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Boundedness of Local Minimizers of Anisotropic Scalar Integral Functionals with General Growth Conditions

Tiziano Granucci

Via Carlo Pisacane 16, Firenze, Italy

Dedicated to Fiorella

Abstract

In this paper we study the local boundedness of local minima of anisotropic scalar integral functionals of Calculus of Variations with general growth conditions.

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1. Introduction

In this paper we study the local boundedness of local minima of anisotropic scalar integral functionals of Calculus of Variations with general growth conditions.

Let us consider the open subset $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ we stady the local boundedness of local minimaof the following functional

$$J[u, \Omega] = \int_{\Omega} \sum_{i=1}^{N} \Phi_i \left(|\partial_{x_i} u| \right) \, d\mathcal{L}^N \tag{1.1}$$

where $\Phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are N-functions [Definition 4] and $\Phi_i \in \triangle_2$ [Definition 5], for $i = 1, ..., N$. Using Remark 2 we can suppose $\Phi_i \in \triangle_2^{m_i}$ with $m_i > 1$, for

$i = 1, ..., N$. The space on which the functional (1.1) is defined is the following

$$X = \left\{u \in W^{1,1}(\Omega) : J[u,\Omega] < +\infty\right\}, \tag{1.2}$$

Our first result is the following Caccioppoli's Inequality.

Theorem 1. *(Caccioppoli's Inequality) If u is a local minimum of (1.1) then there exist two positive real numebers $C_{1,Cacc}$ and R_0 such that for every $x_0 \in \Omega$, every ϱ, R with $0 < \varrho < R < \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}\right)$ and every $k \in \mathbb{R}$ we have*

$$\int\limits_{A(k,\varrho)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i} u|\right) d\mathcal{L}^N \leq C_{1,Cacc} \int\limits_{A(k,R)} \mathcal{H}\left(\frac{u-k}{R-\varrho}\right) d\mathcal{L}^N \tag{1.3}$$

where $\mathcal{H}(t) = \sum_{i=1}^{N} \Phi_i(t)$, for $t \geq 0$.

To get the boundedness of the local minimizers of (1.1) we need the fully anisotropic Sobolev inequality introduced in [7]. If $\Phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are N-functions we define

$$\bar{\Phi}^{-1}(t) = \left[\prod_{i=1}^{N} \Phi_i^{-1}(t)\right]^{\frac{1}{N}} \qquad \text{for } t \geq 0 \tag{1.4}$$

If we assume

$$\int_0^t \left(\frac{s}{\bar{\Phi}(s)}\right)^{1^*-1} ds < +\infty \qquad \text{for } t \geq 0, \tag{1.5}$$

we define

$$H(t) = \left[\int_0^t \left(\frac{s}{\bar{\Phi}(t)}\right)^{1^*-1} ds\right]^{\frac{1}{1^*}} \qquad \text{for } t \geq 0 \tag{1.6}$$

and

$$\bar{B}(t) = \bar{\Phi}\left(H^{-1}(t)\right) \qquad \text{for } t \geq 0 \tag{1.7}$$

where $1^* = \frac{N}{N-1}$.

Proposition 1. *If $\Phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are N-functions and $\Phi_i \in \nabla_2^{r_i} \cap \triangle_2^{m_i}$ with $1 \leq r_i < m_i < N$ for $i = 1, ..., N$; then $\bar{B} \in \nabla_2^{\bar{r}^*} \cap \triangle_2^{\bar{m}^*}$ where $\frac{1}{\bar{r}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}$, $\bar{r}^* = \frac{N\bar{r}}{N-\bar{r}}$, $\frac{1}{\bar{m}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{m_i}$ and $\bar{m}^* = \frac{N\bar{m}}{N-\bar{m}}$.*

Proof. Since $\Phi_i \in \nabla_2^{r_i} \cap \triangle_2^{m_i}$ with $1 \leq r_i < m_i < N$ for $i = 1, ..., N$; then by by Remark 3 and Proposition 2 $\lambda^{\frac{1}{m_i}} \Phi_i^{-1}(t) \leq \Phi_i^{-1}(\lambda t)$ and $\Phi_i^{-1}(\lambda t) \leq \lambda^{\frac{1}{r_i}} \Phi_i^{-1}(t)$ for every $\lambda > 1$, $t > 0$ and $i = 1, ..., N$. By (1.4) it follows

$$\lambda^{\frac{1}{\bar{m}}} \bar{\Phi}^{-1}(t) \leq \bar{\Phi}^{-1}(\lambda t) \leq \lambda^{\frac{1}{\bar{r}}} \bar{\Phi}^{-1}(t)$$

for every $\lambda > 1$, $t > 0$.where $\frac{1}{\bar{r}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}$, and $\frac{1}{\bar{m}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{m_i}$. Moreover by Remark 3 and Proposition 2 we get

$$\lambda^{\bar{r}}\bar{\Phi}(t) \leq \bar{\Phi}(\lambda t) \leq \lambda^{\bar{m}}\bar{\Phi}(t)$$

and

$$\frac{1}{\lambda^{\bar{m}}\bar{\Phi}(t)} \leq \frac{1}{\bar{\Phi}(\lambda t)} \leq \frac{1}{\lambda^{\bar{r}}\bar{\Phi}(t)}$$

for every $\lambda > 1$, $t > 0$.where $\frac{1}{\bar{r}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}$, and $\frac{1}{\bar{m}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{m_i}$. By (1.6) we get

$$H(\lambda t) = \left[\int_0^{\lambda t}\left(\frac{s}{\bar{\Phi}(t)}\right)^{1^*-1} ds\right]^{\frac{1}{1^*}}$$

if we put $s = \lambda\xi$ we have

$$\lambda^{\frac{N-\bar{m}}{N}} H(t) \leq H(\lambda t) \leq \lambda^{\frac{N-\bar{r}}{N}} H(t)$$

and

$$\lambda^{\frac{N}{N-\bar{r}}} H^{-1}(t) \leq H^{-1}(\lambda t) \leq \lambda^{\frac{N}{N-\bar{m}}} H^{-1}(t)$$

for every $\lambda > 1$, $t > 0$.where $\frac{1}{\bar{r}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}$, and $\frac{1}{\bar{m}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{m_i}$. Using (1.7) it follows $\bar{B} \in \nabla_2^{\bar{r}^*} \cap \triangle_2^{\bar{m}^*}$ where $\frac{1}{\bar{r}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}$, $\bar{r}^* = \frac{N\bar{r}}{N-\bar{r}}$, $\frac{1}{\bar{m}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{m_i}$ and $\bar{m}^* = \frac{N\bar{m}}{N-\bar{m}}$. □

The following theorem is given in [7].

