set.

Theorem 3.5 Let $G$ and $H$ be point distinguishing graphs of orders $m \geq 3$ and $n \geq 3$, respectively, with $\Delta(G) \leq m - 2$ and $\Delta(H) \leq n - 2$. Then $S \subseteq V(G + H)$ is a restrained differentiating-dominating set of $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets of $G$ and $H$, respectively, where $S_G$ or $S_H$ is a strictly differentiating set and one of the following holds:

(i) $S_G = V(G)$ and $S_H$ is a restrained differentiating set of $H$;

(ii) $S_H = V(H)$ and $S_G$ is a restrained differentiating set of $G$;

(iii) $S_G \neq V(G)$ and $S_H \neq V(H)$.

Proof: Let $S \subseteq V(G + H)$ be a restrained differentiating-dominating set of $G + H$. Then by Theorem 3.4, $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating set of $G$ and $H$, respectively, where $S_G$ or $S_H$ is a strictly differentiating set. To show that (i), (ii) and (iii) hold, we consider the following cases:

Case 1: $S_G = V(G)$.

Suppose that $S_H \neq V(H)$. Since $V(G + H) \setminus S = V(H) \setminus S_H$ and $\langle V(G + H) \setminus S \rangle$ has no isolated vertex, it follows that $\langle V(H) \setminus S_H \rangle$ has no isolated vertex. Thus, $S_H$ is a restrained differentiating set of $H$. Hence, (i) holds.

Case 2: $S_G \neq V(G)$.

If $S_H \neq V(H)$, then (iii) holds. If $S_H = V(H)$, then $\langle V(G + H) \setminus S \rangle = \langle V(G) \setminus S_G \rangle$ has no isolated vertex. Thus, $S_G$ is a restrained differentiating set of $G$. Hence, (ii) holds.

For the converse, suppose that $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets of $G$ and $H$, respectively, where $S_G$ or $S_H$ is a strictly differentiating set. Then by Theorem 3.4, $S = S_G \cup S_H$ is a differentiating-dominating set of $G + H$. Suppose that $S_G = V(G)$. If $S_H = V(H)$, then $S = V(G + H)$ is a restrained dominating set of $G + H$. If $S_H \neq V(H)$, then $\langle V(G + H) \setminus S \rangle = \langle V(H) \setminus S_H \rangle$ has no isolated vertex, by assumption. Thus, $S$ is a restrained dominating set of $G + H$. Therefore, $S$ is a restrained differentiating-dominating set of $G + H$. Similarly, if (ii) holds, $S$ is a restrained differentiating-dominating set of $G + H$. Finally, suppose that $S_G \neq V(G)$ and $S_H \neq V(H)$. Then clearly, $S$ is a restrained differentiating-dominating set of $G + H$. □

Theorem 3.6 [2] Let $G$ be a point distinguishing graph of order $n \geq 3$ with $\Delta(G) \leq n - 2$ such that $dn(G) < sdn(G)$. Then $1 + dn(G) = sdn(G)$. 
Corollary 3.7 Let $G$ and $H$ be point distinguishing graphs of orders $m \geq 3$ and $n \geq 3$, respectively, with $\Delta(G) \leq m - 2$ and $\Delta(H) \leq n - 2$. Then

$$\gamma_{rD}(G + H) = \begin{cases} m + n, & \text{if } sdn(G) = m \text{ and } sdn(H) = n \\ \min \{ sdn(G) + dn(H), sdn(H) + dn(G) \}, & \text{otherwise.} \end{cases}$$

Proof: Suppose that $sdn(G) = m$ and $sdn(H) = n$. Then $V(G)$ and $V(H)$ are the only strictly differentiating sets of $G$ and $H$, respectively. Since $dn(G) \leq m - 1$, it follows that by Theorem 3.6, $1 + dn(G) = sdn(G)$. Thus, $dn(G) = m - 1$. Since $dn(G) \leq rdn(G)$ and $rdn(G)$ cannot be equal to $m - 1$, it follows that $rdn(G) = m$. Similarly, $rdn(H) = n$. Thus, $V(G)$ and $V(H)$ are the only restrained differentiating sets of $G$ and $H$, respectively. Hence, by Theorem 3.5, if $S$ is a minimum restrained differentiating-dominating set of $G + H$, then $S = V(G) \cup V(H)$. Therefore, $\gamma_{rD}(G + H) = m + n$.

