A Semi-Riemannian Manifold of Quasi-Constant Curvature with Half Lightlike Submanifolds

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Abstract

In this paper, we study the geometry of a semi-Riemannian manifold \( \bar{M} \) of quasi-constant curvature. The main result is two characterization theorems for \( \bar{M} \) endowed with a statical, screen homothetic or screen totally umbilical half lightlike submanifold \( M \).

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1 Introduction

B.Y. Chen and K. Yano [1] introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifold \((\bar{M}, \bar{g}, \bar{\nabla})\) equipped with the curvature tensor \(\bar{R}\) of the Levi-Civita connection \(\bar{\nabla}\) satisfying

\[
\bar{g}(\bar{R}(X,Y)Z,W) = \alpha \{ \bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W) \} + \beta \{ \bar{g}(X,W)\theta(Y)\theta(Z) - \bar{g}(X,Z)\theta(Y)\theta(W) \\
+ \bar{g}(Y,Z)\theta(X)\theta(W) - \bar{g}(Y,W)\theta(X)\theta(Z) \},
\]

where \(\alpha\) and \(\beta\) are scalar functions, and \(\theta\) is a 1-form defined by \(\theta(X) = \bar{g}(X,\zeta)\) and \(\zeta\) is a unit vector field on \(\bar{M}\), which called the curvature vector field of \(\bar{M}\). If \(\beta = 0\), then \(\bar{M}\) is a space of constant curvature \(\alpha\).
Recently D.H. Jin and J.W. Lee studied half lightlike submanifolds [8] and lightlike submanifolds \( M \) [9] of a semi-Riemannian manifold \( \bar{M} \) of quasi-
constant curvature subject to the conditions; (1) \( \zeta \) is tangent to \( M \), (2) the
screen distribution \( S(TM) \) is totally umbilical in \( M \) and (3) the co-screen
distribution \( S(TM^\perp) \) is conformal Killing distribution. Each of this papers
proved two characterization theorems for their lightlike submanifolds.

In this paper, we study the curvature of semi-Riemannian manifold \( \bar{M} \) of
quasi-constant curvature admits either a screen homothetic or a screen totally
umbilical, and stactical half lightlike submanifold \( M \). We prove the following
two characterization theorems for such a semi-Riemannian manifold \( \bar{M} \):

**Theorem 1.1.** Let \( \bar{M} \) be a semi-Riemannian manifold of quasi-constant
curvature admits a screen homothetic and stactical half lightlike submanifold \( M \) satisfying one of the following two conditions;

(1) the curvature vector field \( \zeta \) is tangent to \( M \), or
(2) \( \zeta \) is parallel with respect to \( \bar{\nabla} \), the local screen second fundamental form
\( D \) is parallel and the lightlike transversal connection is flat,

Then the function \( \alpha \) and \( \beta \), defined by (1.1), vanish and \( \bar{M} \) is flat manifold.

**Theorem 1.2.** Let \( \bar{M} \) be a semi-Riemannian manifold of quasi-constant
curvature such that \( \dim \bar{M} > 4 \) admits a screen totally umbilical and stactical half lightlike submanifold \( M \) satisfying one of the following two conditions;

(1) \( \zeta \) is tangent to \( M \) and \( M \) is lightlike totally umbilical, or
(2) \( \zeta \) is parallel with respect to \( \bar{\nabla} \), the local screen second fundamental form
\( D \) is parallel and the lightlike transversal connection is flat,

Then the function \( \alpha \) and \( \beta \), defined by (1.1), vanish and \( \bar{M} \) is flat manifold.

To discuss the curvature of such a semi-Riemannian manifold, we need
the following structure equations: It is well-known [3] that the radical dis-
btribution \( \text{Rad}(TM) = TM \cap TM^\perp \) of half lightlike submanifold \((M, g)\) of a
semi-Riemannian manifold \((\bar{M}, \bar{g})\) of codimension 2 is a vector subbundle of
the tangent bundle \( TM \) and the normal bundle \( TM^\perp \), of rank 1. Therefore
there exist complementary non-degenerate distributions \( S(TM) \) and \( S(TM^\perp) \)
of \( \text{Rad}(TM) \) in \( TM \) and \( TM^\perp \) respectively, which called the screen and co-
screen distribution on \( M \), such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp), \quad (1.2)
\]

where \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. We denote such a half lightlike
submanifold by \( M = (M, g, S(TM), S(TM^\perp)) \). Denote by \( F(M) \) the algebra
of smooth functions on \( M \) and by \( \Gamma(E) \) the \( F(M) \) module of smooth sections
of any vector bundle \( E \) over \( M \). Consider the orthogonal complementary
distribution \( S(TM)^\perp \) to \( S(TM) \) in \( TM \). Certainly \( TM^\perp \) is a vector subbundle
of \( S(TM^\perp) \). As \( S(TM^\perp) \) is a non-degenerate subbundle of \( S(TM)^\perp \), the orthogonal complementary distribution \( S(TM^\perp)^\perp \) of \( S(TM^\perp) \) in \( S(TM)^\perp \) is also a non-degenerate distribution such that

\[
S(TM)^\perp = S(TM^\perp)^\perp \oplus_{\text{orth}} S(TM^\perp)^\perp.
\]