Theorem 2. *Assume (1.5) then there exists a positive constant K, depending only on N, such that*

$$\int_{\mathbb{R}^N}\bar{B}\left(\frac{|u|}{K\left(\int_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}u|\right)\, d\mathcal{L}^N\right)^{\frac{1}{N}}}\right) d\mathcal{L}^N \leq \int_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}u|\right)\, d\mathcal{L}^N \quad (1.8)$$

for every $u \in X_0\left(\mathbb{R}^N\right)$, where $X_0\left(\mathbb{R}^N\right) = \left\{u \in W^{1,1}\left(\mathbb{R}^N\right) : J\left[u, \mathbb{R}^N\right] < +\infty\right\}$.

Besides we will make the following hypothesis.

H-1): $m_{\max} < \bar{r}^*$ and there exist two real positive constant s γ and c_2 such that $1 < \gamma < \frac{m_{\max}}{\bar{r}^*}$ and

$$(\mathcal{H}(t))^{\gamma} \leq c\bar{B}(t) \qquad \text{for } t \geq 0 \quad (1.9)$$

where $m_{\max} = \max\{m_1, ..., m_N\}$.

Theorem 3. *($L^{\mathcal{H}} - L^{\infty}$ inequality) Assume (1.5) and (H-1); if u is a local minimum of (1.1), then there exist two positive real numebers $\mathcal{C}_1$ and R_0 such that for every $x_0 \in \Omega$, every R with $0 < R < \min\left(R_0, \frac{d(x_0,\partial\Omega)}{2\sqrt{2^N}}\right)$ we have*

$$\sup_{Q_{\frac{R}{2}}}(u) \leq 2R\mathcal{H}^{-1}\left(\mathcal{C}_1 \int_{A(0,R)} \mathcal{H}\left(\frac{u}{R}\right) d\mathcal{L}^N\right) \tag{1.10}$$

where $\mathcal{H}(t) = \sum_{i=1}^{N} \Phi_i(t)$ for $t \geq 0$.

The class to which the precedent result can be applied it is very vast and it improves the results gotten in [11 - 14, 22 - 32, 35 - 39, 41, 42 and 44]. Not only, as shown in [23, 24 and 26 - 28] Theorem 1 and Theorem 3 are the starting points to get results of regularity for the local minimizers of the functional (1.1), such results are object of papers in preparation [30 and 31].

Theorem 3 is applied to the following examples:

$$J_{p_i}[u,\Omega] = \int_{\Omega} \sum_{i=1}^{N} (|\partial_{x_i} u| + 1)^{p_i} \, d\mathcal{L}^N \tag{1.11}$$

with $1 < p_i < \min\{N, \bar{p}^*\}$, where $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$.and $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$;

$$J_{p_i,q_i,}[u,\Omega] = \int_{\Omega} \sum_{i=1}^{N} (|\partial_{x_i} u| + 1)^{p_i} \ln^{q_i}(e + |\partial_{x_i} u|) \, d\mathcal{L}^N \tag{1.12}$$

with $1 < p_i < \min\{N, \bar{p}^*\}$, $q_i \geq 0$, $1 < p_i + \varepsilon_i q_i < \min\{N, \bar{p}^*\}$ and $0 < \varepsilon_i < \frac{\min\{N,\bar{p}^*\} - p_1}{q_i}$ for $i = 1, ..., N$ where $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$.and $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$;

$$J_{p_i,q_i,2}[u,\Omega] = \int_{\Omega} \sum_{i=1}^{N} (|\partial_{x_i} u| + 1)^{p_i} \ln^{q_i}(e + |\partial_{x_i} u|) \, d\mathcal{L}^N \tag{1.13}$$

with $1 \leq p_i < \min\{N, \bar{p}^*\}$, $q_i > 0$, $1 < p_i + \varepsilon_i q_i < \min\{N, \bar{p}^*\}$ and $0 < \varepsilon_i < \frac{\min\{N,\bar{p}^*\} - p_1}{q_i}$ for $i = 1, ..., N$ where $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$.and $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$;

$$J_{1,q_i}[u,\Omega] = \int_{\Omega} \sum_{i=1}^{N} (|\partial_{x_i} u| + 1) \ln^{q_i}(e + |\partial_{x_i} u|) \, d\mathcal{L}^N \tag{1.14}$$

with $1 = p_i$, $q_i > 0$, $1 < 1 + \varepsilon_i q_i < \frac{N}{N-1}$ and $0 < \varepsilon_i < \frac{1}{(N-1)q_i}$ for $i = 1, ..., N$.

2. Definitions

Definition 1. *A continuous and convex function* $\Phi : [0, +\infty) \to [0, +\infty)$ *is called N-function (or Young function) if it satisfies*

$$\begin{aligned} &\Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ if } t > 0; \\ &\lim_{t \to o^+} \tfrac{\Phi(t)}{t} = 0; \\ &\lim_{t \to +\infty} \tfrac{\Phi(t)}{t} = +\infty. \end{aligned} \tag{2.1}$$

For exemple the function $\Phi_{p,\beta}(t) = t^p \ln^\beta (1+t)$, for $p > 1$ and $\beta \geq 0$ or $p = 1$ and $\beta > 0$, is a N-function.

Actually, only the growth at infinity really matters in the definition of N-function.

Indeed, given a continuous and convex function $A : [0, +\infty) \to [0, +\infty)$ satisfying

$$\lim_{t \to +\infty} \frac{A(t)}{t} = +\infty$$

there exist a N-function Φ and $t_0 > 0$ such that for every $t > t_0$ there holds

$$A(t) = \Phi(t).$$

The function A is called principal part of the N-function Φ. For exemple there exists a N-function Φ such that $\Phi(t) = t^{\ln(t)}$ near infinity or there exists a N-function Φ such that $\Phi(t) = t \ln(t)$ near infinity.

Definition 2. *If* Φ_1 *and* Φ_2 *are two N-functions we say that* Φ_1 *dominates* Φ_2 *near infinity if there exists positive constants* $\varkappa$ *and* t_0 *such that*

$$\Phi_2(t) \leq \Phi_1(\varkappa t)$$

for all $t \geq t_0$.