Suppose that $sdn(G) \neq m$ or $sdn(H) \neq n$. Consider the following cases:

**Case 1.** Suppose that $sdn(G) = m$ and $sdn(H) \neq n$ or $sdn(G) \neq m$ and $sdn(H) = n$.

Suppose first that $sdn(G) = m$ and $sdn(H) \neq n$. Let $S_1$ and $S_2$ be minimum differentiating set and minimum strictly differentiating set of $G$ and $H$, respectively. Then by Theorem 3.5, $S = S_1 \cup S_2$ is a restrained differentiating-dominating set of $G + H$. Thus, $\gamma_{rD}(G + H) \leq |S| = sdn(H) + dn(G)$.

Now, suppose that $S'$ is a minimum restrained differentiating-dominating set of $G + H$. Then by Theorem 3.5, $S' = S_1' \cup S_2'$, where $S_1' \neq V(G)$ is a differentiating set of $G$ and $S_2' \neq V(H)$ is a strictly differentiating set of $H$. Hence, $\gamma_{rD}(G + H) = |S'| \geq dn(G) + sdn(H)$. Therefore, $\gamma_{rD}(G + H) = dn(G) + sdn(H)$. Similarly, if $sdn(G) \neq m$ and $sdn(H) = n$, then $\gamma_{rD}(G + H) = sdn(G) + dn(H)$.

**Case 2.** Suppose that $sdn(G) \neq m$ and $sdn(H) \neq n$.

Let $S$ be a minimum restrained differentiating-dominating set of $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Then by Theorem 3.5, $S_G$ and $S_H$ are differentiating sets of $G$ and $H$, respectively, where $S_G$ or $S_H$ is a strictly differentiating set. If $S_G = V(G)$, then by Theorem 3.5(i), $S_H$ is a restrained differentiating set of $H$. Thus, $\gamma_{rD}(G + H) = m + rdn(H) \geq sdn(G) + dn(H)$. Similarly, by Theorem 3.5(ii), if $S_H = V(H)$, then $\gamma_{rD}(G + H) = n + rdn(G) \geq sdn(H) + dn(G)$. Suppose that $S_G \neq V(G)$ and $S_H \neq V(H)$. Assume first that $S_G$ is a strictly differentiating set of $G$. Then $sdn(G) + dn(H) \leq |S_G| + |S_H| = |S| = \gamma_{rD}(G + H)$. If $S_H$ is a strictly differentiating set of $H$, then $sdn(H) + dn(G) \leq |S_H| + |S_G| = |S| = \gamma_{rD}(G + H)$. Thus, $\gamma_{rD}(G + H) \geq \min \{ sdn(G) + dn(H), sdn(H) + dn(G) \}$.

Now, suppose that $sdn(G) + dn(H) \leq sdn(H) + dn(G)$. Let $S_G$ be a minimum strictly differentiating set of $G$ and $S_H$ be a minimum differentiating set of $H$. Then $S = S_G \cup S_H$ is a restrained differentiating-dominating set of
G + H by Theorem 3.5. Thus, \( \gamma_{rD}(G + H) \leq |S| = |S_G| + |S_H| = sdn(G) + dn(H) \).

Therefore, \( \gamma_{rD}(G + H) = \min \{ sdn(G) + dn(H), sdn(H) + dn(G) \} \). \( \square \)

**Theorem 3.8** [2] Let \( G = K_1 = \langle v \rangle \) and \( H \) a point distinguishing graph of order \( n \geq 3 \) with \( \Delta(H) \leq n - 2 \). Then \( S \subseteq V(G + H) \) is a differentiating-dominating set of \( G + H \) if and only if \( v \in S \) and \( V(H) \cap S \) is a strictly differentiating set of \( H \) or \( v \notin S \) and \( S \) is strictly differentiating-dominating set of \( H \).