Clearly \( \text{Rad}(TM) \) is a vector subbundle of \( S(TM^\perp)^\perp \). Choose \( L \in \Gamma(S(TM^\perp)) \) as a unit vector field with \( \bar{g}(L, L) = \pm 1 \). In this paper we may assume that \( \bar{g}(L, L) = 1 \) without loss of generality. For any null section \( \xi \) of \( \text{Rad}(TM) \), there exists a uniquely defined null vector field \( N \in \Gamma(S(TM^\perp)^\perp) \) satisfying

\[
\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).
\]

Denote by \( \text{ltr}(TM) \) the subbundle of \( S(TM^\perp)^\perp \) locally spanned by \( N \). Then we show that \( S(TM^\perp)^\perp = \text{Rad}(TM) \oplus \text{ltr}(TM) \). Let \( \text{tr}(TM) = S(TM^\perp)^\perp \oplus_{\text{orth}} \text{ltr}(TM) \). We call \( N, \text{ltr}(TM) \) and \( \text{tr}(TM) \) the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of \( M \) with respect to the screen distribution \( S(TM) \) respectively. Then the tangent bundle \( TM \) of \( M \) is decomposed as

\[
TM = TM \oplus \text{tr}(TM) = \{ \text{Rad}(TM) \oplus \text{tr}(TM) \} \oplus_{\text{orth}} S(TM) \quad (1.3)
\]

\[
= \{ \text{Rad}(TM) \oplus \text{ltr}(TM) \} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).
\]

Let \( P \) be the projection morphism of \( TM \) on \( S(TM) \). Then the local Gauss and Weingarten formulas of \( M \) and \( S(TM) \) are given respectively by

\[
\nabla_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \quad (1.4)
\]

\[
\nabla_X N = -A_N X + \tau(X)N + \rho(X)L, \quad (1.5)
\]

\[
\nabla_X L = -A_L X + \phi(X)N; \quad (1.6)
\]

\[
\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \quad (1.7)
\]

\[
\nabla_X \xi = -A_\xi X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM), \quad (1.8)
\]

where \( \nabla \) and \( \nabla^* \) are induced connections on \( TM \) and \( S(TM) \) respectively, \( B \) and \( D \) are called the local lightlike and screen second fundamental forms of \( M \), \( C \) is called the local second fundamental form on \( S(TM) \) respectively. \( A_N, A_\xi \) and \( A_L \) are linear operators on \( TM \) and \( \tau, \rho \) and \( \phi \) are 1-forms on \( TM \).

Since \( \nabla \) is torsion-free, \( \nabla \) is also torsion-free, and \( B \) and \( D \) are symmetric. The above three local second fundamental forms of \( M \) and \( S(TM) \) are related to their shape operators by

\[
B(X, Y) = g(A_\xi X, Y), \quad \bar{g}(A_\xi X, N) = 0, \quad (1.9)
\]

\[
C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0, \quad (1.10)
\]

\[
D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \bar{g}(A_L X, N) = \rho(X). \quad (1.11)
\]
where $\eta$ is a 1-form on $TM$ such that
\[
\eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM).
\]
From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we know that $B$ and $D$ are independent of the choice of a screen distribution and
\[
B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \quad \forall X \in \Gamma(TM). \tag{1.12}
\]

The induced connection $\nabla$ on $M$ is not metric and satisfies
\[
(\nabla_X g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y), \tag{1.13}
\]
for all $X, Y, Z \in \Gamma(TM)$. But the connection $\nabla^*$ on $M^*$ is metric. By (1.9) and (1.10), we show that $A^*_\xi$ and $A^*_N$ are $S(TM)$-valued shape operators related to $B$ and $C$ respectively and $A^*_\xi$ is self-adjoint on $TM$ and
\[
A^*_\xi \xi = 0. \tag{1.14}
\]

**Definition 1.** A half lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *statical* if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (1.6) and (1.11), we show that this definition is equivalent to the following two conditions: $\phi = 0$ ($M$ is *irrotational*) [10] and $\rho = 0$ ($M$ is *solenoidal*). By $M$ is *statical* we shall mean not only $M$ is *irrotational* but also $M$ is *solenoidal*.

