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A Semi-Riemannian Manifold of Quasi-Constant Curvature with Half Lightlike Submanifolds

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Abstract

In this paper, we study the geometry of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. The main result is two characterization theorems for $\bar{M}$ endowed with a statical, screen homothetic or screen totally umbilical half lightlike submanifold M.

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1 Introduction

B.Y. Chen and K. Yano [1] introduced the notion of a *Riemannian manifold of quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g}, \bar{\nabla})$ equipped with the curvature tensor $\bar{R}$ of the Levi-Civita connection $\bar{\nabla}$ satisfying

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,W) &= \alpha\{\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)\} \\ &\quad + \beta\{\bar{g}(X,W)\theta(Y)\theta(Z) - \bar{g}(X,Z)\theta(Y)\theta(W) \\ &\quad\quad + \bar{g}(Y,Z)\theta(X)\theta(W) - \bar{g}(Y,W)\theta(X)\theta(Z)\},\end{aligned} \tag{1.1}$$

where α and β are scalar functions, and θ is a 1-form defined by $\theta(X) = \bar{g}(X, \zeta)$ and ζ is a unit vector field on $\bar{M}$, which called the *curvature vector field* of $\bar{M}$. If $\beta = 0$, then $\bar{M}$ is a space of constant curvature α.

Recently D.H. Jin and J.W. Lee studied half lightlike submanifolds [8] and lightlike submanifolds M [9] of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature subject to the conditions; (1) ζ is tangent to M, (2) the screen distribution $S(TM)$ is totally umbilical in M and (3) the co-screen distribution $S(TM^{\perp})$ is conformal Killing distribution. Each of this papers proved two characterization theorems for their lightlike submanifolds.

In this paper, we study the curvature of semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature admits either a screen homothetic or a screen totally umbilical, and statical half lightlike submanifold M. We prove the following two characterization theorems for such a semi-Riemannian manifold $\bar{M}$:

Theorem 1.1. *Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits a screen homothetic and statical half lightlike submanifold M satisfying one of the following two conditions*;

(1) *the curvature vector field ζ is tangent to M, or*
(2) *ζ is parallel with respect to $\bar{\nabla}$, the local screen second fundamental form D is parallel and the lightlike transversal connection is flat,*

Then the function α and β, defined by (1.1), *vanish and $\bar{M}$ is flat manifold.*

Theorem 1.2. *Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature such that $\dim \bar{M} > 4$ admits a screen totally umbilical and statical half lightlike submanifold M satisfying one of the following two conditions*;

(1) *ζ is tangent to M and M is lightlike totally umbilical, or*
(2) *ζ is parallel with respect to $\bar{\nabla}$, the local screen second fundamental form D is parallel and the lightlike transversal connection is flat,*

Then the function α and β, defined by (1.1), *vanish and $\bar{M}$ is flat manifold.*

To discuss the curvature of such a semi-Riemannian manifold, we need the following structure equations: It is well-known [3] that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifold (M, g) of a semi-Rimannian manifold $(\bar{M}, \bar{g})$ of codimension 2 is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Therefore there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \quad (1.2)$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly $TM^{\perp}$ is a vector subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate distribution such that

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}.$$

Clearly $Rad(TM)$ is a vector subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \pm 1$. In this paper we may assume that $\bar{g}(L, L) = 1$ without loss of generality. For any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \quad (1.3)\\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \quad (1.4)$$
$$\bar{\nabla}_X N = -A_{N} X + \tau(X)N + \rho(X)L, \quad (1.5)$$
$$\bar{\nabla}_X L = -A_{L} X + \phi(X)N; \quad (1.6)$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \quad (1.7)$$
$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \quad \forall\, X,\, Y \in \Gamma(TM), \quad (1.8)$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local lightlike* and *screen second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$ respectively. A_{N}, A^*_{ξ} and A_{L} are linear operators on TM and τ, ρ and ϕ are 1-forms on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_{\xi} X, Y), \qquad \bar{g}(A^*_{\xi} X, N) = 0, \quad (1.9)$$
$$C(X, PY) = g(A_{N} X, PY), \qquad \bar{g}(A_{N} X, N) = 0, \quad (1.10)$$
$$D(X, Y) = g(A_{L} X, Y) - \phi(X)\eta(Y), \quad \bar{g}(A_{L} X, N) = \rho(X). \quad (1.11)$$

where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N), \quad \forall\, X \in \Gamma(TM).$$