**Theorem 3.9** Let \( G \) be a point distinguishing graph of order \( n \geq 3 \) such that \( \Delta(G) \leq n - 2 \) and let \( K_1 = \langle v \rangle \). Then \( S \subseteq V(G + K_1) \) is a restrained differentiating-dominating set of \( G + K_1 \) if and only if either \( S = S_G \cup \{v\} \), where \( S_G \) is a restrained strictly differentiating set of \( G \) or \( v \notin S \) and \( S \) is a strictly differentiating-dominating set of \( G \) with \( V(G) \neq S \).

**Proof:** Suppose \( S \) is a restrained differentiating-dominating set of \( G + K_1 \). Let \( S_G = V(G) \cap S \). Then by Theorem 3.8, either \( S = S_G \cup \{v\} \), where \( S_G \) is a strictly differentiating set of \( G \) or \( v \notin S \) and \( S \) is a strictly differentiating-dominating set of \( G \). Suppose first that \( S_G \cup \{v\} \) and suppose \( S_G = V(G) \). Then \( S_G \) is a restrained strictly differentiating set of \( G \). If \( G \neq V(G) \), then \( \langle V(G + K_1) \setminus S \rangle = \langle V(G) \setminus S_G \rangle \). Since \( S \) is a restrained dominating set of \( G + K_1 \), it follows that \( \langle V(G) \setminus S_G \rangle \) has no isolated vertex. Hence, \( S_G \) is a restrained strictly differentiating set of \( G \). Next, suppose that \( v \notin S \). Then \( \langle V(G + K_1) \setminus S \rangle = \langle \{v\} \cup (V(G) \setminus S) \rangle \) has no isolated vertex. Hence, \( S \neq V(G) \).

For the converse, assume first that \( S = S_G \cup \{v\} \), where \( S_G \) is a restrained strictly differentiating set of \( G \). Then by Theorem 3.8, \( S \) is a differentiating-dominating set of \( G + K_1 \). If \( S_G = V(G) \), then \( S = V(G + K_1) \) is a restrained differentiating-dominating set of \( G + K_1 \). If \( S_G \neq V(G) \), then \( \langle V(G + K_1) \setminus S \rangle = \langle V(G) \setminus S_G \rangle \) has no isolated vertex, by assumption. It follows that \( S \) is a restrained differentiating-dominating set of \( G + K_1 \). Finally, suppose that \( v \notin S \) and \( S \) is a strictly differentiating-dominating set of \( G \) with \( V(G) \neq S \). Then again by Theorem 3.8, \( S \) is a differentiating-dominating set of \( G + K_1 \). Since \( \langle V(G + K_1) \setminus S \rangle = \langle \{v\} \cup (V(G) \setminus S) \rangle \) and \( V(G) \setminus S \neq \emptyset \), it follows that \( \langle V(G + K_1) \setminus S \rangle \) has no isolated vertex. Therefore, \( S \) is a restrained differentiating-dominating set of \( G + K_1 \). \( \square \)

**Corollary 3.10** Let \( G \) be a point distinguishing graph with \( \Delta(G) \leq |V(G)| - 2 \). Then

\[
\gamma_{rD}(G + K_1) = \begin{cases} 
|V(G)| + 1, & \text{if } \gamma_{SD}(G) = |V(G)| \text{ and } rsdn(G) = |V(G)| \\
\min \{ \gamma_{SD}(G), rsdn(G) + 1 \}, & \text{otherwise.}
\end{cases}
\]
Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding the join $\langle \{v\}\rangle + H^v$.

