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A Present Position-Dependent Conditional Fourier-Feynman Transform and Convolution Product over Continuous Paths

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Abstract

Let $C[0,t]$ denote the function space of all real-valued continuous paths on $[0,t]$. Define $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ by $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$. In the present paper, using a simple formula for the conditional expectation with the conditioning function X_{n+1}, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions which have the form $f((v_1, x), \cdots, (v_r, x))$ for $x \in C[0,t]$, where $\{v_1, \cdots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and $f \in L_p(\mathbb{R}^r)$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions.

Mathematics Subject Classification: 28C20, 44A20

Keywords: analogue of Wiener measure, analytic conditional Feynman integral, analytic conditional Fourier-Feynman transform, analytic conditional Wiener integral, conditional convolution product

1 Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, that is, the space of the real-valued continuous functions x on the closed interval $[0,t]$ with $x(0) = 0$. On the space $C_0[0,t]$, the concept of an analytic Fourier-Feynman transform was introduced by Brue [1]. Huffman, Park and Skoug [9] defined a convolution product on $C_0[0,t]$ and then, established various relationships between the analytic Fourier-Feynman transform and the convolution product. Furthermore Chang and Skoug [4] introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product on the Wiener space $C_0[0,t]$. In that paper they also examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were studied by Chang, Cho, Kim, Song and Yoo [3, 7]. In fact Cho and his coauthors [3] introduced the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and then, established the relationships between them of certain cylinder type functions. Cho [7] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le 2)$-analytic conditional Fourier-Feynman transform of the same kind of cylinder type functions. Moreover, on $C[0,t]$, the space of the real-valued continuous paths on $[0,t]$, Kim [12] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [2]. Cho [5] introduced the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product over $C[0,t]$ and then, established the relationships between them of certain cylinder type functions.

In this paper we further develop the relationships in [3, 5, 7, 12] on the more generalized space $(C[0,t], w_\varphi)$, the analogue of the Wiener space associated with the probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ [10, 14, 15]. For the conditioning function $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ given by $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$, we proceed to study the relationships between the conditional convolution product and the analytic conditional Fourier-Feynman transform of the cylinder functions defined on $C[0,t]$. In fact, using a simple formula for the conditional w_φ-integrals given X_{n+1} [6], we evaluate the $L_p(1 \le p \le \infty)$-

analytic conditional Fourier-Feynman transforms and the conditional convolution products for the cylinder functions of the form $f((v_1, x), \cdots, (v_r, x))$ for w_φ-a.e. $x \in C[0, t]$, where $\{v_1, \cdots, v_r\}$ is an orthonormal set in $L_2[0, t]$ and $f \in L_p(\mathbb{R}^r)$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions. Finally we show that the L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[[(F*G)_q|X_{n+1}](\cdot, \vec{\xi}_n)|X_{n+1}]$ of the conditional convolution product for the cylinder functions F and G, can be expressed by the formula

$$\begin{aligned} & T_q^{(p)}[[(F*G)_q|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](y, \vec{\zeta}_{n+1}) \\ = & \left[T_q^{(p)}[F|X_{n+1}]\left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}) \right) \right] \\ & \times \left[T_q^{(p)}[G|X_{n+1}]\left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}) \right) \right] \end{aligned}$$

for a nonzero real q, w_φ-a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on the Borel class of $\mathbb{R}^{n+2}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the cylinder functions, can be interpreted as the product of the conditional analytic Fourier-Feynman transform of each function.

Throughout this paper let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and the set of complex numbers with positive real parts, respectively.

Now we introduce the concrete form of the probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$. For a positive real t let $C = C[0, t]$ be the space of all real-valued continuous functions on the closed interval $[0, t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$ let $J_{\vec{t}} : C[0, t] \to \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n))$. For $B_j\,(j = 0, 1, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0, t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let

$$\begin{aligned} m_\varphi\left[J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j \right) \right] = & \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^n B_j} \\ & \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d(u_1, \cdots, u_n) d\varphi(u_0). \end{aligned}$$

Then $\mathcal{B}(C[0, t])$, the Borel σ-algebra of $C[0, t]$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [10, 14, 15, 17].

Let $\{e_k : k = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For v in $L_2[0,t]$ and x in $C[0,t]$ let $(v,x) = \lim_{n\to\infty}\sum_{k=1}^{n}\langle v, e_k\rangle \int_0^t e_k(s)dx(s)$ if the limit exists, where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (v,x) is called the Paley-Wiener-Zygmund integral of v according to x. We note that the dot product on the r-dimensional Euclidean space $\mathbb{R}^r$ is also denoted by $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$.

Applying Theorem 3.5 in [10], we can easily prove the following theorem.

