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A Reduction Formula for the Appell Series F_3 in Two Variables

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Abstract

A large number of double series in two variables, in association with the theory of hypergeometric series, have been classified and fully developed with named after their discoverers, for example, Appell series and their confluent cases, and various generalizations of these series in two variables. Here, in this paper, by starting with the classical Beta function, we show how one can derive a (presumably) new and (potentially) useful reduction formula for the Appell series F_3 in two variables. Among many things, some interesting special cases of our main result are considered and relevant connections of some results presented here with those earlier ones are also pointed out.

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Keywords: Gamma function; Beta function; Pochhammer symbol; Generalized binomial series; Appell series; Hypergeometric series; Double series manipulations; Combinatorial identities

1 Introduction

Motivated by the enormous success of the theory of hypergeometric series in a single variable, a large number of double series in two variables have been classified and fully developed with named after their discoverers, for example, Appell series and their confluent cases, and various generalizations of these series in two variables (see, *e.g.*, [1, 2, 3, 7, 8, 9, 11, 12]). In this paper, by starting with the classical Beta function, we aim to show how nicely one can derive a (presumably) new and (potentially) useful reduction formula for the Appell series F_3 in two variables. Among many things, some interesting special cases of our main result are considered and relevant connections of some results presented here with those earlier ones are also pointed out.

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, integers, and positive integers, respectively, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For our purpose, we begin by recalling the Gamma function $\Gamma(z)$ developed by Euler which is usually defined by (see, *e.g.*, [10, Section 1.1])

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (\Re(z) > 0). \tag{1}$$

The Beta function $B(\alpha, \beta)$ is a function of two complex variables α and β, defined by (see, *e.g.*, [10, Section 1.1])

$$B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0)\\[2ex] \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\Re(\alpha)<0;\ \Re(\beta)<0;\ \alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases} \tag{2}$$

The widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\,\nu\in\mathbb{C})$ is defined, in general, by (see, for details, [12]; see also [10, p. 2 and p. 5])

$$\begin{aligned}(\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad \big(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\big)\\ &=\begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}).\end{cases}\end{aligned} \tag{3}$$

The following generalized binomial series is well-known:

$$(1-z)^{-\alpha}=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\,z^k \quad (|z|<1). \tag{4}$$

For an easier reference, the following familiar double series manipulations are recalled (see, *e.g.*, [4]; see also [8, pp. 56-57]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{5}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k). \tag{6}$$

2 A Reduction Formula for a Double Series in two variables

Here we present an interesting and (presumably) new formula for a double series in two variables, which is decomposed into two single series, asserted by the following theorem.

Theorem 2.1. *Let* $x,\,y\in\mathbb{C}\setminus\{0\}$, *and* $p,\,q\in\mathbb{C}$ *with* $\max\{|x|,\,|y|\}<1$ *and* $\min\{\Re(p),\,\Re(q)\}>0$. *Then the following formula holds true:*

$$\begin{aligned}&\sum_{k,\,n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n\\ &\quad=\frac{1}{x+y-xy}\left(\sum_{k=0}^{\infty}\frac{(p)_k}{(p+q)_k}x^{k+1}+\sum_{n=0}^{\infty}\frac{(q)_n}{(p+q)_n}y^{n+1}\right).\end{aligned} \tag{7}$$

Proof. We begin by using (2) to find that

$$B(p+k,q+n)=\frac{\Gamma(p+k)\Gamma(q+n)}{\Gamma(p+q+k+n)}=\int_0^1 t^{p+k-1}(1-t)^{q+n-1}dt, \tag{8}$$

where p and q satisfy the stated conditions, and k, $n \in \mathbb{N}_0$. Multiplying both sides of (8) by $x^k\, y^n$, where x, y, k and n satisfy the stated and given conditions, we have

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n=\int_0^1 t^{p-1}(1-t)^{q-1}(xt)^k[y(1-t)]^n dt. \tag{9}$$

Taking double summations of (9) about the variables k and n from 0 to ∞, we obtain

$$\begin{aligned}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}&\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n\\ &=\int_0^1 t^{p-1}(1-t)^{q-1}\sum_{k=0}^{\infty}(xt)^k\sum_{n=0}^{\infty}\{y(1-t)\}^n dt,\end{aligned} \tag{10}$$

in which, with the stated conditions, the interchange of summations and integration may be verified. Interpreting the two inner series on the right-side of (10) into the well-known geometric series, we get

$$\begin{aligned}\mathcal{L}(p,q;x,y):&=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(p)_k(q)_n}{(p+q)_{k+n}}x^k y^n\\ &=\int_0^1\frac{t^{p-1}(1-t)^{q-1}}{(1-xt)\{1-y(1-t)\}}dt.\end{aligned} \tag{11}$$