**Theorem 3.11** Let $G$ be non-trivial connected graph and $H$ a point distinguishing graph of order $n \geq 3$, such that $\Delta(H) \leq n - 2$. Then $C \subseteq G \circ H$ is a restrained differentiating-dominating set of $G \circ H$ if and only if for every $v \in V(G)$, one of the following statements is true:

(i) $v \in C$, $N_G(v) \cap C \neq \emptyset$ and $C \cap V(H^v)$ is a restrained differentiating set of $H^v$;

(ii) $v \in C$, $N_G(v) \cap C = \emptyset$ and $C \cap V(H^v)$ is a restrained strictly differentiating set of $H^v$;

(iii) $v \notin C$, $N_G(v) \cap C \neq \emptyset$ and $C_1 = C \cap V(H^v)$ is a differentiating-dominating set of $H^v$ with $v \in N_G(V(G) \setminus C)$ if $V(H^v) = C_1$;

(iv) $v \notin C$, $N_G(v) \cap C = \emptyset$ and $C_1 = C \cap V(H^v)$ is a strictly differentiating-dominating set of $H^v$.

**Proof:** Suppose $C$ is a restrained differentiating-dominating set of $G \circ H$. Let $v \in V(G)$, $C_1 = V(H^v) \cap C$ and let $x, y \in V(H^v)$ with $x \neq y$. Then $N_{G \circ H}[x] \cap C = (N_{H^v}[x] \cap C_1) \cup (C \cap \{v\})$, $N_{G \circ H}[y] \cap C = (N_{H^v}[y] \cap C_1) \cup (C \cap \{v\})$ and $N_{G \circ H}[v] \cap C = (N_G(v) \cap C_2) \cup (N_{v+H^v}[v] \cap C_1) \cup (C \cap \{v\})$, where $C_2 = C \cap V(G)$.

Suppose first that $v \in C$. If $N_G(v) \cap C \neq \emptyset$, then, since $C$ is a differentiating set of $G \circ H$, $(N_{H^v}[x] \cap C_1) \cup \{v\} = N_{G \circ H}[x] \cap C \neq N_{G \circ H}[y] \cap C = (N_{H^v}[y] \cap C_1) \cup \{v\}$. Thus, $N_{H^v}[x] \cap C_1 \neq N_{H^v}[y] \cap C_1$. Hence, $C_1$ is a differentiating set of $H^v$. Now, since $C$ is a restrained differentiating-dominating set of $G \circ H$ and $v \in C$, it follows that either $V(H^v) = C_1$ or $\langle V(H^v) \setminus C_1 \rangle$ has no isolated vertex. Thus, $C_1$ is a restrained differentiating set of $H^v$. Hence, (i) holds. Suppose that $N_G(v) \cap C = \emptyset$. Then, again, since $C$ is a differentiating set of $G \circ H$, $C_1$ is a differentiating set in $H^v$. Also, since $N_{G \circ H}[v] \cap C_1 = N_{v+H^v}[v] \cap C_1 = \{v\} \cup C_1$, it follows that $C_1$ must be a strictly differentiating set of $H^v$. Moreover, since $C$ is a restrained differentiating set of $G \circ H$ and $v \in C$, it follows that either $V(H^v) = C_1$ or $\langle V(H^v) \setminus C_1 \rangle$ has no isolated vertex. Hence, $C_1$ is a restrained differentiating set of $H^v$. Thus, (ii) holds.