Theorem 1.1 *Let $\{h_1, h_2, \cdots, h_r\}$ be an orthonormal subset of $L_2[0,t]$. For $i = 1, 2, \cdots, r$, let $Z_i(x) = (h_i, x)$ on $C[0,t]$. Then $Z_1, Z_2, \cdots, Z_r$ are independent and each Z_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^r \to \mathbb{R}$ is Borel measurable, then*

$$\begin{aligned}&\int_C f(Z_1(x), Z_2(x), \cdots, Z_r(x))dw_\varphi(x)\\ \stackrel{*}{=}\ &\left(\frac{1}{2\pi}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r} f(u_1, u_2, \cdots, u_r)\exp\left\{-\frac{1}{2}\sum_{j=1}^{r}u_j^2\right\}d(u_1, u_2, \cdots, u_r),\end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and X be a random vector on $C[0,t]$ assuming that the value space of X is a normed space equipped with the Borel σ-algebra. Then we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory [13]. Furthermore there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi\circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0,t]$ unless otherwise specified. For any x in $C[0,t]$ define the polygonal function $[x]$ of x by

$$\begin{aligned}[x](s) \ =\ &\sum_{j=1}^{n+1}\chi_{(t_{j-1},t_j]}(s)\left(\frac{t_j - s}{t_j - t_{j-1}}x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}x(t_j)\right)\\ &+\chi_{\{t_0\}}(s)x(t_0) \qquad (1)\end{aligned}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by (1), where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \cdots, n+1$.

In the following theorem we introduce a simple formula for the conditional w_φ-integrals on $C[0,t]$ [6].

Theorem 1.2 *Let $F : C[0,t] \to \mathbb{C}$ be integrable and let $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ be given by*

$$X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1})). \tag{2}$$

Then, for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$E[F|X_{n+1}](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])], \tag{3}$$

where the expectation is taken over the variable x and $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$.

For a function $F : C[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ and $X^\lambda_{n+1}(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$, where X_{n+1} is given by (2). Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ-integral and (3), $E[F^\lambda|X^\lambda_{n+1}](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$ for $P_{X^\lambda_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X^\lambda_{n+1}}$ is the probability distribution of X^λ_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. Throughout this paper, for $y \in C[0,t]$ let $I^\lambda_F(y, \vec{\xi}_{n+1}) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$ unless otherwise specified, where the expectation is taken over the variable x. If $I^\lambda_F(0, \vec{\xi}_{n+1})$ has the analytic extension $J^*_\lambda(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the analytic conditional Wiener w_φ-integral of F given X_{n+1} with parameter λ and denoted by $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = J^*_\lambda(F)(\vec{\xi}_{n+1})$ for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic conditional Feynman w_φ-integral of F given X_{n+1} with parameter q and denoted by $E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda\to -iq} E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$.

2 The conditional Fourier-Feynman transform

For a given extended real number p with $1 < p \le \infty$ suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n\to\infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\underset{n\to\infty}{\text{l.i.m.}}(w^{p'})(F_n) = F$ and call F the limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

We now define the analytic conditional Fourier-Feynman transform of the functions on $C[0,t]$.

Definition 2.1 *Let F be defined on $C[0,t]$ and X_{n+1} be given by (2). For $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, let*

$$T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anw_\lambda}[F(y + \cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$*-a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ *if it exists. For a nonzero real* q *and* w_φ*-a.e.* $y \in C[0,t]$*, define the* L_1*-analytic conditional Fourier-Feynman transform* $T_q^{(1)}[F|X_{n+1}]$ *of* F *given* X_{n+1} *by the formula*

$$T_q^{(1)}[F|X_{n+1}](y,\vec{\xi}_{n+1}) = E^{anf_q}[F(y+\cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$*-a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ *if it exists. For* $1 < p \leq \infty$*, define the* L_p*- analytic conditional Fourier-Feynman transform* $T_q^{(p)}[F|X_{n+1}]$ *of* F *given* X_{n+1} *by the formula*

$$T_q^{(p)}[F|X_{n+1}](\cdot,\vec{\xi}_{n+1}) = \underset{\lambda\to -iq}{\text{l.i.m.}}(w^{p'})(T_\lambda[F|X_{n+1}](\cdot,\vec{\xi}_{n+1}))$$

for $P_{X_{n+1}}$*-a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*, where* λ *approaches to* $-iq$ *through* $\mathbb{C}_+$*.*

For each $j = 1, \cdots, n+1$, let $\alpha_j = \frac{1}{\sqrt{t_j - t_{j-1}}}\chi_{(t_{j-1},t_j]}$ on $[0,t]$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \cdots, \alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively. Throughout this paper let $\{v_1, v_2, \cdots, v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ is an independent set. Let $\{e_1, \cdots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \cdots, r$, let

$$\mathcal{P}^\perp v_l = \sum_{j=1}^r \alpha_{lj} e_j \tag{4}$$

be the linear combinations of the e_js and let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{r1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{1r} & \alpha_{2r} & \cdots & \alpha_{rr} \end{bmatrix}$$

be the transpose of the coefficient matrix of the combinations. We can also regard A as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by

$$T_A \vec{z} = \vec{z}A, \tag{5}$$

where $\vec{z}$ is any row-vector in $\mathbb{R}^r$. We note that A is invertible so that T_A is an isomorphism.

For convenience let $(\vec{v}, x) = ((v_1, x), \cdots, (v_r, x))$ for $x \in C[0,t]$. For $1 \leq p \leq \infty$ let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions F_r of the form given by

$$F_r(x) = f_r(\vec{v}, x) \tag{6}$$

for w_φ-a.e. $x \in C[0,t]$, where $f_r \in L_p(\mathbb{R}^r)$. We note that, without loss of generality, we can take f_r to be Borel measurable.