## 2 The Ricci and scalar curvatures

Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of the Levi-Civita connection $\nabla$ on $\bar{M}$, the induced connection $\nabla$ on $M$ and the induced connection $\nabla^*$ on $S(TM)$ respectively. Using the Gauss-Weingarten equations (1.4)–(1.8) for $M$ and $S(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$:

\[
\bar{g}(\bar{R}(X, Y)Z, PW) = g(R(X, Y)Z, PW)
\]
\[
+ B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW)
\]
\[
+ D(X, Z)D(Y, PW) - D(Y, Z)D(X, PW), \tag{2.1}
\]
\[
\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
\]
\[
+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y)
\]
\[
+ D(Y, Z)\phi(X) - D(X, Z)\phi(Y), \tag{2.2}
\]
\[
\bar{g}(\bar{R}(X, Y)Z, N) = \bar{g}(R(X, Y)Z, N)
\]
\[
+ D(X, Z)\rho(Y) - D(Y, Z)\rho(X), \tag{2.3}
\]
\[
\bar{g}(\bar{R}(X, Y)Z, L) = (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \tag{2.4}
\]
\[
+ D(X, Z)\rho(Y) - D(Y, Z)\rho(X), \tag{2.3}
\]
\[
\bar{g}(\bar{R}(X, Y)Z, L) = (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \tag{2.4}
\]
\[
+ D(X, Z)\rho(Y) - D(Y, Z)\rho(X), \tag{2.3}
\]
\[
\bar{g}(\bar{R}(X, Y)Z, L) = (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \tag{2.4}
\]
A semi-Riemannian manifold of quasi-constant curvature

\[ + \rho(X)B(Y, Z) - \rho(Y)B(X, Z), \]
\[ g(\bar{R}(X, Y)\xi, N) = g(A^\xi X, A^N Y) - g(A^\xi Y, A^N X) \]  
\[ - 2d\tau(X, Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X), \]
\[ g(R(X, Y)PZ, PW) = g(R^*(X, Y)PZ, PW) \]
\[ + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW), \]
\[ g(R(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \]
\[ + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \]

for all \( X, Y, Z, W \in \Gamma(TM) \). In case \( \bar{R} = 0 \), we say that \( \bar{M} \) is flat.

The Ricci curvature tensor, denoted by \( \bar{\text{Ric}} \), of \( \bar{M} \) is defined by

\[ \bar{\text{Ric}}(X, Y) = \text{trace}\{Z \rightarrow \bar{R}(Z, X)Y\}, \]

for any \( X, Y \in \Gamma(TM) \). Let \( \dim \bar{M} = m + 3 \). Locally, \( \bar{\text{Ric}} \) is given by

\[ \bar{\text{Ric}}(X, Y) = \sum_{i=1}^{m+3} \epsilon_i \bar{g}(\bar{R}(E_i, X)Y, E_i), \]  
\[ \bar{r} = \sum_{i=1}^{m+3} \epsilon_i \bar{\text{Ric}}(E_i, E_i). \]

Consider an induced quasi-orthonormal frame field \( \{\xi, W_a, N, L\} \) on \( \bar{M} \) such that \( \text{Rad}(TM) = \text{Span}\{\xi\}, S(TM) = \text{Span}\{W_a\} \) and \( \text{tr}(TM) = \text{Span}\{N, L\} \).

Using this frame field, the equations (2.8) and (2.9) reduce respectively to

\[ \bar{\text{Ric}}(X, Y) = \sum_{a=1}^{m} \epsilon_a \bar{g}(\bar{R}(W_a, X)Y, W_a) + \bar{g}(\bar{R}(\xi, X)Y, N) \]
\[ + \bar{g}(\bar{R}(N, X)Y, \xi) + \bar{g}(\bar{R}(L, X)Y, L), \quad \forall X, Y \in \Gamma(TM), \]
\[ \bar{r} = \bar{\text{Ric}}(\xi, \xi) + \bar{\text{Ric}}(N, N) + \bar{\text{Ric}}(L, L) + \sum_{a=1}^{m} \epsilon_a \bar{\text{Ric}}(W_a, W_a). \]

**Definition 2.** For any \( X, Y \in \Gamma(TM) \), let \( \nabla_X^\perp N = \pi(\nabla_X N) \), where \( \pi \) is the projection morphism of \( TM \) on \( ltr(TM) \). Then \( \nabla^\perp \) is a linear connection on \( ltr(TM) \). We say that \( \nabla^\perp \) is the lightlike transversal connection of \( M \). We define the curvature tensor \( R^\perp \) on \( ltr(TM) \) by

\[ R^\perp(X, Y)N = \nabla_X^\perp \nabla_Y^\perp N - \nabla_Y^\perp \nabla_X^\perp N + \nabla^\perp_{[X,Y]}N. \]

The lightlike transversal connection \( \nabla^\perp \) is said to be flat \([5, 6]\) if \( R^\perp = 0 \).
Theorem 2.1 [5, 6]. Let $M$ be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then the lightlike transversal connection of $M$ is flat if and only if the 1-form $\tau$ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $U \subset M$.