Next, suppose that $v \notin C$. If $N_G(v) \cap C \neq \emptyset$, then, since $C$ is a restrained differentiating-dominating set of $G \circ H$, it follows that $N_{H^v}[x] \cap C_1 = \emptyset$. Then, $\langle N_{H^v}[x] \cap C_1 \rangle = \emptyset$ and $\langle N_{H^v}[y] \cap C_1 \rangle = \emptyset$. Thus, (iii) holds. If $N_G(v) \cap C = \emptyset$, then, again, since $C$ is a differentiating set of $G \circ H$, $C_1$ is a differentiating set in $H^v$. Also, since $N_{G \circ H}[v] \cap C_1 = \emptyset$, it follows that $C_1$ must be a strictly differentiating set of $H^v$. Moreover, since $C$ is a restrained differentiating set of $G \circ H$ and $v \in C$, it follows that either $V(H^v) = C_1$ or $\langle V(H^v) \setminus C_1 \rangle$ has no isolated vertex. Hence, $C_1$ is a restrained differentiating set of $H^v$. Thus, (iv) holds.
Case 1: Suppose that $u = v$. If $c, d \in V(H^w)$, then, $N_{H^w}[c] \cap C_1 \neq N_{H^w}[d] \cap C_1$, since $C_1 = C \cap V(H^w)$ is a differentiating set of $H^w$ by (i), (ii), (iii) and (iv). Hence, $(N_{G[H]}[c] \cap C) \setminus \{v\} \cap C \neq (N_{G[H]}[d] \cap C) \setminus \{v\} \cap C$. Thus, $N_{G[H]}[c] \cap C \neq N_{G[H]}[d] \cap C$. Suppose $c = v$ and $d \in V(H^w)$. If $N_G[v] \cap C \neq \emptyset$, say $w \in N_G[v] \cap C$, then $w \in [N_{G[H]}[c] \cap C] \setminus [N_{G[H]}[d] \cap C]$. Hence, $N_{G[H]}[c] \cap C \neq N_{G[H]}[d] \cap C$. If $N_G(v) \cap C = \emptyset$, then $V(H^w) \cap C$ is a strictly differentiating set of $H^w$ by (ii) and (iv). Thus, there exists $z \in V(H^w) \cap C$ such that $z \notin N_{G[H]}[d] \cap C$.

Case 2: Suppose that $u \neq v$. Since $V(H^w) \cap C$ and $V(H^v) \cap C$ are non-empty disjoint sets and $(V(H^w) \cap C) \setminus (N_{G[H]}[c] \cap C) \neq \emptyset$ and $(V(H^v) \cap C) \setminus (N_{G[H]}[d] \cap C) \neq \emptyset$, it follows that $N_{G[H]}[c] \cap C \neq N_{G[H]}[d] \cap C$.

Therefore, $C$ is a differentiating-dominating set of $G \circ H$.

Finally, suppose that $V(G \circ H) = C$, then $C$ is a restrained differentiating-dominating set of $G \circ H$. Suppose that $V(G \circ H) \neq C$. Let $w \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $w \in V(v + H^w)$. Consider the following cases:

Case 1: Suppose that $w = v$.

Then $w = v \notin C$. If $N_G(w) \cap C \neq \emptyset$, then $C_1 = V(H^w) \cap C$ is a differentiating-dominating set of $H^w$. If $V(H^w) \cap C \neq V(H^w)$, then there exists $x \in V(H^w) \cap C \subseteq V(G \circ H) \setminus C$ such that $xw \in E(G \circ H)$. If $V(H^w) = C_1$, then by (iii), $w \in N_G(V(G) \setminus C)$. Thus, there exists $y \in V(G) \setminus C \subseteq V(G \circ H) \setminus C$ such that $wy \in E(G \circ H)$. Suppose that $N_G(w) \cap C = \emptyset$. Since $G$ is connected and non-trivial, there exists $a \in V(G) \setminus C$ such that $wa \in E(G \circ H)$.

Case 2: Suppose that $w \neq v$.

Then $w \in V(H^v) \setminus C_1$, where $C_1 = V(H^v) \cap C$. If $v \in C$, then by (i) and (ii), $C_1$ is a restrained differentiating set of $H^v$. If $v \notin C$, then $vw \in E(G \circ H)$.

Hence, in all cases, $C$ has no isolated vertex. Therefore, $C$ is a restrained differentiating-dominating set of $G \circ H$. □
Lemma 3.12 [2] Let G be a point distinguishing graph of order \( n \geq 3 \) such that \( dn(G) < \gamma_D(G) \). Then \( 1 + dn(G) = \gamma_D(G) \).

Corollary 3.13 Let G be a non-trivial connected graph and H a point distinguishing graph of order \( n \geq 3 \) such that \( \Delta(H) \leq n - 2 \). Then \( |V(G)| \gamma_D(H) \leq \gamma_{rD}(G \circ H) \leq |V(G)| \gamma_{SD}(H) \).