Definition 3. *If* Φ_1 *and* Φ_2 *are two N-functions we say that* Φ_1 *and* Φ_2 *are equivalent near infinity (*$\Phi_1 \sim \Phi_2$*) if and only if there exists positive constants* $\varkappa_1$, $\varkappa_2$ *and* t_0 *such that*

$$\Phi_1(\varkappa_1 t) \leq \Phi_2(t) \leq \Phi_1(\varkappa_2 t)$$

for all $t \geq t_0$.

Remark 1. *If* $0 < \lim\limits_{t \to +\infty} \frac{\Phi_1(t)}{\Phi_2(t)} < +\infty$ *then* Φ_1 *and* Φ_2 *are equivalent near infinity. Let us introduce two important classes of N-functions.*

Definition 4. *A N-function* Φ *is of class* $\triangle_2$ *globally in* $(0, +\infty)$ *if exists* $k > 1$ *such that*

$$\Phi(2t) \leq k\Phi(t) \qquad \forall t \in (0, +\infty). \tag{2.2}$$

Definition 5. *A N-function* Φ *is of class* $\triangle_2^m$ *globally in* $(0, +\infty)$*, with* $m \geq 1$*, if for every* $\lambda > 1$

$$\Phi(\lambda t) \leq \lambda^m \Phi(t) \qquad \forall t \in (0, +\infty). \tag{2.3}$$

Definition 6. *A N-function Φ is of class ∇_2 globally in $(0,+\infty)$ if exists $l>1$ such that*

$$\Phi(t) \le \frac{\Phi(lt)}{2l} \qquad \forall t \in (0,+\infty). \tag{2.4}$$

Definition 7. *A N-function Φ is of class ∇_2^r globally in $(0,+\infty)$, with $r \ge 1$, if for every $\lambda > 1$*

$$\lambda^r \Phi(t) \le \Phi(\lambda t) \qquad \forall t \in (0,+\infty). \tag{2.5}$$

The N-functions $\Phi \in \triangle_2^m$ are characterized by the following result.

Lemma 1. *Let Φ be a N-function and let $\dot{\Phi}_+$ be its right derivative. For $m > 1$ the following properties are equivalent:*

- **(i):** $\Phi(\lambda t) \le \lambda^m \Phi(t)$, *for every $t \ge 0$, for every $\lambda > 1$;*
- **(ii):** $t\dot{\Phi}_+(t) \le m\Phi(t)$, *for every $t \ge 0$;*
- **(iii):** *the function $\frac{\Phi(t)}{t^m}$ is nonincreasing on $(0,+\infty)$.*

Proof. Refer to [33] and [43]. □

The N-functions $\Phi \in \nabla_2^r$ are characterized by the following result.

Lemma 2. *Let Φ be a N-function and let $\dot{\Phi}_-$ be its left derivative. For $r > 1$ the following properties are equivalent:*

- **(i)':** $\Phi(\lambda t) \ge \lambda^r \Phi(t)$, *for every $t \ge 0$, for every $\lambda > 1$;*
- **(ii)':** $t\dot{\Phi}_-(t) \ge r\Phi(t)$, *for every $t \ge 0$;*
- **(iii)':** *the function $\frac{\Phi(t)}{t^r}$ is nondecreasing on $(0,+\infty)$.*

Proof. Refer to [33] and [43]. □

Remark 2. *We observe that*

$$\Delta_2 = \bigcup_{m>1} \triangle_2^m$$

and

$$\nabla_2 = \bigcup_{r>1} \nabla_2^r.$$

Remark 3. *If Φ is a N-function of class $\triangle_2^m$ globally in $(0,+\infty)$, then we have $\Phi(\lambda t) \le \lambda^m \Phi(t)$ for every $t \in (0,+\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^m} \le \Phi\left(\frac{s}{\lambda}\right)$ and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^m}\right) \le \frac{s}{\lambda}$ for every $s \in (0,+\infty)$ and $\lambda > 1$. Let us put $s = \Phi^{-1}(w)$ then we have $\Phi^{-1}\left(\frac{w}{\lambda^m}\right) \le \frac{\Phi^{-1}(w)}{\lambda}$ for every $w \in (0,+\infty)$ and $\lambda > 1$. Let us put $\frac{1}{\lambda^m} = a$ then we have $\Phi^{-1}(aw) \le a^{\frac{1}{m}} \Phi^{-1}(w)$ for every $w \in (0,+\infty)$ and $a \in (0,1)$. Moreover if $\Phi(\lambda t) \le \lambda^m \Phi(t)$ for every $t \in (0,+\infty)$ and $\lambda > 1$ we get $\lambda t \le \Phi^{-1}(\lambda^m \Phi(t))$ then if $t = \Phi^{-1}(s)$ it follows $\lambda \Phi^{-1}(s) \le \Phi^{-1}(\lambda^m s)$ for every $s \in (0,+\infty)$ and $\lambda > 1$. If Φ is a N-function of class ∇_2^r globally in $(0,+\infty)$, then we have $\lambda^r \Phi(t) \le \Phi(\lambda t)$ for every $t \in (0,+\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^r} \ge \Phi\left(\frac{s}{\lambda}\right)$*

and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^r}\right) \geq \frac{s}{\lambda}$ *for every* $s \in (0,+\infty)$ *and* $\lambda > 1$*. Let us put* $s = \Phi^{-1}(w)$ *then we have* $\Phi^{-1}\left(\frac{w}{\lambda^r}\right) \geq \frac{\Phi^{-1}(w)}{\lambda}$ *for every* $w \in (0,+\infty)$ *and* $\lambda > 1$*. Let us put* $\frac{1}{\lambda^r} = a$ *then we have* $\Phi^{-1}(aw) \geq a^{\frac{1}{r}}\Phi^{-1}(w)$ *for every* $w \in (0,+\infty)$ *and* $a \in (0,1)$*. Moreover if* $\lambda^r \Phi(t) \leq \Phi(\lambda t)$ *for every* $t \in (0,+\infty)$ *and* $\lambda > 1$ *we get* $\Phi^{-1}(\lambda^r \Phi(t)) \leq \lambda t$ *then if* $t = \Phi^{-1}(s)$ *it follows* $\Phi^{-1}(\lambda^r s) \leq \lambda \Phi^{-1}(s)$ *for every* $s \in (0,+\infty)$ *and* $\lambda > 1$*.*