Proof. From (1.5), (2.12) and the definition of the connection $\nabla^\perp$, we have

$$\nabla^\perp_X N = \tau(X)N, \quad R^\perp(X,Y)N = 2d\tau(X,Y)N.$$ 

From this equations, we have our assertion.

Note 1 [3, 5]. In case $d\tau = 0$, by the cohomology theory there exist a smooth function $f$ such that $\tau = df$. Thus we get $\tau(X) = X(f)$. If we take $\xi = \gamma\xi$, then we have $\tau(X) = \bar{\tau}(X) + X(\ln \gamma)$. Setting $\gamma = \exp(f)$ in this equation, we get $\bar{\tau}(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form $\tau$ vanishes the canonical null pair of $M$. Although $S(TM)$ is not unique but it is canonically isomorphic to the factor vector bundle $S(TM) \cong TM/Rad(TM)$ considered by Kupeli [10]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, in case $d\tau = 0$ we deal with only half lightlike submanifolds $M$ equipped with the canonical null pair $\{\xi, N\}$.

3 Proof of Theorem 1.1

Let $M$ be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. In the entire discussion of this article, we shall assume the curvature vector field $\zeta$ of $\bar{M}$ is a unit spacelike one without loss of generality. Let $\lambda$, $\mu$ and $\nu$ are the smooth functions defined by $\lambda = \theta(N)$, $\mu = \theta(\xi)$ and $\nu = \theta(L)$. Substituting (1.1) into (2.7), we have

$$\bar{\text{Ric}}(X,Y) = \{(m+2)\alpha + \beta\}\bar{g}(X,Y) + (m+1)\beta \theta(X)\theta(Y),$$

for any $X, Y \in \Gamma(TM)$. Substituting (3.1) into (2.9) and (2.11), we have

$$\bar{r} = (m+2)\{(m+3)\alpha + 2\beta\},$$

$$\bar{r} = (m+1)\{(m+2)\alpha + \beta\} + (m+1)\beta(\lambda^2 + \mu^2 + 1 - 2\lambda\mu),$$

respectively. Comparing the last two equations, we obtain

$$2(m+2)\alpha + (m+3)\beta = (m+1)\beta(\lambda^2 + \mu^2 + 1 - 2\lambda\mu).$$ (3.2)

Replacing $W$ by $N$ to (1.1), for any $X, Y, Z \in \Gamma(TM)$, we have

$$\bar{g}(\bar{R}(X,Y)Z, N) = \{\alpha\eta(X) + \lambda\beta\theta(X)\}g(Y, Z)$$

$$- \{\alpha\eta(Y) + \lambda\beta\theta(Y)\}g(X, Z) + \beta\theta(Y)\eta(X) - \theta(X)\eta(Y)\theta(Z).$$ (3.3)

Taking $X = W = L$ to (1.1) and using the fact $\bar{g}(L, L) = 1$, we have

$$\bar{g}(\bar{R}(L,Y)L, N) = (\alpha + \nu^2)g(X, Y) + \beta \theta(X)\theta(Y),$$

(3.4)
for all \(X, Y \in \Gamma(TM)\). From (3.2), we have the following result:

**Theorem 3.1.** Let \(\bar{M}\) be semi-Riemannian manifold of quasi-constant curvature admits a half lightlike submanifold \(M\). If the function \(\beta\), defined by (1.1), vanishes, then the function \(\alpha\) also vanishes and \(\bar{M}\) is a flat manifold.

**Definition 3.** A half lightlike submanifold \(M\) of a semi-Riemannian manifold \(\bar{M}\) is screen homothetic [4] if the shape operators \(A_N\) and \(A^*_\xi\) of \(M\) and \(S(TM)\) respectively are related by \(A_N = \varphi A^*_\xi\), or equivalently, the second fundamental forms \(B\) and \(C\) of \(M\) and \(S(TM)\) respectively satisfy

\[
C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{3.5}
\]

where \(\varphi\) is a non-zero constant on any coordinate neighborhood \(U\) in \(M\). In particular, if \(\varphi = 0\), i.e., \(C = A_N = 0\), then \(M\) is called screen totally geodesic.

**Proof of Theorem 1.1.** Assume that \(M\) is statical and screen homothetic. Replacing \(Z\) by \(\xi\) to (3.3) and using the fact \(\mu = \theta(\xi)\), we get

\[
g(\bar{R}(X,Y)\xi, N) = \mu \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}. \tag{3.6}
\]

Comparing this with (2.5) and using the facts \(A_N = \varphi A^*_\xi\) and \(\phi = 0\), we get

\[
2d\tau(X,Y) = \mu \beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}, \quad \forall X, Y \in \Gamma(TM). \tag{3.7}
\]

(1) In case \(\zeta\) is tangent to \(M\): It is known [8] that if \(M\) is statical and screen homothetic, then \(\bar{M}\) is a flat manifold. Here we sketch out this proof briefly: If \(\zeta\) belongs to \(Rad(TM)\), then we have \(\zeta = \lambda \xi\). This implies \(1 = g(\zeta, \zeta) = \lambda^2 g(\xi, \xi) = 0\). It is a contradiction. This enables one to choose a screen distribution \(S(TM)\) which contains \(\zeta\). This implies that if \(\zeta\) is tangent to \(M\), then it belongs to \(S(TM)\) which we assume in this case. Thus we get \(\lambda = 0\). As \(\mu = 0\), we have \(d\tau = 0\) by (3.6). Therefore the lightlike transversal connection of \(M\) is flat by Theorem 2.1. We can take \(\tau = 0\) by Note 1.