Proof: Let S be a minimum restrained differentiating-dominating set of \( G \circ H \). Then \( \gamma_{rD}(G \circ H) = |S| = \sum_{v \in V(G) \cap S} (1 + |V(H^v) \cap S|) + \sum_{v \in V(G) \setminus S} |V(H^v) \cap S| \). By Lemma 3.12 and Theorem 3.11(i) and (ii) and the fact that \( rdn(H) \geq dn(H) \), \( (1 + |V(H^v) \cap S|) \geq 1 + rdn(H) \geq 1 + dn(H) \geq \gamma_D(H) \) for every \( v \in V(G) \cap S \). By Theorem 3.11(iii) and (iv), \( |V(H^v) \cap S| \geq \gamma_D(H) \) for every \( v \in V(G) \setminus S \). Thus, \( \gamma_{rD}(G \circ H) = |S| \leq |V(G)| \gamma_{SD}(H) \).

Now, let \( F \) be a minimum strictly differentiating-dominating set of \( H \). For each \( v \in V(G) \), pick \( F_v \subseteq V(H^v) \), where \( \langle F_v \rangle \cong \langle F \rangle \). Then \( S = \bigcup_{v \in V(G)} F_v \) is a restrained differentiating-dominating set of \( G \circ H \) by Theorem 3.11. Hence, \( \gamma_{rD}(G \circ H) \leq |S| \leq |V(G)| \gamma_{SD}(H) \). \( \square \)

The composition (lexicographic product) \( G[H] \) of two graphs \( G \) and \( H \) is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u, u')(v, v') \in E(G[H]) \) if and only if either \( uv \in E(G) \) or \( u = v \) and \( u' v' \in E(H) \).

Theorem 3.14 [2] Let G be a connected non-trivial graph and H a point distinguishing graph of order \( n \geq 3 \) with \( \Delta(H) \leq n - 2 \). Then \( C = \bigcup_{x \in S} \{x\} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a differentiating dominating set of \( G[H] \) if and only if

(i) \( S = V(G) \);

(ii) \( T_x \) is a differentiating set of \( H \) for every \( x \in V(G) \);

(iii) \( T_x \) or \( T_y \) is a strictly differentiating set of \( H \) whenever \( x \) and \( y \) are adjacent vertices of \( G \) with \( N_G[x] = N_G[y] \); and

(iv) \( T_x \) or \( T_y \) is (differentiating) dominating of \( H \) whenever \( x \) and \( y \) are distinct non-adjacent vertices of \( G \) with \( N_G(x) = N_G(y) \).

Theorem 3.15 Let G be a connected non-trivial graph and H a point distinguishing graph of order \( n \geq 3 \) with \( \Delta(H) \leq n - 2 \). Then \( C = \bigcup_{x \in S} \{x\} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a restrained differentiating-dominating set of \( G[H] \) if and only if

(i) \( S = V(G) \);
(ii) \( T_x \) is a differentiating set of \( H \) for every \( x \in V(G) \), where \( T_x \) is a restrained differentiating set of \( H \) for all \( x \in S_1 \setminus N_G(S_1) \) with \( S_1 = \{ y \in V(G) : T_y \neq V(H) \} \);

(iii) \( T_x \) or \( T_y \) is a strictly differentiating set of \( H \) whenever \( x \) and \( y \) are adjacent vertices of \( G \) with \( N_G(x) = N_G(y) \);

(iv) \( T_x \) or \( T_y \) is (differentiating) dominating of \( H \) whenever \( x \) and \( y \) are distinct non-adjacent vertices of \( G \) with \( N_G(x) = N_G(y) \).