We evaluate the conditional Fourier-Feynman transform of cylinder function in the following theorem.

Theorem 2.2 *Let X_{n+1} be given by (2). Let $F_r \in \mathcal{A}_r^{(p)} (1 \le p \le \infty)$ and f_r be related by (6). Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $T_\lambda[F_r|X_{n+1}](y, \vec{\xi}_{n+1})$ exists and it is given by*

$$\begin{aligned} & T_\lambda[F_r|X_{n+1}](y,\vec{\xi}_{n+1}) \\ = \; & \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f_r((\vec{v},y)+(\vec{v},[\vec{\xi}_{n+1}])+T_A\vec{z})\exp\left\{-\frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\right\}d\vec{z}, \end{aligned} \tag{7}$$

where T_A is given by (5). As a function of y, $T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in \mathcal{A}_r^{(p)}$.

Proof. For $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$I^\lambda_{F_r}(y,\vec{\xi}_{n+1}) \overset{*}{=} \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f_r((\vec{v},y)+(\vec{v},[\vec{\xi}_{n+1}])+T_A\vec{z})\exp\left\{-\frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\right\}d\vec{z}$$

by Lemma 2.1 of [5]. We note that if $1 \le p < \infty$, then by the change of variable theorem

$$\int_{\mathbb{R}^r} |f_r((\vec{v},y)+(\vec{v},[\vec{\xi}_{n+1}])+T_A\vec{z})|^p d\vec{z} = |\det(A^{-1})| \, \|f_r\|_p^p < \infty, \tag{8}$$

where A^{-1} is the inverse of the matrix A given by (5). Now, by Morera's theorem with aids of Hölder's inequality and the dominated convergence theorem, we have (7) for $\lambda \in \mathbb{C}_+$. To prove the remainder of the theorem, for $\lambda \in \mathbb{C}_+$ and for $\vec{u} \in \mathbb{R}^r$, let

$$\Psi(\lambda,\vec{u}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \exp\left\{-\frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\right\}. \tag{9}$$

By the change of variable theorem

$$\begin{aligned} \gamma(\lambda,\vec{u},\vec{\xi}_{n+1}) \; &\equiv \; \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f_r(\vec{u}+(\vec{v},[\vec{\xi}_{n+1}])+T_A\vec{z})\exp\left\{-\frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\right\}d\vec{z} \\ &= \int_{\mathbb{R}^r} f_r(T_A(((\vec{v},[\vec{\xi}_{n+1}])+\vec{u})A^{-1}-\vec{z}))\Psi(\lambda,\vec{z})d\vec{z} \\ &= (f_r(T_A\cdot) * \Psi(\lambda,\cdot))(((\vec{v},[\vec{\xi}_{n+1}])+\vec{u})A^{-1}). \end{aligned} \tag{10}$$

By (8), $f_r(T_A\cdot) \in L_p(\mathbb{R}^r)$ and $\Psi(\lambda,\cdot) \in L_1(\mathbb{R}^r)$ so that $f_r(T_A\cdot) * \Psi(\lambda,\cdot) \in L_p(\mathbb{R}^r)$ by Young's inequality [8, p.232]. Now $\gamma(\lambda,\cdot,\vec{\xi}_{n+1}) = (f_r(T_A\cdot) * \Psi(\lambda, \cdot))(((\vec{v}, [\vec{\xi}_{n+1}]) + \cdot)A^{-1}) \in L_p(\mathbb{R}^r)$ by the change of variable theorem which completes the proof. □

Theorem 2.3 *Let X_{n+1} be given by (2). Let $F_r \in \mathcal{A}_r^{(p)}(1 \le p \le 2)$ and f_r be related by (6). Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $T_q^{(p)}[F_r|X_{n+1}](y,\vec{\xi}_{n+1})$ exists and it is given by (7), where λ is replaced by $-iq$. Furthermore, as a function of y, $T_q^{(p)}[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in \mathcal{A}_r^{(p')}$, where $\frac{1}{p}+\frac{1}{p'}=1$ if $1<p\le 2$ and $p'=\infty$ if $p=1$.*

Proof. When $p=1$, the results follow from Theorem 2.2 of [5]. Suppose that $1<p\le 2$. For either $\lambda \in \mathbb{C}_+$ or $\lambda=-iq$, for $\vec{u}\in\mathbb{R}^r$ and $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$, let $\Psi(\lambda,\vec{u})$ and $\gamma(\lambda,\vec{u},\vec{\xi}_{n+1})$ be given by (9) and (10), respectively. By (8), $f_r(T_A\cdot)$ belongs to $L_p(\mathbb{R}^r)$ so that $f_r(T_A\cdot)*\Psi(\lambda,\cdot)\in L_{p'}(\mathbb{R}^r)$ by Lemma 1.1 of [11]. Now $\gamma(\lambda,\cdot,\vec{\xi}_{n+1}) = (f_r(T_A\cdot)*\Psi(\lambda,\cdot))(((\vec{v},[\vec{\xi}_{n+1}])+\cdot)A^{-1})$ belongs to $L_{p'}(\mathbb{R}^r)$ by the change of variable theorem. Let $T_q^{(p)}[F_r|X_{n+1}](y,\vec{\xi}_{n+1}) = \gamma(-iq,(\vec{v},y),\vec{\xi}_{n+1})$ formally for w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$. Then