From the facts $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$ and $D(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, L)$, we know that B and D are independent of the choice of a screen distribution and

$$B(X,\xi) = 0, \quad D(X,\xi) = -\phi(X), \quad \forall\, X \in \Gamma(TM). \tag{1.12}$$

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.13}$$

for all $X,\ Y,\ Z \in \Gamma(TM)$. But the connection ∇^* on M^* is metric. By (1.9) and (1.10), we show that A^*_ξ and A_N are $S(TM)$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{1.14}$$

Definition 1. A half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *statical* if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (1.6) and $(1.11)_2$, we show that this definition is equivalent to the following two conditions: $\phi = 0$ (M is *irrotational*) [10] and $\rho = 0$ (M is *solenoidal*). By M is *statical* we shall mean not only M is *irrotational* but also M is *solenoidal.*

2 The Ricci and scalar curvatures

Denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations (1.4)∼(1.8) for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,\, PW) &= g(R(X,Y)Z,\, PW) \\ &\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) \\ &\quad + D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW),\end{aligned} \tag{2.1}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,\, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ &\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,\, N) &= \bar{g}(R(X,Y)Z,\, N) \\ &\quad + D(X,Z)\rho(Y) - D(Y,Z)\rho(X),\end{aligned} \tag{2.3}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, L) = (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) \tag{2.4}$$

$$
\begin{aligned}
&+\rho(X)B(Y,Z)-\rho(Y)B(X,Z),\\
\bar{g}(\bar{R}(X,Y)\xi,\, N) &= g(A^{*}_{\xi}X,\, A_{N}Y)-g(A^{*}_{\xi}Y,\, A_{N}X) \qquad (2.5)\\
&-2d\tau(X,Y)+\rho(X)\phi(Y)-\rho(Y)\phi(X),\\
\bar{g}(R(X,Y)PZ,\, PW) &= g(R^{*}(X,Y)PZ,\, PW) \qquad (2.6)\\
&+C(X,PZ)B(Y,PW)-C(Y,PZ)B(X,PW),\\
g(R(X,Y)PZ,\, N) &= (\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ) \qquad (2.7)\\
&+C(X,PZ)\tau(Y)-C(Y,PZ)\tau(X).
\end{aligned}
$$

for all $X,\, Y,\, Z,\, W \in \Gamma(TM)$. In case $\bar{R}=0$, we say that $\bar{M}$ is *flat*.

The *Ricci curvature tensor*, denoted by $\bar{Ric}$, of $\bar{M}$ is defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(Z,X)Y\},$$

for any $X,\, Y \in \Gamma(T\bar{M})$. Let $\dim \bar{M} = m+3$. Locally, $\bar{Ric}$ is given by

$$\bar{Ric}(X,Y) = \sum_{i=1}^{m+3} \epsilon_i\, \bar{g}(\bar{R}(E_i,X)Y,\, E_i), \qquad (2.8)$$

where $\{E_1,\, \ldots,\, E_{m+3}\}$ is an orthonormal frame field of $T\bar{M}$ and $\epsilon_i\, (=\pm 1)$ denotes the causal character of respective vector field E_i. In case $\bar{Ric}=0$, we say that $\bar{M}$ is *Ricci flat*. The *scalar curvature* $\bar{r}$ is defined by

$$\bar{r} = \sum_{i=1}^{m+3} \epsilon_i\, \bar{Ric}(E_i,\, E_i). \qquad (2.9)$$

Consider an induced quasi-orthonormal frame field $\{\xi, W_a, N, L\}$ on $\bar{M}$ such that $Rad(TM) = Span\{\xi\}$, $S(TM) = Span\{W_a\}$ and $tr(TM) = Span\{N, L\}$. Using this frame field, the equations (2.8) and (2.9) reduce respectively to

$$
\begin{aligned}
\bar{Ric}(X,Y) &= \sum_{a=1}^{m} \epsilon_a\, \bar{g}(\bar{R}(W_a,X)Y,\, W_a)+\bar{g}(\bar{R}(\xi,X)Y,\, N) \qquad (2.10)\\
&+\ \bar{g}(\bar{R}(N,X)Y,\, \xi)+\bar{g}(\bar{R}(L,X)Y,\, L), \quad \forall\, X,\, Y \in \Gamma(T\bar{M}),\\
\bar{r} &= \bar{Ric}(\xi,\, \xi)+\bar{Ric}(N,\, N)+\bar{Ric}(L,\, L)+\sum_{a=1}^{m} \epsilon_a\, \bar{Ric}(W_a,\, W_a). \qquad (2.11)
\end{aligned}
$$