Replacing \(X\) by \(\xi\) to (3.3) and the fact \(\mu = 0\), we have

\[
g(\bar{R}(\xi,Y)X, N) = \alpha g(X,Y) + \beta \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.8}
\]

Replacing \(W\) by \(\xi\) to (1.1) and using (2.2) and the fact \(\tau = \phi = 0\), we get

\[
(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{3.9}
\]

Substituting (3.5) into (2.7) and using (2.3) with \(\rho = 0\) and (3.8), we obtain

\[
g(\bar{R}(X,Y)PZ, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).
\]

From this and the fact \(g(\bar{R}(X,Y)\xi, N) = 0\), we have

\[
g(\bar{R}(X,Y)Z, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).
\]
Replacing $X$ by $\xi$ and $Z$ by $X$ to this and then, comparing with (3.7), we have

$$\alpha g(X,Y) + \beta \theta(X)\theta(Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$, we get $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.2) and using the fact $\lambda = \mu = 0$, we have $2(m+1)\alpha = 0$. Thus $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

(2) In case $\zeta$ is parallel with respect to $\bar{\nabla}$, $D$ is parallel and the lightlike transversal connection is flat: If $\zeta$ is tangent to $M$, then, by Case (1) we show that $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold. Thus we may assume that $\zeta$ is not tangent to $M$. In this case, we show that $(\mu, \nu) \neq (0, 0)$.

(i) In case $\mu \neq 0$: As the lightlike transversal connection is flat, by Theorem 2.1, we get $d\tau = 0$. From this result, (3.6) and the fact $\mu \neq 0$, we have $2(m+1)\alpha = 0$. Thus $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

As $\zeta$ is parallel with respect to $\bar{\nabla}$, applying $\bar{\nabla}_X$ to (3.9) and using (1.4)~(1.8), (1.12) and the facts $\tau = \phi = \rho = 0$ and $A_N^\iota = \varphi A^\iota_N$, we show that the functions $\lambda$, $\mu$ and $\nu$ are constants and $(\lambda + \mu \varphi)A^\iota_N + \nu A_L = 0$. As $D$ is parallel and $\rho = 0$, (2.4) reduce

$$\bar{g}(\bar{R}(X,Y)Z, L) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

On the other hand, replacing $W$ by $L$ to (1.1), we get

$$\bar{g}(\bar{R}(X,Y)Z, L) = \nu \beta \{\theta(X)g(Y,Z) - \theta(Y)g(X,Z)\},$$

for all $X, Y, Z \in \Gamma(TM)$. From the last two equations, we obtain

$$\nu \beta \{\theta(X)g(Y,Z) - \theta(Y)g(X,Z)\} = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing $X$ by $\xi$ to this, we get $\mu \nu \beta = 0$. As $\mu \nu$ is a constant and $\beta \neq 0$, we have $\mu \nu = 0$. As $\mu$ is a non-zero constant, we have $\nu = 0$. Therefore we get

$$\zeta = \lambda \xi + \mu N, \quad (\lambda + \mu \varphi)A^\iota_N = 0.$$
In case $\lambda + \mu \phi = 0$: Replacing $W$ by $\xi$ to (1.1) and using (2.2) and the fact $\tau = \phi = 0$, for any $X, Y, Z \in \Gamma(TM)$, we have
\[(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \mu \beta \{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}.\] (3.10)
Substituting (3.5) into (2.7) with $\tau = 0$ and using (2.3) with $\rho = 0$, (3.10) and the fact $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we have
\[\bar{g}(\bar{R}(X, Y)Z, N) = \mu \beta \{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\},\] (3.11)
for all $X, Y, Z \in \Gamma(TM)$. Replacing $X$ by $\xi$ and $Z$ by $X$ to (3.3), we get
\[\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda \mu \beta)g(X, Y), \quad \forall X, Y \in \Gamma(TM).\] (3.12)
Replacing $X$ by $\xi$ and $Z$ by $X$ to (3.11) and then, comparing this result and (3.12), we have $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.2), we have
\[\alpha(1 + \lambda^2 + \mu^2) = 0.\]
This implies $\alpha = 0$. Consequently $\beta = 0$ and $\bar{M}$ is a flat manifold.