$$
\begin{aligned}
&\int_C |T_\lambda[F_r|X_{n+1}](y,\vec{\xi}_{n+1}) - T_q^{(p)}[F_r|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'} dw_\varphi(y)\\
=\ &\int_C |\gamma(\lambda,(\vec{v},y),\vec{\xi}_{n+1}) - \gamma(-iq,(\vec{v},y),\vec{\xi}_{n+1})|^{p'} dw_\varphi(y)\\
=\ &\left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} |\gamma(\lambda,\vec{u},\vec{\xi}_{n+1}) - \gamma(-iq,\vec{u},\vec{\xi}_{n+1})|^{p'} \exp\left\{-\frac{1}{2}\|\vec{u}\|_{\mathbb{R}^r}^2\right\} d\vec{u}\\
=\ &\left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} |(f_r(T_A\cdot)*\Psi(\lambda,\cdot))(((\vec{v},[\vec{\xi}_{n+1}])+\vec{u})A^{-1})\\
&-(f_r(T_A\cdot)*\Psi(-iq,\cdot))(((\vec{v},[\vec{\xi}_{n+1}])+\vec{u})A^{-1})|^{p'} \exp\left\{-\frac{1}{2}\|\vec{u}\|_{\mathbb{R}^r}^2\right\} d\vec{u}
\end{aligned}
$$

by Theorems 1.1 and 2.2. Let $\vec{z} = ((\vec{v},[\vec{\xi}_{n+1}])+\vec{u})A^{-1}$. Then, by the change of variable theorem,

$$
\begin{aligned}
&\int_C |T_\lambda[F_r|X_{n+1}](y,\vec{\xi}_{n+1}) - T_q^{(p)}[F_r|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'} dw_\varphi(y)\\
\le\ &|\det(A)| \int_{\mathbb{R}^r} |(f_r(T_A\cdot)*\Psi(\lambda,\cdot))(\vec{z}) - (f_r(T_A\cdot)*\Psi(-iq,\cdot))(\vec{z})|^{p'} d\vec{z}
\end{aligned}
$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by Lemma 1.2 of [11]. Now the proof is completed. □

Theorem 2.4 *Let X_{n+1} be given by (2). Let $F_r \in \mathcal{A}_r^{(p)}(1\le p\le\infty)$ and f_r be related by (6). For w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1},\vec{\zeta}_{n+1}\in\mathbb{R}^{n+2}$, let $F_{r1}(y,\vec{\xi}_{n+1},\vec{\zeta}_{n+1}) = f_r((\vec{v},y)+(\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}]))$. Then, for a nonzero real q,*

$$
\int_C |T_{\bar{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) - F_{r1}(y,\vec{\xi}_{n+1},\vec{\zeta}_{n+1})|^p dw_\varphi(y) \to 0
$$

for $1 \le p < \infty$*, and for* $1 \le p \le \infty$

$$T_{\overline{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \longrightarrow F_{r1}(y,\vec{\xi}_{n+1},\vec{\zeta}_{n+1})$$

as λ *approaches to* $-iq$ *through* $\mathbb{C}_+$*.*

Proof. Note that $T_{\overline{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$ is well-defined by Theorem 2.2. For $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1},\vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned}
&T_{\overline{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
=\ &\left(\frac{|\lambda|^2}{4\pi\mathrm{Re}\lambda}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r} f_r(T_A(((\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])+(\vec{v},y))A^{-1}-\vec{\eta}))\\
&\times\exp\left\{-\frac{|\lambda|^2}{4\mathrm{Re}\lambda}\|\vec{\eta}\|^2_{\mathbb{R}^r}\right\}d\vec{\eta}\\
=\ &\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda}\right)^{\frac{r}{2}}\left(f_r(T_A\cdot)*\Psi\left(1,\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda}\right)^{\frac12}\cdot\right)\right)(((\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])\\
&+(\vec{v},y))A^{-1})
\end{aligned}$$

by Theorem 2.3 of [5], where Ψ is given by (9). We have $\int_{\mathbb{R}^r}\Psi(1,\vec{u})d\vec{u}=1$ and hence $\Psi(1,\cdot)\in L_1(\mathbb{R}^r)$, and $\operatorname{ess\,sup}\{|\Psi(1,\vec{z})| : \|\vec{z}\|_{\mathbb{R}^r}\ge\|\vec{u}\|_{\mathbb{R}^r}\}=\Psi(1,\vec{u})$ is an L_1-function of $\vec{u}$. Furthermore $f_r(T_A\cdot)$ is in $L_p(\mathbb{R}^r)$ by (8). Let $\epsilon=(\frac{2\mathrm{Re}\lambda}{|\lambda|^2})^{\frac12}>0$. By Theorem 1.1 and Theorem 1.18 of [16]