Definition 2. For any $X,\, Y \in \Gamma(TM)$, let $\nabla^{\perp}_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $ltr(TM)$. Then $\nabla^{\perp}$ is a linear connection on $ltr(TM)$. We say that $\nabla^{\perp}$ is the *lightlike transversal connection* of M. We define the curvature tensor $R^{\perp}$ on $ltr(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X\nabla^{\perp}_Y N-\nabla^{\perp}_Y\nabla^{\perp}_X N-\nabla^{\perp}_{[X,Y]}N. \qquad (2.12)$$

The lightlike transversal connection $\nabla^{\perp}$ is said to be *flat* [5, 6] if $R^{\perp}=0$.

Theorem 2.1 [5, 6]. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then the lightlike transversal connection of M is flat if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

Proof. From (1.5), (2.12) and the definition of the connection $\nabla^{\perp}$, we have

$$\nabla^{\perp}_X N = \tau(X)N, \quad R^{\perp}(X, Y)N = 2d\tau(X, Y)N.$$

From this equations, we have our assertion.

Note 1 [3, 5]. In case $d\tau = 0$, by the cohomology theory there exist a smooth function f such that $\tau = df$. Thus we get $\tau(X) = X(f)$. If we take $\tilde{\xi} = \gamma\xi$, then we have $\tau(X) = \tilde{\tau}(X) + X(\ln \gamma)$. Setting $\gamma = \exp(f)$ in this equation, we get $\tilde{\tau}(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form τ vanishes the *canonical null pair* of M. Although $S(TM)$ is not unique but it is canonically isomorphic to the factor vector bundle $S(TM)^{\sharp} = TM/Rad(TM)$ considered by Kupeli [10]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, in case $d\tau = 0$ we deal with only half lightlike submanifolds M equipped with the canonical null pair $\{\xi, N\}$.

3 Proof of Theorem 1.1

Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. In the entire discussion of this article, we shall assume the curvature vector field ζ of $\bar{M}$ is a unit spacelike one without loss of generality. Let λ, μ and ν are the smooth functions defined by $\lambda = \theta(N)$, $\mu = \theta(\xi)$ and $\nu = \theta(L)$. Substituting (1.1) into (2.7), we have

$$\bar{Ric}(X, Y) = \{(m+2)\alpha + \beta\}\bar{g}(X, Y) + (m+1)\beta\, \theta(X)\theta(Y), \tag{3.1}$$

for any $X, Y \in \Gamma(T\bar{M})$. Substituting (3.1) into (2.9) and (2.11), we have

$$\begin{aligned} \bar{r} &= (m+2)\{(m+3)\alpha + 2\beta\}, \\ \bar{r} &= (m+1)\{(m+2)\alpha + \beta\} + (m+1)\beta(\lambda^2 + \mu^2 + 1 - 2\lambda\mu), \end{aligned}$$

respectively. Comparing the last two equations, we obtain

$$2(m+2)\alpha + (m+3)\beta = (m+1)\beta(\lambda^2 + \mu^2 + 1 - 2\lambda\mu). \tag{3.2}$$

Replacing W by N to (1.1), for any $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, N) &= \{\alpha\eta(X) + \lambda\beta\theta(X)\}g(Y, Z) \\ &- \{\alpha\eta(Y) + \lambda\beta\theta(Y)\}g(X, Z) + \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z). \end{aligned} \tag{3.3}$$

Taking $X = W = L$ to (1.1) and using the fact $\bar{g}(L, L) = 1$, we have

$$\bar{g}(\bar{R}(L, Y)X, L) = (\alpha + \beta\nu^2)g(X, Y) + \beta\, \theta(X)\theta(Y), \tag{3.4}$$

for all $X,\, Y \in \Gamma(TM)$. From (3.2), we have the following result:

Theorem 3.1. *Let $\bar{M}$ be semi-Riemannian manifold of quasi-constant curvature admits a half lightlike submanifold M. If the function β, defined by* (1.1)*, vanishes, then the function α also vanishes and $\bar{M}$ is a flat manifold.*

Definition 3. A half lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is *screen homothetic* [4] if the shape operators A_N and A^*_ξ of M and $S(TM)$ respectively are related by $A_N = \varphi A^*_\xi$, or equivalently, the second fundamental forms B and C of M and $S(TM)$ respectively satisfy

$$C(X, PY) = \varphi B(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{3.5}$$

where φ is a non-zero constant on any coordinate neighborhood $\mathcal{U}$ in M. In particular, if $\varphi = 0$, i.e., $C = A_N = 0$, then M is called *screen totally geodesic.*

Proof of Theorem 1.1. Assume that M is statical and screen homothetic. Replacing Z by ξ to (3.3) and using the fact $\mu = \theta(\xi)$, we get

$$\bar{g}(\bar{R}(X, Y)\xi,\, N) = \mu\beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}.$$

Comparing this with (2.5) and using the facts $A_N = \varphi A^*_\xi$ and $\phi = 0$, we get

$$2d\tau(X, Y) = \mu\beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.6}$$

(1) In case ζ is tangent to M: It is known [8] that if M is statical and screen homothetic, then $\bar{M}$ is a flat manifold. Here we sketch out this proof briefly: If ζ belongs to $Rad(TM)$, then we have $\zeta = \lambda\xi$. This implies $1 = \bar{g}(\zeta, \zeta) = \lambda^2 g(\xi, \xi) = 0$. It is a contradiction. This enables one to choose a screen distribution $S(TM)$ which contains ζ. This implies that if ζ is tangent to M, then it belongs to $S(TM)$ which we assume in this case. Thus we get $\lambda = 0$. As $\mu = 0$, we have $d\tau = 0$ by (3.6). Therefore the lightlike transversal connection of M is flat by Theorem 2.1. We can take $\tau = 0$ by Note 1. Replacing X by ξ to (3.3) and the fact $\mu = 0$, we have

$$\bar{g}(\bar{R}(\xi, Y)X,\, N) = \alpha g(X, Y) + \beta\, \theta(X)\theta(Y), \;\; \forall\, X,\, Y \in \Gamma(TM). \tag{3.7}$$

Replacing W by ξ to (1.1) and using (2.2) and the fact $\tau = \phi = 0$, we get

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{3.8}$$

Substituting (3.5) into (2.7) and using (2.3) with $\rho = 0$ and (3.8), we obtain

$$\bar{g}(\bar{R}(X, Y)PZ, N) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

From this and the fact $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we have

$$\bar{g}(\bar{R}(X, Y)Z, N) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Replacing X by ξ and Z by X to this and then, comparing with (3.7), we have

$$\alpha g(X,Y)+\beta\theta(X)\theta(Y)\;=\;0,\quad \forall\, X,\, Y\in\Gamma(TM).$$

Taking $X=Y=\zeta$, we get $\alpha=-\beta$. Substituting $\beta=-\alpha$ into (3.2) and using the fact $\lambda=\mu=0$, we have $2(m+1)\alpha=0$. Thus $\alpha=\beta=0$ and $\bar{M}$ is a flat manifold.

(2) In case ζ is parallel with respect to $\bar{\nabla}$, D is parallel and the lightlike transversal connection is flat: If ζ is tangent to M, then, by Case (1) we show that $\alpha=\beta=0$ and $\bar{M}$ is a flat manifold. Thus we may assume that ζ is not tangent to M. In this case, we show that $(\mu,\,\nu)\neq(0,\,0)$.