In case $M$ is screen totally geodesic: From (2.3) and (2.7), we get
\[\bar{g}(\bar{R}(X, Y)PZ, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).\]
From this equation and the fact $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we show that
\[\bar{g}(\bar{R}(X, Y)Z, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).\] (3.13)
Replacing $X$ by $\xi$ and $Z$ by $X$ to (3.3), we get
\[\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda \mu \beta)g(X, Y), \quad \forall X, Y \in \Gamma(TM).\] (3.14)
Using (3.13) and (3.14) and the fact $S(TM)$ is non-degenerate, we have $\alpha + \lambda \mu \beta = 0$, i.e., $\beta = -2\alpha$. Substituting $\beta = -2\alpha$ into (3.2), we have
\[\alpha\{1 - (m + 1)(\lambda^2 + \mu^2)\} = 0.\]
As $\{1 - (m + 1)(\lambda^2 + \mu^2)\}$ is constant, if $\alpha \neq 0$, then $1 = (m + 1)(\lambda^2 + \mu^2)$. As $\lambda^2 + \mu^2 < (m + 1)(\lambda^2 + \mu^2) = 1$. Using $2\lambda \mu = 1$, we have
\[(\lambda - \mu)^2 = \lambda^2 + \mu^2 - 2\lambda \mu < 1 - 1 = 0.\]
It is a contradiction. This implies $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

(ii) In case $\mu = 0$: As $(\mu, \nu) \neq (0, 0)$, we have $\nu \neq 0$. Since $\mu = 0$, by (3.6) we get $d\tau = 0$. Thus we can take $\tau = 0$. Replacing $X$ by $\xi$ to (3.3), we get
\[\bar{g}(\bar{R}(\xi, Y)X, N) = \alpha g(X, Y) + \beta \theta(X)\theta(Y).\] (3.15)
Replacing $W$ by $\xi$ to (1.1) and using (2.2) and the fact $\tau = \phi = \mu = 0$, we get
\[
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = 0, \quad \forall \, X, Y, Z \in \Gamma(TM).
\] (3.16)
Substituting (3.5) into (2.7) and using (2.3) satisfying $\rho = 0$ and (3.16), we get
\[
g(\bar{\nabla}(X, Y)PZ, N) = 0. \quad \forall \, X, Y, Z \in \Gamma(TM).
\]
Replacing $X$ by $\xi$ to this and then, comparing with (3.15), we have
\[
\alpha g(X, Y) + \beta \theta(X)\theta(Y) = 0, \quad \forall \, X, Y \in \Gamma(TM).
\]
Taking $X = Y = \zeta$ to this equation, we have $\alpha = -\beta$. Substituting $\alpha = -\beta$ into (3.2), we get $(m + 1)(2 + \lambda^2)\alpha = 0$. Thus we have $\alpha = 0$. Consequently we also have $\beta = 0$ and $\bar{M}$ is a flat manifold.

**Corollary 1.** Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits a screen totally geodesic and statical half lightlike submanifold $M$ satisfying one of the following two conditions:

1. the curvature vector field $\zeta$ is tangent to $M$, or

2. $\zeta$ is parallel with respect to $\bar{\nabla}$, the local screen second fundamental form $D$ is parallel and the lightlike transversal connection is flat,

Then the function $\alpha$ and $\beta$, defined by (1.1), vanish and $\bar{M}$ is flat manifold.

**Corollary 2.** Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits either a screen homothetic or a screen totally geodesic and statical half lightlike submanifold $M$ such that $\mu = 0$. Then the function $\alpha$ and $\beta$, defined by (1.1), vanish and $\bar{M}$ is flat manifold.

**Proof.** In case $\mu = 0$. By the same method of Case (1) of above theorem, we have $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.2), we have $(m + 1)(2 + \lambda^2)\alpha = 0$. Thus we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

### 4 Proof of Theorem 1.2

**Definition 4.** (1) We say that the half lightlike submanifold $M$ is **screen totally umbilical**[2] if there exist a smooth function $\gamma$ on any coordinate neighborhood $U \subset M$ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,
\[
C(X, PY) = \gamma g(X, Y), \quad \forall \, X, Y \in \Gamma(TM).
\] (4.1)

(2) We say that $M$ is **lightlike totally umbilical** if there is a smooth function $\sigma$ on any coordinate neighborhood $U$ such that
\[
B(X, Y) = \sigma g(X, Y), \quad \forall \, X, Y \in \Gamma(TM).
\] (4.2)
Note that, in case $\gamma = 0$ on $\mathcal{U}$, $M$ is screen totally geodesic.