$$\begin{aligned}
&\int_C|T_{\overline{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})-F_{r1}(y,\vec{\xi}_{n+1},\vec{\zeta}_{n+1})|^p dw_\varphi(y)\\
=\ &\int_C\bigg|\epsilon^{-r}\left(f_r(T_A\cdot)*\Psi\left(1,\frac{\cdot}{\epsilon}\right)\right)(((\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])+(\vec{v},y))A^{-1})-f_r(T_A((\vec{v},\\
&[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])+(\vec{v},y))A^{-1})\bigg|^p dw_\varphi(y)\\
\le\ &|\det(A)|\int_{\mathbb{R}^r}\left|\epsilon^{-r}\left(f_r(T_A\cdot)*\Psi\left(1,\frac{\cdot}{\epsilon}\right)\right)(\vec{u})-f_r(T_A\vec{u})\right|^p d\vec{u}
\end{aligned}$$

which converges to 0 if $1\le p<\infty$, and by Theorem 1.25 of [16]

$$\begin{aligned}
&T_{\overline{\lambda}}[T_\lambda[F_r|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
=\ &\epsilon^{-r}\left(f_r(T_A\cdot)*\Psi\left(1,\frac{\cdot}{\epsilon}\right)\right)(((\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])+(\vec{v},y))A^{-1})
\end{aligned}$$

which converges to

$$f_r(T_A((\vec{v},[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])+(\vec{v},y))A^{-1})=F_{r1}(y,\vec{\xi}_{n+1},\vec{\zeta}_{n+1})$$

if $1\le p\le\infty$, as λ approaches $-iq$ through $\mathbb{C}_+$. □

3 The conditional convolution product

In this section we derive the conditional convolution product of cylinder functions and investigate various relationships between the conditional Fourier-Feynman transform and convolution product. To do this we need the following definition.

Definition 3.1 *Let X_{n+1} be given by (2), and F and G be defined on $C[0,t]$. Define the conditional convolution product $[(F*G)_\lambda|X_{n+1}]$ of F and G given X_{n+1} by the formula, for w_φ-a.e. $y\in C[0,t]$*

$$
\begin{aligned}
&[(F*G)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})\\
&= \begin{cases} E^{anw_\lambda}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda\in\mathbb{C}_+;\\ E^{anf_q}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda=-iq;\quad q\in\mathbb{R}-\{0\}\end{cases}
\end{aligned}
$$

if they exist for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$. If $\lambda=-iq$, we can replace $[(F*G)_\lambda|X_{n+1}]$ by $[(F*G)_q|X_{n+1}]$.

Theorem 3.2 *Let $F_r\in\mathcal{A}_r^{(p_1)}$, $G_r\in\mathcal{A}_r^{(p_2)}$ and f_r, g_r be related by (6), respectively, where $1\le p_1,p_2\le\infty$. Furthermore let $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $\frac{1}{p_2}+\frac{1}{p_2'}=1$ and X_{n+1} be given by (2). Then for $\lambda\in\mathbb{C}_+$, w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$, $[(F_r*G_r)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})$ exists and it is given by*

$$
\begin{aligned}
[(F_r*G_r)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) &= \int_{\mathbb{R}^r} f_r\left(\frac{1}{\sqrt{2}}[(\vec{v},y+[\vec{\xi}_{n+1}])+T_A\vec{z}]\right)\\
&\quad\times g_r\left(\frac{1}{\sqrt{2}}[(\vec{v},y-[\vec{\xi}_{n+1}])-T_A\vec{z}]\right)\Psi(\lambda,\vec{z})d\vec{z},
\end{aligned}
$$

where T_A and Ψ are given by (5) and (9), respectively. Furthermore, as functions of y, $[(F_r*G_r)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})\in\mathcal{A}_r^{(1)}$ if either $p_2\le p_1'$ or $p_1\le p_2'$, $[(F_r*G_r)_\lambda|X_{n+1}](\cdot,\,\vec{\xi}_{n+1})\in\mathcal{A}_r^{(p_2)}$ if $p_2\ge p_1'$ and $[(F_r*G_r)_\lambda|X_{n+1}](\cdot,\,\vec{\xi}_{n+1})\in\mathcal{A}_r^{(p_1)}$ if $p_1\ge p_2'$.

Proof. Using the same method as used in the proof of Theorem 2.4 of [5], for $\lambda>0$, w_φ-a.e. $y\in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$,

$$
\begin{aligned}
[(F_r*G_r)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) &\overset{*}{=} \int_{\mathbb{R}^r} f_r\left(\frac{1}{\sqrt{2}}[(\vec{v},y+[\vec{\xi}_{n+1}])+T_A\vec{z}]\right)\\
&\quad\times g_r\left(\frac{1}{\sqrt{2}}[(\vec{v},y-[\vec{\xi}_{n+1}])-T_A\vec{z}]\right)\Psi(\lambda,\vec{z})d\vec{z}
\end{aligned}
$$

by Lemma 2.1 of [5]. Now, for $\lambda \in \mathbb{C}_+$ and $\vec{u} \in \mathbb{R}^r$, let

$$\begin{aligned} \gamma_1(\vec{u}) &= \int_{\mathbb{R}^r} f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{z}]\right) \\ &\quad \times g_r\left(\frac{1}{\sqrt{2}}[\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{z}]\right)\Psi(\lambda, \vec{z})d\vec{z} \end{aligned} \tag{11}$$