Considering the following partial fraction:

$$\frac{1}{(1-xt)\{1-y(1-t)\}}=\frac{1}{x+y-xy}\left(\frac{x}{1-xt}+\frac{y}{1-y(1-t)}\right)$$

into the right-hand side of (11) and applying the geometric expansion, we find

$$\begin{aligned}\mathcal{L}(p,q;x,y)&=\frac{1}{x+y-xy}\int_0^1\left(\frac{x}{1-xt}+\frac{y}{1-y(1-t)}\right)t^{p-1}(1-t)^{q-1}dt\\ &=\frac{1}{x+y-xy}\left(\sum_{k=0}^{\infty}x^{k+1}\int_0^1 t^{p+k-1}(1-t)^{q-1}dt\right.\\ &\qquad\left.+\sum_{n=0}^{\infty}y^{n+1}\int_0^1 t^{p-1}(1-t)^{q+n-1}dt\right).\end{aligned} \tag{12}$$

Applying (2) to (12), we obtain

$$\begin{aligned}\mathcal{L}(p,q;x,y) &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty} B(p+k,q)x^{k+1} + \sum_{n=0}^{\infty} B(p,q+n)y^{n+1}\right] \\ &= \frac{1}{x+y-xy}\left[\sum_{k=0}^{\infty} \frac{\Gamma(p+k)\Gamma(q)}{\Gamma(p+q+k)}x^{k+1} + \sum_{n=0}^{\infty} \frac{\Gamma(p)\Gamma(q+n)}{\Gamma(p+q+n)}y^{n+1}\right],\end{aligned}$$

which, in view of (3), is easily seen to yield the desired result (7). □

The identity in (7) is observed to be expressed in terms of some known special functions given in the following theorem.

Theorem 2.2. *Let $x,\ y \in \mathbb{C}\setminus\{0\}$ with $\max\{|x|,\ |y|\} < 1$ and $p,\ q \in \mathbb{C}$. Then the following decomposition formula holds true:*

$$\begin{aligned} &F_3\,[p,\ q,\ 1,\ 1\,;\ p+q\,;\ x,\ y] \\ &\quad = \frac{1}{x+y-xy}\Big(x\,{}_2F_1\,(p,1;p+q;x) + y\,{}_2F_1\,(q,1;p+q;y)\Big), \end{aligned} \tag{13}$$

or, equivalently,

$$\begin{aligned} &F_3\,[p,\ q,\ 1,\ 1\,;\ p+q\,;\ x,\ y] \\ &\quad = \frac{x\,(1-x)^{-p}}{x+y-xy}\,{}_2F_1\left(p,p+q-1;p+q;\frac{x}{x-1}\right) \\ &\qquad + \frac{y\,(1-y)^{-q}}{x+y-xy}\,{}_2F_1\left(q,p+q-1;p+q;\frac{y}{y-1}\right), \end{aligned} \tag{14}$$

where $F_3\,[\cdot]$ is one of the four Appell series in two variables (see, e.g.*, [11, p. 23]) defined by*

$$\begin{aligned} &F_3\,[a,\ a',\ b,\ b'\,;\ c\,;\ x,\ y] \\ &\quad = \sum_{m,n=0}^{\infty} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n}}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \quad (\max\{|x|,\ |y|\} < 1) \end{aligned} \tag{15}$$

and ${}_2F_1\,(\cdot)$ is the familiar hypergeometric series defined by

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta; \\ \gamma;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} = {}_2F_1(\alpha,\ \beta;\ \gamma;\ z) \quad (|z|<1). \tag{16}$$

Proof. From (7), it is easy to prove (13). For (14), an application of the following Euler transformation for ${}_2F_1$:

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ z) = (1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\ \gamma-\beta;\ \gamma;\ \frac{z}{z-1}\right) \tag{17}$$

to the two ${}_2F_1$ in (13) is easily seen to yield (14).

□

3 Special Cases and Remarks

We begin by recalling a known reduction formula for the Appell series F_3 (see, *e.g.*, [11, p. 301, Entry (88)]):

$$\begin{aligned} &F_3\left[p,\, q,\, 1,\, 1\,;\, p+q\,;\, x,\, y\right] \\ &\quad= \frac{x\,(1-x)^{-p}}{x+y-xy}\, {}_2F_1\left(p,\, q\,;\, p+q\,;\, \frac{x}{x-1}\right) \\ &\quad+ \frac{y\,(1-y)^{-q}}{x+y-xy}\, {}_2F_1\left(p,\, q\,;\, p+q\,;\, \frac{y}{y-1}\right). \end{aligned} \tag{18}$$

The formulas (14) and (18) are immediately seen to be same except for the second numerator parameters in ${}_2F_1$. The formula (7) (and so (13) and (14)) is sure to be correct, since the process of its proof is correct and, further, some special cases of it yield certain known formulas, as shown in examples below. So the formula (18) should be corrected as the one in (14).

A large number of special cases of (7) can be presented by suitably choosing the parameters p, q and the variables x, y. Here we choose to consider only some examples.