**Proof of Theorem 1.2.** Assume that $M$ is statical and screen totally umbilical. From (2.5), (3.3), (4.1) and the fact $A^* \xi$ is self-adjoint, we get

$$2d\tau(X,Y) = \mu \beta \{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}, \quad \forall X, Y \in \Gamma(TM). \quad (4.3)$$

(1) In case $\zeta$ is tangent to $M$: In this case we show that $\mu = \nu = 0$ and $\zeta$ belongs to $S(TM)$ by Case 1 of Theorem 1.1. Thus $\lambda = 0$. As $\mu = 0$, we get $d\tau = 0$. Therefore the transversal connection is flat and $\bar{g}(\bar{R}(X,Y)\xi, N) = 0$. Replacing $X$ by $\xi$ to (3.3) and using the fact $\mu = 0$, we have

$$\bar{g}(\bar{R}(\xi,Y)X, N) = \alpha g(X,Y) + \beta \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM). \quad (4.4)$$

As $d\tau = 0$, we can take $\tau = 0$. Assume that $M$ is lightlike totally umbilical. Substituting (4.1) into (2.7) and using (2.3) and (4.2), we get

$$\bar{g}(\bar{R}(X,Y)Z, N) = \{X[\gamma] - \sigma \gamma \eta(X)\}g(Y,Z) - \{Y[\gamma] - \sigma \gamma \eta(Y)\}g(X,Z).$$

Replacing $X$ by $\xi$ and $Z$ by $X$ to this equation, we have

$$\bar{g}(\bar{R}(\xi,Y)X, N) = \{\xi[\gamma] - \sigma \gamma\}g(X,Y), \quad \forall X, Y \in \Gamma(TM).$$

Comparing this equation and (4.4), we obtain

$$\{\xi[\gamma] - \sigma \gamma - \alpha\}g(X,Y) = \beta \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this equation, we have $\beta = \xi[\gamma] - \sigma \gamma - \alpha$ and

$$\beta g(X,Y) = \beta \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM). \quad (4.5)$$

Substituting (4.5) into (3.1), (3.7) and (3.4), we obtain

$$\bar{R}ic(X,Y) = (m + 2)(\alpha + \beta)g(X,Y), \quad \forall X, Y \in \Gamma(T\bar{M}), \quad (4.6)$$

$$\bar{g}(\bar{R}(\xi,X)Y, N) = (\alpha + \beta)g(X,Y), \quad \forall X, Y \in \Gamma(TM), \quad (4.7)$$

$$\bar{g}(\bar{R}(\xi,X)Y, L) = (\alpha + \beta)g(X,Y), \quad \forall X, Y \in \Gamma(TM). \quad (4.8)$$

Substituting (4.5) into (1.1), for any $X, Y, Z, W \in \Gamma(TM)$, we have

$$\bar{g}(\bar{R}(X,Y)Z, W) = (\alpha + 2\beta)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \quad (4.9)$$

Substituting (4.7), (4.8) and (4.9) into (2.9), we also have

$$\bar{R}ic(X,Y) = \{(m + 2)\alpha + (2m + 1)\beta\}g(X,Y). \quad (4.10)$$

Comparing (4.6) and (4.10), we obtain $\beta = 0$ as $m > 1$. As $\beta = 0$, from (3.2) we have $\alpha = 0$. Thus $\bar{M}$ is a flat manifold.
(2) In case $\zeta$ is parallel with respect to $\bar{\nabla}$, $D$ is parallel and the lightlike transversal connection is flat: If $\zeta$ is tangent to $M$, then, by Case (1) of this theorem, we show that $\alpha = \beta = 0$ and $M$ is a flat manifold. Thus we may assume that $\zeta$ is not tangent to $M$. In this case, we find $(\mu, \nu) \neq (0, 0)$.

(i) In case $\mu \neq 0$: As the lightlike transversal connection is flat, we get $d\tau = 0$ by Theorem 2.1. Thus, from (3.6), we have
\[ \beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\} = 0, \quad \forall X, Y \in \Gamma(TM). \] (4.11)
Replacing $Y$ by $\xi$ to (4.11), we have
\[ \beta\theta(PX) = \beta\{\theta(X) - f\eta(X)\} = 0. \]
By Theorem 3.1, we set $\beta \neq 0$. As $\theta(PX) = 0$ for all $X \in \Gamma(TM)$, we get
\[ \zeta = \lambda\xi + \mu N + \nu L. \] (4.12)