Proposition 2. $\Phi \in \nabla_2^r \cap \triangle_2^m$ *with* $1 \leq r \leq m$ *if and only if* $a^{\frac{1}{r}}\Phi^{-1}(t) \leq \Phi^{-1}(at)$*.and* $\Phi^{-1}(at) \leq a^{\frac{1}{m}}\Phi^{-1}(t)$ *for every* $0 < a < 1$, $t > 0$*.*

Proof. It follows using Lemma 1, Lemma 2 and Remark 3. □

Proposition 3. $\Phi \in \nabla_2^r \cap \triangle_2^m$ *with* $1 \leq r \leq m$ *if and only if* $\Phi^{-1}(\lambda t) \leq \lambda^{\frac{1}{r}}\Phi^{-1}(t)$*.and* $\lambda^{\frac{1}{m}}\Phi^{-1}(t) \leq \Phi^{-1}(\lambda t)$ *for every* $\lambda > 1$, $t > 0$*.*

Proof. It follows using Lemma 1, Lemma 2 and Remark 3. □

Remark 4. *Let* $\dot{\Phi}$ *be the weak derivative of* Φ *then it follows that*

$$\dot{\Phi}(t) \leq \dot{\Phi}_+(t)$$

for every $t > 0$*. Moreover, using Lemma 1 (ii) and Lemma 3 of [27] we get*

$$\begin{aligned} a\dot{\Phi}(b) &\leq a\dot{\Phi}(a) + b\dot{\Phi}(b) \\ &\leq a\dot{\Phi}_+(a) + b\dot{\Phi}_+(b) \\ &\leq m\left[\Phi(a) + \Phi(b)\right] \end{aligned}$$

for every $a, b > 0$*.*

Definition 8. *We say that the N-function* Φ *satisfies the* $\triangle'$*- condition if there exist positive constants* c_9 *and* t_0 *such that*

$$\Phi(ts) \leq c_9 \Phi(t)\Phi(s) \tag{2.6}$$

for every $t, s \geq t_0$*. If* $t_0 = 0$ *we say that* Φ *satisfies globally the* $\triangle'$*- condition (*$\Phi \in \triangle'$ *in* $(0,+\infty)$*).*

The functions $\Phi \in \triangle_2 \cap \nabla_2$ and the functions $\Phi \in \triangle'$ in $(0,+\infty)$ are somehow almost-homogeneous, this ownership has been used in [23, 26, 39 and 41]. Let us consider the N-functions

$$\begin{array}{ll} \Phi_1(t) = t^p & \text{with } p > 1; \\ \Phi_2(t) = t^p(|\ln(t)| + 1) & \text{with } p > 1; \\ \Phi_3(t) = (1+t)\ln(1+t) - t; & \\ \Phi_4(t) = \frac{t^2}{1+\ln(1+t)}; & \\ \Phi_5(t) = e^t - t - 1. & \end{array} \tag{2.7}$$

We observe that Φ_1 and Φ_2 satisfy the $\triangle'$-condition globally in $[0,+\infty)$; moreover Φ_1 and Φ_2 belong to the class $\triangle_2 \cap \nabla_2$ globally in $[0,+\infty)$. The function Φ_3 satisfy $\triangle'$-condition for all $t \geq t_0$ but $\Phi_3 \notin \nabla_2$. The function $\Phi_5 \in \nabla_2$ but $\Phi_5 \notin \triangle_2$. Finally $\Phi_4 \in \triangle_2 \cap \nabla_2$ but $\Phi_4 \notin \triangle'$. For further details refer to [1], [4 - 7], [10 - 12], [15 - 17], [22 - 41] and [43 - 44]. Now we can introduce Orlicz spaces and Orlicz Sobolev Spaces, L^{Φ} and W^1L^{Φ}. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and open set, the Orlicz class $K^{\Phi}(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ (equivalence classes modulo equality $\mathcal{L}^N$ a.e. in Ω) satisfying $\int_{\Omega} \Phi(|u|)\, d\mathcal{L}^N < +\infty$. The Orlicz space $L^{\Phi}(\Omega)$ is defined to be the linear hull of $K^{\Phi}(\Omega)$, thus it consists of all measurable functions u such that $\lambda u \in K^{\Phi}(\Omega)$ for some $\lambda > 0$. Moreover, the equality $K^{\Phi}(\Omega) \equiv L^{\Phi}(\Omega)$ holds if and only if $\Phi \in \triangle_2$.

Definition 9. *If $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\Phi \in \triangle_2$ then we define*

$$W^1L^{\Phi}(\Omega) = \left\{u \in L^{\Phi}(\Omega) : \partial_i u \in L^{\Phi}(\Omega) \text{ for } i = 1, ..., N\right\}$$

where $\partial_i u$ are the weak derivatives of u for $i = 1, ..., N$.

Theorem 4. *Let $\Phi \in \triangle_2$, then $L^{\Phi}(\Omega)$ and $W^1L^{\Phi}(\Omega)$ are Banach spaces with the following norms*

$$\|u\|_{\Phi,\Omega} = \inf \left(k > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{k}\right) d\mathcal{L}^N \leq 1 \right)$$

and

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \sum_{i=1}^{N} \|\partial_i u\|_{\Phi,\Omega} .$$

We observe that if $\Phi(t) = t^p$, with $p > 1$, then $\|u\|_{\Phi,\Omega} = \|u\|_{p,\Omega}$, where $\|u\|_{p,\Omega} = \int_{\Omega} |u|^p\, d\mathcal{L}^N$. In general, however, so simple relationships don't be had among the Luxemburg norm $\|u\|_{\Phi,\Omega}$ and the integral $\int_{\Omega} \Phi(|u|)\, d\mathcal{L}^N$, this creates some difficulties to use the Luxemburg norm and the Hölder inequality especially in the Decay Lemma [see Lemma 12 of [28]] where we are forced to introduce some suitable tricks to proceed. For greater details on Orlicz spaces, Orlicz-Sobolev spaces and Luxemburg norm we refer to [1], [4 - 7], [10 - 12], [15 - 17], [22 - 41] and [43 - 44]. Let Φ be a N-function then there exists a real valued function p defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right continuous on $(0, +\infty)$ such that

$$\Phi(t) = \int_0^t p(s)\, ds \qquad \text{for every } t \in (0, +\infty)$$

and

$$\dot{\Phi}_+(t) = p(t) \qquad \text{a.e. on } (0, +\infty) .$$

Definition 10. *Let p be a real valued function defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right* *continuous on* $(0,+\infty)$*. We define*