Example 1. Let $x,\, y \in \mathbb{C} \setminus \{0\}$ with $\max\{|x|,\, |y|\} < 1$. Then we have

$$\sum_{k,\, n=0}^{\infty} \frac{1}{(k+n+1)\binom{k+n}{k}}\, x^k\, y^n = -\frac{\log\{(1-x)\,(1-y)\}}{x+y-xy}. \tag{19}$$

Proof. Setting $p = q = 1$ in (7) gives

$$\begin{aligned} \sum_{k,\, n=0}^{\infty} \frac{(1)_k(1)_n}{(2)_{k+n}}\, x^k y^n &= \frac{1}{x+y-xy}\left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1}\right) \\ &= \frac{1}{x+y-xy}\left(\sum_{k=1}^{\infty} \frac{x^k}{k} + \sum_{n=1}^{\infty} \frac{y^n}{n}\right) \\ &= \frac{-\log(1-x) - \log(1-y)}{x+y-xy}. \end{aligned}$$

□

Example 2. Let $x,\, y \in \mathbb{C} \setminus \{0\}$ with $\max\{|x|,\, |y|\} < 1$ and $p \in \mathbb{C}$ with $0 < \Re(p) < 1$. Then we have

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(p)_k(1-p)_n}{(k+n)!} x^k y^n &= F_3\left[p,\, 1-p,\, 1,\, 1\,;\, 1\,;\, x,\, y\right] \\ &= \frac{1}{x+y-xy}\left(x(1-x)^{-p} + y(1-y)^{-1+p}\right). \end{aligned} \tag{20}$$

Proof. Setting $q = 1 - p$ in (7) yields

$$\sum_{k,\,n=0}^{\infty} \frac{(p)_k (1-p)_n}{(k+n)!} x^k y^n \quad = \frac{1}{x+y-xy} \left(x \sum_{k=0}^{\infty} \frac{(p)_k}{k!} x^k + y \sum_{n=0}^{\infty} \frac{(1-p)_n}{n!} y^n \right),$$

which, in view of (4), proves the desired result.

□

Example 3. Let p, $q \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then the following formula holds true:

$$\frac{1}{(p+q)_n} \sum_{k=0}^{n} (p)_k (q)_{n-k} = \sum_{k=0}^{n} 2^{-n+k-1} \frac{(p)_k + (q)_k}{(p+q)_k}. \tag{21}$$

Proof. Taking $y = x$ in (7) and using the geometric series expansion, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(p)_k (q)_n}{(p+q)_{k+n}} x^{k+n} &= \frac{1}{2(1-\frac{x}{2})} \sum_{k=0}^{\infty} \frac{(p)_k + (q)_k}{(p+q)_k} x^k \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} \sum_{k=0}^{\infty} \frac{(p)_k + (q)_k}{(p+q)_k} x^k. \end{aligned}$$

Then we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(p)_k (q)_n}{(p+q)_{k+n}} x^{k+n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(p)_k + (q)_k}{2^{n+1} (p+q)_k} x^{n+k}.$$

Applying the double series manipulation (5) to both sides of the last identity, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(p+q)_n} \sum_{k=0}^{n} (p)_k (q)_{n-k}\, x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2^{n-k+1}} \frac{(p)_k + (q)_k}{(p+q)_k} x^n,$$

which, upon equating the coefficient of x^n, yields the desired identity. □

Example 4. The following combinatorial identity holds true:

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} \quad (n \in \mathbb{N}), \tag{22}$$

which is a known formula recorded, for example, in [6, Entry (2.25)].

Proof. Taking $q = p$ in (21), we get

$$\frac{1}{(2p)_n}\sum_{k=0}^{n}(p)_k(p)_{n-k} = \sum_{k=0}^{n} 2^{-n+k}\frac{(p)_k}{(2p)_k}, \tag{23}$$

Setting $p = 1$ in (23) yields

$$\frac{1}{(2)_n}\sum_{k=0}^{n}(1)_k(1)_{n-k} = \sum_{k=0}^{n} 2^{-n+k}\frac{(1)_k}{(2)_k}.$$

Or, equivalently,

$$\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} 2^{-n+k}\frac{1}{k+1},$$

which, after a little simplification, is led to the formula (22). □

Example 5. Let $p,\, q \in \mathbb{C}$ with $p + q \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $n \in \mathbb{N}_0$. Then the following formula holds true:

$$\sum_{k=0}^{n}(-1)^{n-k}(p)_k(q)_{n-k} = \frac{(p)_{n+1} + (-1)^n (q)_{n+1}}{p+q+n}. \tag{24}$$

Proof. The identity here can be established in parallel with the proof of (21), by taking $y = -x$ in (7). So its detailed account is omitted. □

Example 6. Let $p \in \mathbb{C}$ with $p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $n \in \mathbb{N}_0$. Then the following formula holds true:

$$\sum_{k=0}^{2n}(-1)^k(p)_k(p)_{2n-k} = \frac{(p)_{2n+1}}{p+n}, \tag{25}$$

whose special case $p = 1$ yields the following combinatorial identity:

$$\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}} = \frac{2n+1}{n+1} \quad (n \in \mathbb{N}_0)\,. \tag{26}$$

It is noted that the identity (26) is a special case of a known formula [6, Entry (2.1)].

Proof. Setting $q = p$ in (24) and replacing n by $2n$ in the resulting identity proves (25). □

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