As $D$ is parallel and $\rho = 0$, from (1.1) and (2.4) we have
\[ \nu\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\} = 0, \quad \forall X, Y, Z \in \Gamma(TM). \]
Replacing $X$ by $\xi$, we get $\mu \nu \beta = 0$. As $\mu \nu$ is a non-zero constant and $\beta \neq 0$, we have $\mu \nu = 0$. As $\mu$ is a non-zero constant, we have $\nu = 0$. Thus we get
\[ \zeta = \lambda\xi + \mu N, \quad A^*_\xi X = \sigma PX, \quad \text{where} \quad \sigma = -2\mu^2\gamma. \] (4.13)
As $\bar{g}(\zeta, \zeta) = 1$, we get $2\lambda \mu = 1$. Replacing $X$ by $\xi$ to (3.3), we get
\[ \bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda\mu\beta)g(X, Y), \quad \forall X, Y \in \Gamma(TM). \] (4.14)
Substituting (4.1) into (2.6) and using (2.3), the second equation of (4.13) and the fact that $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we get
\[ \bar{g}(\bar{R}(X, Y)Z, N) = \{X[\gamma] - \sigma\gamma\eta(X)\}g(Y, Z) - \{Y[\gamma] - \sigma\gamma\eta(Y)\}g(X, Z). \]
Replacing $X$ by $\xi$ and $Z$ by $X$ to this equation, we have
\[ \bar{g}(\bar{R}(\xi, Y)X, N) = \{\xi[\gamma] - \sigma\gamma\}g(X, Y), \quad \forall X, Y \in \Gamma(TM). \]
Comparing this and (4.14) and using $\sigma = -2\mu^2\gamma$ and $2\lambda \mu = 1$, we get
\[ 2\{\xi[\gamma] + 2\mu^2\gamma^2\} = 2\alpha + \beta. \] (4.15)
Replacing $W$ by $\xi$ to (1.1) and using (2.2) and the fact $\tau = \phi = 0$, we have
\[(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \mu \beta \{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\} \tag{4.16}\]
Applying $\nabla_X$ to $B(Y, Z) = -2\mu^2 \gamma g(Y, Z)$ and using $\mu$ is a constant, we have
\[(\nabla_X B)(Y, Z) = -2\mu^2 \gamma \{\theta(Y)g(X, Z) - \theta(X)g(Y, Z)\}, \quad \forall X, Y, Z \in \Gamma(TM). \tag{1.9}\]
Substituting this into (4.16) and using (1.9) and the fact $\mu \neq 0$, we have
\[2\mu \{X[\gamma] + 2\mu^2 \gamma^2 \eta(Y)\}g(Y, Z) - 2\mu \{Y[\gamma] + 2\mu^2 \gamma^2 \eta(Y)\}g(X, Z) = \beta \{\theta(Y)g(X, Z) - \theta(X)g(Y, Z)\}, \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.15}\]
Thus we have $\alpha = -\beta$. Substituting this into (3.2), we get
\[\alpha(1 + \lambda^2 + \mu^2) = 0. \tag{4.17}\]
From (4.15) and (4.17), we get $\alpha = -\beta$. Substituting this into (3.2), we get
\[\alpha(1 + \lambda^2 + \mu^2) = 0. \tag{4.18}\]
Thus we have $\alpha = 0$. Consequently we get $\beta = 0$ and $\bar{M}$ is a flat manifold.

(ii) In case $\mu = 0$: As $(\mu, \nu) \neq (0, 0)$, we have $\nu \neq 0$. By Theorem 3.1, we let $\beta \neq 0$. As $D$ is parallel and $\rho = 0$, from (1.1) with $W = L$, we get
\[\theta(X)g(Y, Z) - \theta(Y)g(X, Z) = 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{1.10}\]
Using this and the fact $S(TM)$ is non-degenerate, we obtain
\[\theta(PX)PY = \theta(PY)PX, \quad \forall X, Y \in \Gamma(TM). \tag{1.11}\]
Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $\theta(PX_o) \neq 0$, then $PX = fPX_o$ for any $X \in \Gamma(TM)$, where $f$ is a smooth function. It is a contradiction as rank $S(TM) > 1$. Thus we have $\theta(PX) = 0$ for all $X \in \Gamma(TM)$. From this and $\theta(\xi) = 0$, we have $\theta(X) = 0$ for any $X \in \Gamma(TM)$. Replacing $X$ by $\xi$ to (3.3), we get
\[\bar{g}(\bar{\nabla}(\xi, Y)X, N) = \alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.19}\]
Substituting (4.1) into (2.7) and using (1.13) and (2.3) with $\rho = 0$, we get
\[\bar{g}(\bar{\nabla}(X, Y)Z, N) = X[\gamma]g(Y, Z) - Y[\gamma]g(X, Z) + \gamma \{B(X, Z)\eta(Y) - B(Y, Z)\eta(X)\}. \tag{4.20}\]
Replacing $X$ by $\xi$ and $Z$ by $X$ to this equation, we have
\[\bar{g}(\bar{\nabla}(\xi, Y)X, N) = \xi[\gamma]g(X, Y) - \gamma B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.21}\]
Comparing this equation and (4.18), we obtain
\[\gamma B(X, Y) = (\xi[\gamma] - \alpha)g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.22}\]
Thus $M$ is either lightlike totally umbilical or $M$ is screen totally geodesic. In case $M$ is lightlike totally umbilical, by the method of Case (1) of this theorem, we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold. In case $M$ is screen totally geodesic, by Corollary 2, we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.
References


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