$$\widetilde{p}\,(s)=\sup_{p(t)\leq s}\,(t)$$

33 and 43and

$$\widetilde{\Phi}\,(t)=\int\limits_0^t\widetilde{p}\,(s)\;ds.$$

The N-functions Φ *and* $\widetilde{\Phi}$ *are complementary N-functoins.*

Particularly from the relationship (2.1) of the Definition 2 we get the following Young inequality

$$ab\leq\widetilde{\Phi}\,(a)+\Phi\,(b) \tag{2.8}$$

and

$$\widetilde{\Phi}\left(\frac{\Phi\,(b)}{b}\right)<\Phi\,(b) \tag{2.9}$$

for every $a,b>0$. Moreover we have the following Hölder Inequality

$$\int\limits_{\Omega}u\,(x)\,v\,(x)\;dx\leq\|u\|_{\Phi,\Omega}\,\|v\|_{\tilde{\Phi},\Omega} \tag{2.10}$$

for every $u\in L^{\Phi}\,(\Omega)$ and every $v\in L^{\tilde{\Phi}}\,(\Omega)$, [for dettails refer to 1, 33 and 43].

Proposition 4. *If* Φ *is a N-function and* $\Phi\in\triangle_2$ *then* $\tilde{\Phi}\in\nabla_2$*.*

Proof. Refer to [33] and [43]. □

Proposition 5. *If* Φ *is a N-functionand* $\Phi\in\nabla_2$ *then* $\tilde{\Phi}\in\triangle_2$*.*

Proof. Refer to [33] and [43]. □

Corollary 1. *If* Φ *is a N-function then* $\Phi\in\triangle_2\cap\nabla_2$ *if and only if* $\tilde{\Phi}\in\triangle_2\cap\nabla_2$*.*

Proof. Refer to [33] and [43]. □

Proposition 6. *If* Φ *and d* Φ_1 *are two N-functions, if* $\Phi\sim\Phi_1$ *and* $\Phi\in\triangle_2$ *then* $\Phi_1\in\triangle_2$*.*

Proof. Refer to [33] and [43]. □

3. Caccioppoli Inequalities

In order to proof the Caccioppoli inquality (1.3) we need the following Lemma.

Lemma 3. *Let* $\digamma\,(k,\cdot)$ *be a nonnegative bounded function defined in* $[\tau_0,\tau_1]$, $\tau_0\geq 0$*. Suppose that for all* t*,* s *with* $\tau_0\leq t<s\leq\tau_1$ *we have*

$$\digamma\,(k,t)\leq\theta\digamma\,(k,s)+A\int\limits_{A(k,s)}\Phi\left(\frac{u-k}{s-t}\right)\,dx+B$$

where A, B, θ are nonnegative constants, $0 \le \theta < 1$, Φ is a N-function and $\Phi \in \triangle_2^m$ with $m > 1$. Then for all ϱ, R, $\tau_0 \le \varrho < R \le \tau_1$ we have

$$F(k,\varrho) \le c\left[\int_{A(k,R)} \Phi\left(\frac{u-k}{R-\varrho}\right)dx + B\right]$$

where c is a constant depending only on θ and m.

Proof. See Lemma 9 of [27]. □

Lemmas 3 generalizes the Lemma 6.1 of [21]. If u is a local minimum of (1.1) then

$$\int_\Omega \sum_{i=1}^N \dot\Phi_i\left(|\partial_{x_i}u|\right)\frac{\partial_{x_i}u\cdot\partial_{x_i}\varphi}{|\partial_{x_i}u|}\,d\mathcal{L}^N = 0 \tag{3.1}$$

for every $\varphi \in X$ and $\mathrm{supp}(\varphi) \subset \Omega$. Let us choose $\varphi = \eta\varpi$, where $\eta \in C_0^\infty\left(Q_{\frac{s+t}{2}}\right)$.and $\eta = 1$ on Q_s and $|\nabla\eta| \le \frac{c}{t-s}$.on $Q_{\frac{s+t}{2}}$, then we get

$$\begin{aligned}
0 \le I &\le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N \dot\Phi_i\left(|\partial_{x_i}u|\right)|\partial_{x_i}\eta|\,(u-k)\;d\mathcal{L}^N\\
&\le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N \dot\Phi_i\left(|\partial_{x_i}u|\right)|\nabla\eta|\,(u-k)\;d\mathcal{L}^N\\
&\le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N \dot\Phi_i\left(|\partial_{x_i}u|\right)\tfrac{(u-k)}{t-s}\,d\mathcal{L}^N\\
&\le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N m_i\varepsilon_i\left[\tfrac{\dot\Phi_i\left(|\partial_{x_i}u|\right)}{m_i}\tfrac{(u-k)}{\varepsilon_i(t-s)}\right]d\mathcal{L}^N\\
&\le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N m_i\varepsilon_i\tilde\Phi_i\left(\tfrac{\dot\Phi_i\left(|\partial_{x_i}u|\right)}{m_i}\right)d\mathcal{L}^N\\
&+ \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^N m_i\varepsilon_i\Phi_i\left(\tfrac{(u-k)}{\varepsilon_i(t-s)}\right)d\mathcal{L}^N
\end{aligned} \tag{3.2}$$

where

$$I = \int_{A(k,s)} \sum_{i=1}^N \dot\Phi_i\left(|\partial_{x_i}u|\right)|\partial_{x_i}u|\,\eta\,d\mathcal{L}^N. \tag{3.3}$$

Using lemma 1 (i) it foollows

$$\frac{\dot\Phi_i\left(|\partial_{x_i}u|\right)}{m_i} \le \frac{\dot\Phi_i\left(|\partial_{x_i}u|\right)|\partial_{x_i}u|}{|\partial_{x_i}u|\,m_i} \le \frac{\Phi_i\left(|\partial_{x_i}u|\right)}{|\partial_{x_i}u|} \tag{3.4}$$

for $i = 1, ..., N$, by (2.9) and (3.4) we get

$$\tilde\Phi_i\left(\frac{\dot\Phi_i\left(|\partial_{x_i}u|\right)}{m_i}\right) \le \Phi_i\left(|\partial_{x_i}u|\right). \tag{3.5}$$