formally, and suppose that $p_2 \le p_1'$. Let $\vec{p} = \frac{1}{\sqrt{2}}(\vec{u}+T_A\vec{z})$ and $\vec{q} = \frac{1}{\sqrt{2}}(\vec{u}-T_A\vec{z})$. Then $\vec{u} = \frac{1}{\sqrt{2}}(\vec{p}+\vec{q})$ and $\vec{z} = \frac{1}{\sqrt{2}}(\vec{p}-\vec{q})A^{-1}$. By the change of variable theorem,

$$\begin{aligned} \int_{\mathbb{R}^r} |\gamma_1(\vec{u})|d\vec{u} &\le |\det(A^{-1})| \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \left| f_r\left(\vec{p} + \frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}])\right)\right| \\ &\quad \times \left| g_r\left(\vec{q} - \frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}])\right)\Psi\left(\lambda, \frac{1}{\sqrt{2}}(\vec{p}-\vec{q})A^{-1}\right)\right| d\vec{q}d\vec{p} \\ &= |\det(A^{-1})| \int_{\mathbb{R}^r} |f_{r1}(\vec{p})|(|g_{r1}| * |\Psi_1|)(\vec{p})d\vec{p}, \end{aligned}$$

where $f_{r1}(\vec{p}) = f_r(\vec{p}+\frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}]))$, $g_{r1}(\vec{p}) = g_r(\vec{p}-\frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}]))$ and $\Psi_1(\vec{p}) = \Psi(\lambda, \frac{1}{\sqrt{2}}\vec{p}A^{-1}) = (\frac{\lambda}{2\pi})^{\frac{r}{2}} \exp\{-\frac{\lambda}{4}\|\vec{p}A^{-1}\|^2_{\mathbb{R}^r}\}$. Now let $\frac{1}{p_2} + \frac{1}{q} = \frac{1}{p_1'} + 1$. Since $p_2 \le p_1'$, we have $\frac{1}{q} = 1 + \frac{1}{p_1'} - \frac{1}{p_2} = \frac{p_1'p_2+p_2-p_1'}{p_1'p_2}$ and $0 \le p_1'(p_2-1)+p_2 = p_1'p_2+p_2-p_1' \le p_1'p_2$ if $1 < p_1 \le \infty$ and $1 \le p_2 < \infty$. Hence $1 \le q \le \infty$ for $1 \le p_1, p_2 \le \infty$. On the other hand, $\Psi_1 \in L_q(\mathbb{R}^r)$. Now by the general form of Young's inequality [8, Theorem 8.9] and Hölder's inequality

$$\begin{aligned} \int_{\mathbb{R}^r} |\gamma_1(\vec{u})|d\vec{u} &\le |\det(A^{-1})|\, \|f_{r1}\|_{p_1} \|(|g_{r1}| * |\Psi_1|)\|_{p_1'} \\ &\le |\det(A^{-1})|\, \|f_{r1}\|_{p_1}\|g_{r1}\|_{p_2}\|\Psi_1\|_q < \infty \end{aligned}$$

which shows that $\gamma_1 \in L_1(\mathbb{R}^r)$ and hence $[(F_r * G_r)_\lambda|X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(1)}$. Similarly $[(F_r * G_r)_\lambda|X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(1)}$ if $p_1 \le p_2'$. Suppose that $1 < p_1' \le p_2 < \infty$. Then by Hölder's inequality and the change of variable theorem

$$\begin{aligned} \int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2}d\vec{u} &\le [|\det(A^{-1})|2^{\frac{r}{2}}]^{\frac{p_2}{p_1}}\|f_r\|_{p_1}^{p_2} \int_{\mathbb{R}^r}\left[\int_{\mathbb{R}^r}\left|\Psi(\lambda, \vec{z})g_r\left(\frac{1}{\sqrt{2}}T_A((\vec{u}\right.\right.\right. \\ &\quad \left.\left.\left. -(\vec{v}, [\vec{\xi}_{n+1}]))A^{-1} - \vec{z})\right)\right|^{p_1'} d\vec{z}\right]^{\frac{p_2}{p_1'}} d\vec{u} \\ &= [|\det(A^{-1})|2^{\frac{r}{2}}]^{\frac{p_2}{p_1}}\|f_r\|_{p_1}^{p_2} \int_{\mathbb{R}^r}\left[\left(|\Psi(\lambda, \cdot)|^{p_1'} * \left|g_r\left(\frac{1}{\sqrt{2}}T_A\cdot\right)\right|^{p_1'}\right)\right. \\ &\quad \left.((\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]))A^{-1})\right]^{\frac{p_2}{p_1'}} d\vec{u}. \end{aligned}$$

Let $\vec{z} = (\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]))A^{-1}$. By Young's inequality and the change of variable theorem

$$\begin{aligned}\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2} d\vec{u} &\leq |\det(A)|[|\det(A^{-1})|2^{\frac{r}{2}}]^{\frac{p_2}{p_1}} \|f_r\|_{p_1}^{p_2} \||\Psi(\lambda,\cdot)|^{p_1'}\|_1^{\frac{p_2}{p_1'}} \\ &\quad \times \left\| \left| g_r\left(\frac{1}{\sqrt{2}} T_A \cdot\right)\right|^{p_1'} \right\|_{\frac{p_2}{p_1'}}^{\frac{p_2}{p_1'}} \\ &= |\det(A)|[|\det(A^{-1})|2^{\frac{r}{2}}]^{\frac{p_2}{p_1}+1} \|f_r\|_{p_1}^{p_2} \|\Psi(\lambda,\cdot)\|_{p_1'}^{p_2} \|g_r\|_{p_2}^{p_2} < \infty.\end{aligned}$$