(i) In case $\mu\neq 0$: As the lightlike transversal connection is flat, by Theorem 2.1, we get $d\tau=0$. From this result, (3.6) and the fact $\mu\neq 0$, we have

$$\beta\{\theta(X)\eta(Y)-\theta(Y)\eta(X)\}=0,\quad \forall\, X,\, Y\in\Gamma(TM).$$

Replacing Y by ξ to this equation, we obtain $\beta\theta(PX)=\beta\{\theta(X)-\mu\eta(X)\}=0$. By Theorem 3.1, we set $\beta\neq 0$. As $\theta(PX)=0$ for all $X\in\Gamma(TM)$, we find

$$\zeta=\lambda\xi+\mu N+\nu L. \tag{3.9}$$

As ζ is parallel with respect to $\bar{\nabla}$, applying $\bar{\nabla}_X$ to (3.9) and using (1.4)∼(1.8), (1.12) and the facts $\tau=\phi=\rho=0$ and $A_{_N}=\varphi A^*_\xi$, we show that the functions λ, μ and ν are constants and $(\lambda+\mu\varphi)A^*_\xi+\nu A_{_L}\;=\;0$. As D is parallel and $\rho=0$, (2.4) reduce

$$\bar{g}(\bar{R}(X,Y)Z,\,L)=0,\quad \forall\, X,\, Y,\, Z\in\Gamma(TM).$$

On the other hand, replacing W by L to (1.1), we get

$$\bar{g}(\bar{R}(X,Y)Z,\,L)=\nu\beta\{\theta(X)g(Y,Z)-\theta(Y)g(X,Z)\},$$

for all $X,\,Y,\,Z\in\Gamma(TM)$. From the last two equations, we obtain

$$\nu\beta\{\theta(X)g(Y,Z)-\theta(Y)g(X,Z)\}=0,\quad \forall\, X,\, Y,\, Z\in\Gamma(TM).$$

Replacing X by ξ to this, we get $\mu\nu\beta=0$. As $\mu\nu$ is a constant and $\beta\neq 0$, we have $\mu\nu=0$. As μ is a non-zero constant, we have $\nu=0$. Therefore we get

$$\zeta=\lambda\xi+\mu N,\qquad (\lambda+\mu\varphi)A^*_\xi=0.$$

Since $\bar{g}(\zeta,\zeta)=1$, we get $2\lambda\mu=1$. As $(\lambda+\mu\varphi)$ is a constant on M, we show that either $\lambda+\mu\varphi=0$ or $A^*_\xi=0$. Due to (3.5), the later case is equivalent to the condition: M is totally geodesic and screen totally geodesic.

In case $\lambda + \mu\varphi = 0$: Replacing W by ξ to (1.1) and using (2.2) and the fact $\tau = \phi = 0$, for any $X, Y, Z \in \Gamma(TM)$, we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \mu\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}. \quad (3.10)$$

Substituting (3.5) into (2.7) with $\tau = 0$ and using (2.3) with $\rho = 0$, (3.10) and the fact $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we have

$$\bar{g}(\bar{R}(X, Y)Z, N) = \mu\varphi\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}, \quad (3.11)$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing X by ξ and Z by X to (3.3), we get

$$\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda\mu\beta)g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \quad (3.12)$$

Replacing X by ξ and Z by X to (3.11) and then, comparing this result and (3.12), we have $\alpha = \lambda(\lambda\varphi - \mu)\beta$. As $\mu + \lambda\varphi = 0$ and $2\lambda\mu = 1$, this results imply $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.2), we have

$$\alpha(1 + \lambda^2 + \mu^2) = 0.$$

This implies $\alpha = 0$. Consequently $\beta = 0$ and $\bar{M}$ is a flat manifold.

In case M is screen totally geodesic: From (2.3) and (2.7), we get

$$\bar{g}(\bar{R}(X, Y)PZ, N) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

From this equation and the fact $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we show that

$$\bar{g}(\bar{R}(X, Y)Z, N) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM). \quad (3.13)$$

Replacing X by ξ and Z by X to (3.3), we get

$$\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda\mu\beta)g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \quad (3.14)$$

Using (3.13) and (3.14) and the fact $S(TM)$ is non-degenerate, we have $\alpha + \lambda\mu\beta = 0$, i.e., $\beta = -2\alpha$. Substituting $\beta = -2\alpha$ into (3.2), we have

$$\alpha\{1 - (m + 1)(\lambda^2 + \mu^2)\} = 0.$$

As $\{1 - (m + 1)(\lambda^2 + \mu^2)\}$ is constant, if $\alpha \neq 0$, then $1 = (m + 1)(\lambda^2 + \mu^2)$. As $\lambda^2 + \mu^2 < (m + 1)(\lambda^2 + \mu^2) = 1$. Using $2\lambda\mu = 1$, we have

$$(\lambda - \mu)^2 = \lambda^2 + \mu^2 - 2\lambda\mu \;<\; 1 - 1 = 0.$$

It is a contradiction. This implies $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