Using (3.3), (2.8) and (3.5) we have

$$II \le \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^{N} m_i\varepsilon_i \left[\Phi_i\left(|\partial_{x_i}u|\right) + \Phi_i\left(\tfrac{(u-k)}{\varepsilon_i(t-s)}\right)\right] d\mathcal{L}^N \tag{3.6}$$

where

$$II = \int_{A(k,s)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \tag{3.7}$$

Moreover it follows

$$\begin{aligned} III \;&\le \sum_{i=1}^{N} m_i\varepsilon_i \int_{A(k,t)} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \\ &+ \sum_{i=1}^{N} m_i\varepsilon_i \int_{A(k,t)\setminus A(k,s)} \Phi_i\left(\tfrac{(u-k)}{\varepsilon_i(t-s)}\right) d\mathcal{L}^N \end{aligned} \tag{3.8}$$

where

$$III = \sum_{i=1}^{N}\left[(1+m_i\varepsilon_i)\int_{A(k,s)} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N\right] \tag{3.9}$$

Now we can choose $\varepsilon_i = \frac{m_N}{m_i}\varepsilon_N$ and we get

$$\begin{aligned} (1+m_N\varepsilon_N)\int_{A(k,s)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \;&\le m_N\varepsilon_N \int_{A(k,t)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \\ &+ m_N\varepsilon_N \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^{N} \Phi_i\left(\tfrac{(u-k)}{\frac{m_N}{m_i}\varepsilon_N(t-s)}\right) d\mathcal{L}^N \end{aligned} \tag{3.10}$$

If we choose $\varepsilon_N = m_{\max} = \max(m_1, ..., m_N)$ we have

$$\begin{aligned} \int_{A(k,s)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \;&\le \tfrac{m_N\varepsilon_N}{(1+m_N\varepsilon_N)} \int_{A(k,t)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \\ &+ \tfrac{m_N\varepsilon_N}{(1+m_N\varepsilon_N)} \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^{N} \Phi_i\left(\tfrac{(u-k)}{m_N(t-s)}\right) d\mathcal{L}^N \end{aligned} \tag{3.11}$$

and, since $m_N > 1$, it follows

$$\begin{aligned} \int_{A(k,s)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \;&\le \tfrac{m_N\varepsilon_N}{(1+m_N\varepsilon_N)} \int_{A(k,t)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \\ &+ \tfrac{m_N\varepsilon_N}{(1+m_N\varepsilon_N)} \int_{A(k,t)\setminus A(k,s)} \sum_{i=1}^{N} \Phi_i\left(\tfrac{(u-k)}{(t-s)}\right) d\mathcal{L}^N \end{aligned} \tag{3.12}$$

Using Lemma 3 it follows

$$\int_{A(k,\varrho)} \sum_{i=1}^{N} \Phi_i\left(|\partial_{x_i}u|\right) d\mathcal{L}^N \le C_{1,Cacc} \int_{A(k,R)} \sum_{i=1}^{N} \Phi_i\left(\tfrac{(u-k)}{(t-s)}\right) d\mathcal{L}^N \tag{3.13}$$

Particolary we get the following Caccioppoli inequality

$$\int\limits_{A\left(k,\frac{R}{2}\right)} \sum_{i=1}^{N} \Phi_i\left(\left|\partial_{x_i} u\right|\right)\, d\mathcal{L}^N \quad \leq C_{1,Cacc} \int\limits_{A(k,R)} \mathcal{H}\left(\tfrac{u-k}{R}\right)\, d\mathcal{L}^N \tag{3.14}$$

where $\mathcal{H}(t) = \sum\limits_{i=1}^{N} \Phi_i(t)$, for $t \geq 0$.

Let ϖ_R be the function defined by

$$\varpi_R(y) = \frac{u(Ry)}{R} \tag{3.15}$$

then we have the following Caccioppoli inequality

$$\int\limits_{A\left(\tilde{k},\tau\right)} \sum_{i=1}^{N} \Phi_i\left(\left|\partial_{x_i} \varpi_R\right|\right)\, d\mathcal{L}^N \quad \leq C_{2,Cacc} \int\limits_{A\left(\tilde{k},\sigma\right)} \mathcal{H}\left(\frac{\varpi_R-\tilde{k}}{\sigma-\tau}\right)\, d\mathcal{L}^N \tag{3.16}$$

for $\tilde{k} = \frac{k}{R}$ and $\frac{1}{2} \leq \tau < \sigma \leq 1$.

4. $L^{\mathcal{H}} - L^{\infty}$ inequality

In this paragraph we will show the Theorem 3. Particularly we need the following Lemma.

Lemma 4. *Let both $\lambda > 0$ and $\{x_i\}_{i\in\mathbb{N}}$ a set of positive real numbers, such that*

$$x_{i+1} \leq CB^i x_i^{1+\lambda} \tag{4.1}$$

with $C > 0$ and $B > 1$. Then, if $x_0 \leq C^{-\frac{1}{\lambda}} B^{-\frac{1}{\lambda^2}}$, we have

$$x_i \leq B^{-\frac{i}{\lambda}} x_0 \tag{4.2}$$

and consequently, in particular, we have

$$\lim_{i\to+\infty} x_i = 0. \tag{4.3}$$

Proof. Refer to Lemma 7.1 of [21]. □

Let us consider the function $\mathcal{H}$ defined by

$$\mathcal{H}(s) = \sum_{i=1}^{N} \Phi_i(s) \tag{4.4}$$

for $s > 0$ then

$$\begin{array}{ll} \mathcal{H}(s) & \text{is a convex function} \\ \lim\limits_{s\to+\infty} \frac{\mathcal{H}(s)}{s} = +\infty & \\ \lim\limits_{s\to 0^+} \frac{\mathcal{H}(s)}{s} = 0 & \end{array} \tag{4.5}$$

by Definition 1 $\mathcal{H}(s)$ is a N-function.

Remark 5. *Moreover, if* $\lambda > 1$ *and* $s > 0$ *it follows*

$$\begin{aligned} \mathcal{H}(\lambda s) &= \sum_{i=1}^{N} \Phi_i(\lambda s) \\ &\leq \sum_{i=1}^{N} \lambda^{m_i} \Phi_i(\lambda s) \\ &\leq \lambda^{m_{\max}} \mathcal{H}(s) \end{aligned} \tag{4.6}$$

where $m_{\max} = \max(m_1, ..., m_N)$, *then* $\mathcal{H}(s) \in \triangle_2^{m_{amx}}$.