**Proof:** Let \( C = \bigcup_{x \in V(G)} \{ \{ x \} \times T_x \} \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) be a restrained differentiating-dominating set of \( G[H] \). Thus, by Theorem 3.14, (i), (iii) and (iv) hold and \( T_x \) is a differentiating set of \( H \) for each \( x \in V(G) \). Let \( S_1 = \{ y \in V(G) : T_y \neq V(H) \} \) and \( x \in S_1 \setminus N_G(S_1) \). Suppose that \( T_x \) is not a restrained differentiating set of \( H \). Then \( \langle V(H) \setminus T_x \rangle \) has an isolated vertex, say \( a \). Then \( (x,a) \notin C \) and \( (x,a) \) is an isolated vertex in \( \langle V(G[H]) \setminus C \rangle \), contrary to the assumption that \( S \) is a restrained differentiating-dominating set of \( G[H] \). Hence, \( T_x \) is a restrained differentiating set of \( H \) for all \( x \in S_1 \setminus N_G(S_1) \). Hence, (ii) holds.

For the converse, suppose that (i), (ii), (iii) and (iv) hold. Then by Theorem 3.14, \( C = \bigcup_{x \in V(G)} \{ \{ x \} \times T_x \} \) is a differentiating-dominating set of \( G[H] \). Suppose that \( V(G[H]) = C \). Then \( C \) is a restrained differentiating-dominating set of \( G[H] \). Suppose that \( V(G[H]) \neq C \). Let \( (y,a) \in V(G[H]) \setminus C \). Then \( y \in S_1 \). If \( y \in N_G(S_1) \), then there exists \( z \in S_1 \cap N_G(y) \). Pick any \( b \in V(H) \setminus T_z \). Then \( (y,a)(z,b) \in E(G[H]) \). If \( y \notin N_G(S_1) \), then by (ii), there exists \( c \in V(H) \setminus T_y \cap N_G(a) \). Thus, \( (y,a)(y,c) \in E(G[H]) \). Hence, \( \langle V(G[H]) \setminus C \rangle \) has no isolated vertex. Therefore, \( C \) is a restrained differentiating-dominating set of \( G[H] \). \( \square \)

**Corollary 3.16** Let \( G \) be a totally point determining graph and \( H \) a point distinguishing graph of orders \( m \geq 3 \) and \( n \geq 3 \), respectively, with \( \Delta(H) \leq n - 2 \). Then \( |V(G)| \cdot dn(H) \leq \gamma_{rD}(G[H]) \leq |V(G)| \cdot rdn(H) \).

**Proof:** Let \( D \) be a minimum restrained differentiating set of \( H \). Then by Theorem 3.15, \( C = \bigcup_{x \in V(G)} \{ \{ x \} \times D \} \) is a restrained differentiating-dominating set of \( G[H] \). Hence, \( \gamma_{rD}(G[H]) \leq |V(G)| \cdot rdn(H) \).

Now, let \( C \) be a minimum restrained differentiating-dominating set of \( G[H] \). Then by Theorem 3.15 and the fact that \( dn(H) \leq rdn(H) \),

\[
\gamma_{rD}(G[H]) = |C| = \sum_{x \in V(G) \setminus S_1} |T_x| + \sum_{x \in S_1 \setminus N_G(S_1)} |T_x| + \sum_{x \in S_1 \setminus N_G(S_1)} |T_x|
\]

\[
= |V(G) \setminus S_1| \cdot n + |S_1 \cap N_G(S_1)| \cdot dn(H) + |S_1 \setminus N_G(S_1)| \cdot rdn(H)
\]
\[ \geq |V(G)\setminus S_1|dn(H) + |S_1|dn(H) \]
\[ = |V(G)|dn(H). \]

Therefore,  
\[ |V(G)|dn(H) \leq \gamma_{rD}(G[H]) \leq |V(G)|rdn(H). \]

**Corollary 3.17** Let \( G \) be a totally point determining graph and \( H \) a point distinguishing graph of orders \( m \geq 3 \) and \( n \geq 3 \), respectively, with \( \Delta(H) \leq n-2 \) such that \( dn(H) = rdn(H) \). Then \( \gamma_{rD}(G[H]) = |V(G)|rdn(H) \).

**References**


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