Suppose that $1 = p_1' \leq p_2 < \infty$. Then $p_1 = \infty$. Hence by the change of variable theorem and Young's inequality

$$\begin{aligned}&\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2} d\vec{u} \\ \leq\ & \|f_r\|_\infty^{p_2} \int_{\mathbb{R}^r} \left[\left(|\Psi(\lambda,\cdot)| * \left| g_r\left(\frac{1}{\sqrt{2}} T_A\cdot\right)\right|\right)((\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]))A^{-1})\right]^{p_2} d\vec{u} \\ \leq\ & |\det(A)|\, \|f_r\|_\infty^{p_2} \|\Psi(\lambda,\cdot)\|_1^{p_2} \left\| g_r\left(\frac{1}{\sqrt{2}} T_A\cdot\right)\right\|_{p_2}^{p_2} \\ =\ & 2^{\frac{r}{2}} \|f_r\|_\infty^{p_2} \|\psi(\lambda,\cdot)\|_1^{p_2} \|g_r\|_{p_2}^{p_2} < \infty.\end{aligned}$$

Let $1 < p_1' \leq p_2 = \infty$. Then we have by the change of variable theorem and Hölder's inequality

$$\begin{aligned}|\gamma_1(\vec{u})| &\leq \|g_r\|_\infty \int_{\mathbb{R}^r} \left| f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{z}]\right)\Psi(\lambda,\vec{z})\right| d\vec{z} \\ &\leq \|g_r\|_\infty [|\det(A^{-1})|2^{\frac{r}{2}}]^{\frac{1}{p_1}} \|f_r\|_{p_1} \|\Psi(\lambda,\cdot)\|_{p_1'}\end{aligned}$$

for $\vec{u} \in \mathbb{R}^r$. Let $1 = p_1' \leq p_2 = \infty$. Then $p_1 = \infty$ and we have

$$|\gamma_1(\vec{u})| \leq \|f_r\|_\infty \|g_r\|_\infty \|\Psi(\lambda,\cdot)\|_1$$

for $\vec{u} \in \mathbb{R}^r$. Consequently we have $\gamma_1 \in L_{p_2}(\mathbb{R}^r)$ for $1 \leq p \leq \infty$ and hence $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(p_2)}$. Similarly we can prove $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(p_1)}$ if $p_1 \geq p_2'$. Note that the existence of $[(F_r * G_r)_\lambda | X_{n+1}]$ follows from Morera's theorem with an aid of the dominated convergence theorem. □

Theorem 3.3 *Let X_{n+1} be given by (2) and q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have the followings:*

1. *if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(1)}$,*
2. *if $F_r, G_r \in \mathcal{A}_r^{(2)}$, then $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(\infty)}$,*

3. *if* $F_r \in \mathcal{A}_r^{(1)}$ *and* $G_r \in \mathcal{A}_r^{(2)}$*, then* $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(2)}$*,*
4. *if* $F_r \in \mathcal{A}_r^{(1)}$ *and* $G_r \in \mathcal{A}_r^{(1)} \cap \mathcal{A}_r^{(2)}$*, then* $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(1)} \cap \mathcal{A}_r^{(2)}$*, and*
5. *if* $F_r \in \mathcal{A}_r^{(1)}$ *and* $G_r \in \mathcal{A}_r^{(\infty)}$*, then* $[(F_r * G_r)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(\infty)}$*.*

Proof. Let F_r, G_r and f_r, g_r be related by (6), respectively. Furthermore, for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, let γ_1 be given by (11).

1. The result follows from Theorem 2.4 of [5].
2. We have for $\vec{u} \in \mathbb{R}^r$

$$\begin{aligned} |\gamma_1(\vec{u})|^2 &\leq \|\Psi(\lambda,\cdot)\|_\infty^2 \left[\int_{\mathbb{R}^r} \left| f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{z}]\right)\right|^2 d\vec{z}\right] \\ &\quad \times \left[\int_{\mathbb{R}^r} \left| g_r\left(\frac{1}{\sqrt{2}}[\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{z}]\right)\right|^2 d\vec{z}\right] \\ &= 2^r |\det(A^{-1})|^2 \|\Psi(\lambda,\cdot)\|_\infty^2 \|f_r\|_2^2 \|g_r\|_2^2 < \infty \end{aligned}$$

   by Hölder's inequality and the change of variable theorem. By the dominated convergence theorem, $[(F_r * G_r)_q | X_{n+1}]$ exists and (2) follows.