Let us consider $\int\limits_{A(\tilde{k},\tau)} \mathcal{H}(w)\, d\mathcal{L}^N$ where $w = \eta\left(\varpi_R - \tilde{k}\right)_+$, $\eta \in C_0^\infty\left(Q_{\frac{\varrho+\tau}{2}}\right)$.and $\eta = 1$ on Q_τ and $|\nabla\eta| \leq \frac{c}{\varrho-\tau}$.on $Q_{\frac{\varrho+t}{2}}$, then we get

$$\int\limits_{A(\tilde{k},\tau)} \mathcal{H}(w)\, d\mathcal{L}^N \leq \left|A\left(\tilde{k},\tau\right)\right|^{1-\frac{1}{\gamma}} \mathcal{I}_\gamma \tag{4.7}$$

where $\mathcal{I}_\gamma = \left[\int\limits_{A(\tilde{k},\tau)} [\mathcal{H}(w)]^\gamma\, d\mathcal{L}^N\right]^{\frac{1}{\gamma}}$.

Using (H-1) it follows

$$\left[\int\limits_{A(\tilde{k},\tau)} [\mathcal{H}(w)]^\gamma\, d\mathcal{L}^N\right]^{\frac{1}{\gamma}} \leq c\left[\int\limits_{A(\tilde{k},\tau)} \bar{B}(w)\, d\mathcal{L}^N\right]^{\frac{1}{\gamma}} \tag{4.8}$$

Since $\int\limits_{A(\tilde{k},\tau)} \bar{B}(w)\, d\mathcal{L}^N = \int\limits_{\mathbb{R}^N} \bar{B}(w)\, d\mathcal{L}^N$ we have $\int\limits_{\mathbb{R}^N} \bar{B}(w)\, d\mathcal{L}^N \leq c^a\theta \int\limits_{\mathbb{R}^N} \bar{B}(\frac{w}{a})\, d\mathcal{L}^N$ for every $a \in (0,1)$ and $0 < \theta \leq 1$. Since

$$\int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)} \sum_{i=1}^{N} \Phi_i(|\partial_{x_i} w|)\, d\mathcal{L}^N = \int\limits_{\mathbb{R}^N} \sum_{i=1}^{N} \Phi_i(|\partial_{x_i} w|)\, d\mathcal{L}^N \tag{4.9}$$

there exists $R_0 > 0$ such that

$$K^N \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)} \sum_{i=1}^{N} \Phi_i(|\partial_{x_i} w|)\, d\mathcal{L}^N < K^N \int\limits_{Q_{R_0}} \sum_{i=1}^{N} \Phi_i(|\partial_{x_i} w|)\, d\mathcal{L}^N < 1 \tag{4.10}$$

then if we choose $a=\left[K^N\int\limits_{\mathbb{R}^N}\sum\limits_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N\right]^{\frac{1}{N}}\in(0,1)$ we have

$$\int\limits_{\mathbb{R}^N}\bar{B}(w)\,d\mathcal{L}^N\le cK\left[\int\limits_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N\right]^{\frac{\theta}{N}}\int\limits_{\mathbb{R}^N}\bar{B}\Big(\frac{w}{K\left[\int\limits_{\mathbb{R}^N}\sum\limits_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N\right]^{\frac{1}{N}}}\Big)\,d\mathcal{L}^N. \tag{4.11}$$

Usin the fully anisotropic Sobolev inequality (1.8) it follows

$$\int\limits_{\mathbb{R}^N}\bar{B}(w)\,d\mathcal{L}^N\le cK\left[\int\limits_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N\right]^{\frac{\theta}{N}+1}. \tag{4.12}$$

Since $\eta\left(\varpi_R-\tilde{k}\right)_+$ and $|\partial_{x_i}w|\le|\partial_{x_i}\eta|\left(\varpi_R-\tilde{k}\right)_++\eta\,|\partial_{x_i}\varpi_R|$ on $A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)$ we get

$$\begin{aligned}\int\limits_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N &= \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N\\
&\le \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}\eta|\left(\varpi_R-\tilde{k}\right)_++\eta\,|\partial_{x_i}\varpi_R|\right)\,d\mathcal{L}^N\\
&\le \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(|\nabla\eta|\left(\varpi_R-\tilde{k}\right)\right)+\sum_{i=1}^{N}\Phi_i\left(\eta\,|\partial_{x_i}\varpi_R|\right)\,d\mathcal{L}^N\\
&\le \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(\frac{c\left(\varpi_R-\tilde{k}\right)}{\varrho-\tau}\right)\,d\mathcal{L}^N\\
&+ \int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}\varpi_R|\right)\,d\mathcal{L}^N\end{aligned} \tag{4.13}$$

using Caccioppoli inequality (1.3) it follows

$$\begin{aligned}\int\limits_{\mathbb{R}^N}\sum_{i=1}^{N}\Phi_i\left(|\partial_{x_i}w|\right)\,d\mathcal{L}^N &\le c\int\limits_{A\left(\tilde{k},\frac{\varrho+\tau}{2}\right)}\sum_{i=1}^{N}\Phi_i\left(\frac{\left(\varpi_R-\tilde{k}\right)}{\varrho-\tau}\right)\,d\mathcal{L}^N\\
&+C_{Cacc}\int\limits_{A\left(\tilde{k},\varrho\right)}\sum_{i=1}^{N}\Phi_i\left(\frac{\left(\varpi_R-\tilde{k}\right)}{\varrho-\tau}\right)\,d\mathcal{L}^N\\
&\le (c+C_{Cacc})\int\limits_{A\left(\tilde{k},\varrho\right)}\mathcal{H}\left(\frac{\left(\varpi_R-\tilde{k}\right)}{\varrho-\tau}\right)\,d\mathcal{L}^N\end{aligned} \tag{4.14}$$