(ii) In case $\mu = 0$: As $(\mu, \nu) \neq (0, 0)$, we have $\nu \neq 0$. Since $\mu = 0$, by (3.6) we get $d\tau = 0$. Thus we can take $\tau = 0$. Replacing X by ξ to (3.3), we get

$$\bar{g}(\bar{R}(\xi, Y)X, N) = \alpha g(X, Y) + \beta\theta(X)\theta(Y). \quad (3.15)$$

Replacing W by ξ to (1.1) and using (2.2) and the fact $\tau = \phi = \mu = 0$, we get

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{3.16}$$

Substituting (3.5) into (2.7) and using (2.3) satisfying $\rho = 0$ and (3.16), we get $\bar{g}(\bar{R}(X,Y)PZ,N) = 0$. From this and the fact $\bar{g}(\bar{R}(X,Y)\xi,N) = 0$, we have

$$\bar{g}(\bar{R}(X,Y)Z,N) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Replacing X by ξ to this and then, comparing with (3.15), we have

$$\alpha g(X,Y) + \beta\theta(X)\theta(Y) \;=\; 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this equation, we have $\alpha = -\beta$. Substituting $\alpha = -\beta$ into (3.2), we get $(m+1)(2+\lambda^2)\alpha = 0$. Thus we have $\alpha = 0$. Consequently we also have $\beta = 0$ and $\bar{M}$ is a flat manifold.

Corollary 1. *Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits a screen totally geodesic and statical half lightlike submanifold M satisfying one of the following two conditions*;

(1) *the curvature vector field ζ is tangent to M, or*

(2) *ζ is parallel with respect to $\bar{\nabla}$, the local screen second fundamental form D is parallel and the lightlike transversal connection is flat,*

Then the function α and β, defined by (1.1)*, vanish and $\bar{M}$ is flat manifold.*

Corollary 2. *Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits either a screen homothetic or a screen totally geodesic and statical half lightlike submanifold M such that $\mu = 0$. Then the function α and β, defined by* (1.1)*, vanish and $\bar{M}$ is flat manifold.*

Proof. In case $\mu = 0$. By the same method of Case (1) of above theorem, we have $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.2), we have $(m+1)(2+\lambda^2)\alpha = 0$. Thus we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

4 Proof of Theorem 1.2

Definition 4. (1) We say that the half lightlike submanifold M is *screen totally umbilical* [2] if there exist a smooth function γ on any coordinate neighborhood $\mathcal{U} \subset M$ such that $A_{_N}X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$C(X,PY) = \gamma\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.1}$$

(2) We say that M is *lightlike totally umbilical* if there is a smooth function σ on any coordinate neighborhood $\mathcal{U}$ such that

$$B(X,Y) = \sigma g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.2}$$

Note that, in case $\gamma = 0$ on $\mathcal{U}$, M is screen totally geodesic.

Proof of Theorem 1.2. Assume that M is statical and screen totally umbilical. From (2.5), (3.3), (4.1) and the fact A^*_ξ is self-adjoint, we get

$$2d\tau(X,Y) = \mu\beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.3}$$

(1) In case ζ is tangent to M: In this case we show that $\mu = \nu = 0$ and ζ belongs to $S(TM)$ by Case 1 of Theorem 1.1. Thus $\lambda = 0$. As $\mu = 0$, we get $d\tau = 0$. Therefore the transversal connection is flat and $\bar{g}(\bar{R}(X,Y)\xi,\, N) = 0$. Replacing X by ξ to (3.3) and using the fact $\mu = 0$, we have

$$\bar{g}(\bar{R}(\xi,Y)X,\, N) = \alpha g(X,Y) + \beta\theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.4}$$

As $d\tau = 0$, we can take $\tau = 0$. Assume that M is lightlike totally umbilical. Substituting (4.1) into (2.7) and using (2.3) and (4.2), we get

$$\bar{g}(\bar{R}(X,Y)Z,\, N) = \{X[\gamma] - \sigma\gamma\,\eta(X)\}g(Y,Z) - \{Y[\gamma] - \sigma\gamma\,\eta(Y)\}g(X,Z).$$