Using (4.7), (4.13) and (4.14) it follows

$$\int_{A(\tilde{k},\tau)} \mathcal{H}\left(\varpi_R-\tilde{k}\right) d\mathcal{L}^N \leq \left|A\left(\tilde{k},\tau\right)\right|^{1-\frac{1}{\gamma}} \left[cK\left[(c+C_{Cacc})\int_{A(\tilde{h},\varrho)} \mathcal{H}\left(\frac{\left(\varpi_R-\tilde{h}\right)}{\varrho-\tau}\right) d\mathcal{L}^N\right]^{\frac{\theta}{N}+1}\right]^{\frac{1}{\gamma}} \tag{4.15}$$

where $\tilde{k}>\tilde{h}$. Using (4.6), (2.3) and (4.15) we get

$$\int_{A(\tilde{k},\tau)} \mathcal{H}\left(\varpi_R-\tilde{k}\right) d\mathcal{L}^N \leq \frac{\mathcal{C}_0\left|A\left(\tilde{k},\tau\right)\right|^{1-\frac{1}{\gamma}}}{(\varrho-\tau)^{\left(\frac{\theta}{N}+1\right)\frac{m_{\max}}{\gamma}}} \left[\int_{A(\tilde{h},\varrho)} \mathcal{H}\left(\frac{\left(\varpi_R-\tilde{h}\right)}{\varrho-\tau}\right) d\mathcal{L}^N\right]^{\left(\frac{\theta}{N}+1\right)\frac{1}{\gamma}} \tag{4.16}$$

where $\mathcal{C}_0=(cK)^{\frac{1}{\gamma}}(c+C_{Cacc})^{\left(\frac{\theta}{N}+1\right)\frac{1}{\gamma}}$. If $\tilde{k}>\tilde{h}$ then it follows

$$\left|A\left(\tilde{k},\tau\right)\right| \leq \frac{1}{\mathcal{H}\left(\tilde{k}-\tilde{h}\right)}\int_{A(\tilde{h},\tau)} \mathcal{H}\left(\varpi_R-\tilde{h}\right) d\mathcal{L}^N \tag{4.17}$$

and

$$\int_{A(\tilde{k},\sigma)} \mathcal{H}\left(\varpi_R-\tilde{k}\right) d\mathcal{L}^N \leq \int_{A(\tilde{h},\sigma)} \mathcal{H}\left(\varpi_R-\tilde{h}\right) d\mathcal{L}^N \tag{4.18}$$

Using (4.17), (4.18) and (4.16) we obtain

$$\int_{A(\tilde{k},\tau)} \mathcal{H}\left(\varpi_R-\tilde{k}\right) d\mathcal{L}^N \leq \frac{\mathcal{C}_0}{\left[\mathcal{H}\left(\tilde{k}-\tilde{h}\right)\right]^{1-\frac{1}{\gamma}}(\varrho-\tau)^{\left(\frac{\theta}{N}+1\right)\frac{m_{\max}}{\gamma}}} \left[\int_{A(\tilde{h},\varrho)} \mathcal{H}\left(\frac{\left(\varpi_R-\tilde{h}\right)}{\varrho-\tau}\right) d\mathcal{L}^N\right]^{1+\frac{\theta}{N}\frac{1}{\gamma}}. \tag{4.19}$$

Fix $\theta=N(\gamma-1)$, since $1<\gamma<\frac{N+1}{N}$, then $0<\theta<1$ and $1-\frac{1}{\gamma}=\frac{\theta}{N}\frac{1}{\gamma}=\alpha$.

Let us consider $r_i=\frac{1}{2}\left(1+\frac{1}{2^i}\right)$, $\tilde{k}_0=\frac{d}{R}$ and $\tilde{k}_{i+1}=\tilde{k}_i+\mathcal{H}^{-1}\left(\frac{\mathcal{H}\left(\frac{d}{R}\right)}{2^{m_{\max}i}}\right)$ where d is a positive real number, if we choose $\tau=r_{i+1}$, $\tilde{k}=\tilde{k}_{i+1}$, $\varrho=r_i$ and $\tilde{h}=\tilde{k}_i$ it follows

$$\Lambda_{i+1} \leq \frac{\mathcal{C}_0 2^{2\frac{m_{\max}}{\gamma}\left(1+\frac{\theta}{N}\right)} 2^{m_{\max}\left(1+\frac{\theta}{\gamma N}+\frac{1}{\gamma}\right)i}}{\left[\mathcal{H}\left(\frac{d}{R}\right)\right]^{\alpha}}\left[\Lambda_i\right]^{1+\alpha} \tag{4.20}$$

where $\Lambda_i=\int_{A(\tilde{k}_i,r_i)} \mathcal{H}\left(\varpi_R-\tilde{k}_i\right) d\mathcal{L}^N$.

If we choose

$$\frac{d}{R}=\mathcal{H}^{-1}\left(\mathcal{C}_0^{\frac{1}{\alpha}} 2^{\frac{2m_{\max}}{\alpha\gamma}\left(1+\frac{\theta}{N}\right)+\frac{m_{\max}}{\alpha^2}\left(1+\alpha+\frac{1}{\gamma}\right)} \int_{A(0,1)} \mathcal{H}\left(\varpi_R\right) d\mathcal{L}^N\right) \tag{4.21}$$

then $\Lambda_0 = \int\limits_{A\left(\frac{d}{R},1\right)} \mathcal{H}\left(\varpi_R - \frac{d}{R}\right) d\mathcal{L}^N \leq \mathcal{H}\left(\frac{d}{R}\right) \frac{1}{\mathcal{C}_0^{\frac{1}{\alpha}} 2^{\frac{2m_{\max}}{\alpha\gamma}\left(1+\frac{\theta}{N}\right)+\frac{m_{\max}}{\alpha^2}\left(1+\alpha+\frac{1}{\gamma}\right)}}$ and by Lemma 4 it follows

$$\sup_{Q_{\frac{1}{2}}} (\varpi_R) \leq 2\mathcal{H}^{-1}\left(\mathcal{C}_0^{\frac{1}{\alpha}} 2^{\frac{2m_{\max}}{\alpha\gamma}\left(1+\frac{\theta}{N}\right)+\frac{m_{\max}}{\alpha^2}\left(1+\alpha+\frac{1}{\gamma}\right)} \int\limits_{A(0,1)} \mathcal{H}(\varpi_R)\, d\mathcal{L}^N\right) \quad (4.22)$$

and

$$\sup_{Q_{\frac{R}{2}}} (u) \leq 2R\mathcal{H}^{-1}\left(\mathcal{C}_1 \int\limits_{A(0,R)} \mathcal{H}\left(\frac{u}{R}\right) d\mathcal{L}^N\right) \quad (4.23)$$

where $\mathcal{C}_1 = \mathcal{C}_0^{\frac{1}{\alpha}} 2^{\frac{2m_{\max}}{\alpha\gamma}\left(1+\frac{\theta}{N}\right)+\frac{m_{\max}}{\alpha^2}\left(1+\alpha+\frac{1}{\gamma}\right)}$.

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