Replacing X by ξ and Z by X to this equation, we have

$$\bar{g}(\bar{R}(\xi,Y)X,\, N) = \{\xi[\gamma] - \sigma\gamma\}g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Comparing this equation and (4.4), we obtain

$$\{\xi[\gamma] - \sigma\gamma - \alpha\}g(X,Y) = \beta\theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this equation, we have $\beta = \xi[\gamma] - \sigma\gamma - \alpha$ and

$$\beta g(X,Y) = \beta\theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.5}$$

Substituting (4.5) into (3.1), (3.7) and (3.4), we obtain

$$\bar{Ric}(X,Y) = (m+2)(\alpha+\beta)g(X,Y), \quad \forall\, X,\, Y \in \Gamma(T\bar{M}), \tag{4.6}$$
$$\bar{g}(\bar{R}(\xi,X)Y,\, N) = (\alpha+\beta)g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{4.7}$$
$$\bar{g}(\bar{R}(L,X)Y,\, L) = (\alpha+\beta)g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.8}$$

Substituting (4.5) into (1.1), for any $X,\, Y,\, Z,\, W \in \Gamma(TM)$, we have

$$\bar{g}(\bar{R}(X,Y)Z,\, W) = (\alpha+2\beta)\{g(Y,Z)g(X,W) - g(X,Z)(Y,W)\}. \tag{4.9}$$

Substituting (4.7), (4.8) and (4.9) into (2.9), we also have

$$\bar{Ric}(X,Y) = \{(m+2)\alpha + (2m+1)\beta\}g(X,Y). \tag{4.10}$$

Comparing (4.6) and (4.10), we obtain $\beta = 0$ as $m > 1$. As $\beta = 0$, from (3.2) we have $\alpha = 0$. Thus $\bar{M}$ is a flat manifold.

(2) In case ζ is parallel with respect to $\bar{\nabla}$, D is parallel and the lightlike transversal connection is flat: If ζ is tangent to M, then, by Case (1) of this theorem, we show that $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold. Thus we may assume that ζ is not tangent to M. In this case, we find $(\mu, \nu) \neq (0, 0)$.

(i) In case $\mu \neq 0$: As the lightlike transversal connection is flat, we get $d\tau = 0$ by Theorem 2.1. Thus, from (3.6), we have

$$\beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\} = 0, \quad \forall X, Y \in \Gamma(TM). \tag{4.11}$$

Replacing Y by ξ to (4.11), we have $\beta\theta(PX) = \beta\{\theta(X) - f\eta(X)\} = 0$. By Theorem 3.1, we set $\beta \neq 0$. As $\theta(PX) = 0$ for all $X \in \Gamma(TM)$, we get

$$\zeta = \lambda\xi + \mu N + \nu L. \tag{4.12}$$

Assume that ζ is parallel with respect to $\bar{\nabla}$. Applying $\bar{\nabla}_X$ to (3.9) and using (1.4)$\sim$(1.8), (1.12) and the facts $\tau = \rho = 0$ and $A_{_N}X = \gamma PX$, we show that the functions λ, μ and ν are constants and

$$\mu\gamma PX + \lambda A^*_\xi + \nu A_{_L} = 0.$$

As D is parallel and $\rho = 0$, from (1.1) and (2.4) we have

$$\nu\beta\{\theta(X)g(Y,Z) - \theta(Y)g(X,Z)\} = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing X by ξ, we get $\mu\nu\beta = 0$. As $\mu\nu$ is a non-zero constant and $\beta \neq 0$, we have $\mu\nu = 0$. As μ is a non-zero constant, we have $\nu = 0$. Thus we get

$$\zeta = \lambda\xi + \mu N, \qquad A^*_\xi X = \sigma PX, \quad \text{where} \quad \sigma = -2\mu^2\gamma. \tag{4.13}$$

As $\bar{g}(\zeta, \zeta) = 1$, we get $2\lambda\mu = 1$. Replacing X by ξ to (3.3), we get

$$\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + \lambda\mu\beta)g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.14}$$

Substituting (4.1) into (2.6) and using (2.3), the second equation of (4.13) and the fact that $\bar{g}(\bar{R}(X,Y)\xi, N) = 0$, we get

$$\bar{g}(\bar{R}(X,Y)Z, N) = \{X[\gamma] - \sigma\gamma\,\eta(X)\}g(Y,Z) - \{Y[\gamma] - \sigma\gamma\,\eta(Y)\}g(X,Z).$$

Replacing X by ξ and Z by X to this equation, we have

$$\bar{g}(\bar{R}(\xi, Y)X, N) = \{\xi[\gamma] - \sigma\gamma\}g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Comparing this and (4.14) and using $\sigma = -2\mu^2\gamma$ and $2\lambda\mu = 1$, we get

$$2\{\xi[\gamma] + 2\mu^2\gamma^2\} = 2\alpha + \beta. \tag{4.15}$$

Replacing W by ξ to (1.1) and using (2.2) and the fact $\tau = \phi = 0$, we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \mu\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}. \quad (4.16)$$

Applying ∇_X to $B(Y, Z) = -2\mu^2\gamma g(Y, Z)$ and using μ is a constant, we have

$$(\nabla_X B)(Y, Z) = -2\mu^2 X[\gamma]g(Y, Z) - 2\mu^2\gamma(\nabla_X g)(Y, Z).$$

Substituting this into (4.16) and using (1.9) and the fact $\mu \neq 0$, we have

$$\begin{aligned} &2\mu\{X[\gamma] + 2\mu^2\gamma^2\eta(X)\}g(Y, Z) - 2\mu\{Y[\gamma] + 2\mu^2\gamma^2\eta(Y)\}g(X, Z) \\ &= \beta\{\theta(Y)g(X, Z) - \theta(X)g(Y, Z)\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \end{aligned}$$

Replacing Y by ξ to this equation and using the fact $\theta(\xi) = \mu \neq 0$, we have

$$2\{\xi[\gamma] + 2\mu^2\gamma^2\} = -\beta. \quad (4.17)$$

From (4.15) and (4.17), we get $\alpha = -\beta$. Substituting this into (3.2), we get

$$\alpha(1 + \lambda^2 + \mu^2) = 0.$$

Thus we have $\alpha = 0$. Consequently we get $\beta = 0$ and $\bar{M}$ is a flat manifold.

(ii) In case $\mu = 0$: As $(\mu, \nu) \neq (0, 0)$, we have $\nu \neq 0$. By Theorem 3.1, we let $\beta \neq 0$. As D is parallel and $\rho = 0$, from (1.1) with $W = L$, we get

$$\theta(X)g(Y, Z) - \theta(Y)g(X, Z) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Using this and the fact $S(TM)$ is non-degenerate, we obtain

$$\theta(PX)PY = \theta(PY)PX, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $\theta(PX_o) \neq 0$, then $PX = fPX_o$ for any $X \in \Gamma(TM)$, where f is a smooth function. It is a contradiction as $\operatorname{rank} S(TM) > 1$. Thus we have $\theta(PX) = 0$ for all $X \in \Gamma(TM)$. From this and $\theta(\xi) = 0$, we have $\theta(X) = 0$ for any $X \in \Gamma(TM)$. Replacing X by ξ to (3.3), we get

$$\bar{g}(\bar{R}(\xi, Y)X,\, N) = \alpha g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \quad (4.18)$$

Substituting (4.1) into (2.7) and using (1.13) and (2.3) with $\rho = 0$, we get

$$\bar{g}(\bar{R}(X, Y)Z,\, N) = X[\gamma]g(Y, Z) - Y[\gamma]g(X, Z) + \gamma\{B(X, Z)\eta(Y) - B(Y, Z)\eta(X)\}.$$

Replacing X by ξ and Z by X to this equation, we have

$$\bar{g}(\bar{R}(\xi, Y)X,\, N) = \xi[\gamma]g(X, Y) - \gamma B(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Comparing this equation and (4.18), we obtain

$$\gamma B(X, Y) = (\xi[\gamma] - \alpha)g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Thus M is either lightlike totally umbilical or M is screen totally geodesic. In case M is lightlike totally umbilical, by the method of Case (1) of this theorem, we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold. In case M is screen totally geodesic, by Corollary 2, we have $\alpha = \beta = 0$ and $\bar{M}$ is a flat manifold.

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