3. We have

$$\begin{aligned} &\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^2 d\vec{u} \\ \leq\ & \|\Psi(\lambda,\cdot)\|_\infty^2 \int_{\mathbb{R}^r} \left[\int_{\mathbb{R}^r} \left| f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{z}]\right)\right| \left| g_r\left(\frac{1}{\sqrt{2}}[\vec{u} - \right.\right.\right. \\ & \left.\left.\left.(\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{z}]\right)\right| d\vec{z}\right] \left[\int_{\mathbb{R}^r} \left| f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{w}]\right)\right| \right| g_r \\ & \left.\left(\frac{1}{\sqrt{2}}[\vec{u} - (\vec{v}, [\vec{\xi}_{n+1}]) - T_A\vec{w}]\right)\right| d\vec{w}\right] d\vec{u}. \end{aligned}$$

   Let $\vec{\alpha} = \frac{1}{\sqrt{2}}(\vec{u} + T_A\vec{z})$ and $\vec{\beta} = \frac{1}{\sqrt{2}}(\vec{u} + T_A\vec{w})$. Then we have by the change of variable theorem and Hölder's inequality

$$\begin{aligned} &\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^2 d\vec{u} \\ \leq\ & 2^r |\det(A^{-1})|^2 \|\Psi(\lambda,\cdot)\|_\infty^2 \int_{\mathbb{R}^r} \left| f_r\left(\vec{\alpha} + \frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}])\right)\right| \int_{\mathbb{R}^r} \left| f_r\left(\vec{\beta}\right.\right. \\ & \left.\left. + \frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}])\right)\right| \int_{\mathbb{R}^r} \left| g_r\left(\sqrt{2}\vec{u} - \frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}]) - \vec{\alpha}\right)\right| \left| g_r\left(\sqrt{2}\vec{u} - \right.\right. \\ & \left.\left.\frac{1}{\sqrt{2}}(\vec{v}, [\vec{\xi}_{n+1}]) - \vec{\beta}\right)\right| d\vec{u}\, d\vec{\beta}\, d\vec{\alpha} \\ \leq\ & 2^{\frac{r}{2}} |\det(A^{-1})|^2 \|\Psi(\lambda,\cdot)\|_\infty^2 \|f_r\|_1^2 \|g_r\|_2^2 < \infty. \end{aligned}$$

4. The result follows from (1) and (3).

5. It follows immediately from $F_r \in \mathcal{A}_r^{(1)}$ and the dominated convergence theorem.

Now the proof is completed. □

Now applying the same method as used in the proof of Theorem 4.1 of [5], we have the following theorem from Theorems 2.2 and 3.2.

Theorem 3.4 *Let $F_r, G_r \in \cup_{1\le p\le\infty}\mathcal{A}_r^{(p)}$ and let X_{n+1} be given by (2). Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, we have*

$$\begin{aligned} &T_\lambda[[(F_r * G_r)_\lambda|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ =\ & \left[T_\lambda[F_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1})\right)\right] \\ &\times\left[T_\lambda[G_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1})\right)\right]. \end{aligned}$$

We have the following relationships between the conditional Fourier-Feynman transform and the convolution products from Theorems 2.3, 3.3, 3.4 and Theorem 4.1 of [5].

Theorem 3.5 *Let X_{n+1} be given by (2) and q be nonzero real. Then we have the followings;*

1. *if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then we have for w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned} &T_q^{(1)}[[(F_r * G_r)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ =\ & \left[T_q^{(1)}[F_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1})\right)\right] \\ &\times\left[T_q^{(1)}[G_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1})\right)\right], \end{aligned}$$

2. *if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(2)}$, then we have for w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned} &T_q^{(2)}[[(F_r * G_r)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ =\ & \left[T_q^{(1)}[F_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1})\right)\right] \\ &\times\left[T_q^{(2)}[G_r|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1})\right)\right]. \end{aligned}$$

4 Evaluation formulas for bounded cylinder functions

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{z}\rangle_{\mathbb{R}^r}\} d\rho(\vec{z}), \tag{12}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. For w_φ-a.e. $x \in C[0,t]$, let Φ be given by

$$\Phi(x) = \phi(\vec{v}, x) \tag{13}$$

where ϕ is given by (12).

Now we have the following theorem.

Theorem 4.1 *Let $1 \le p \le \infty$. Let X_{n+1} and Φ be given by (2) and (13), respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1})$ exists and it is given by*

$$\begin{aligned} & T_\lambda[\Phi|X_{n+1}](y, \vec{\xi}_{n+1}) \\ = & \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y) + (\vec{v}, [\vec{\xi}_{n+1}]), \vec{z}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r} \right\} d\rho(\vec{z}), \end{aligned} \tag{14}$$

where A^T is the transpose of A and $T_{A^T}\vec{z} = \vec{z}A^T$ for $\vec{z} \in \mathbb{R}^r$. For nonzero real q, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $T_q^{(p)}[\Phi|X_{n+1}](y, \vec{\xi}_{n+1})$ exists and it is given by (14), where λ is replaced by $-iq$. Furthermore, as functions of y, $T_q^{(p)}[\Phi|X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in \mathcal{A}_r^{(\infty)}$.

Proof. For $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ we have by Theorem 2.2

$$\begin{aligned} & T_\lambda[\Phi|X_{n+1}](y, \vec{\xi}_{n+1}) \\ = & \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \phi((\vec{v}, y + [\vec{\xi}_{n+1}]) + T_A\vec{z}) \exp\left\{-\frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\right\} d\vec{z} \\ = & \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\{i\langle (\vec{v}, y + [\vec{\xi}_{n+1}]), \vec{u}\rangle_{\mathbb{R}^r}\} \int_{\mathbb{R}^r} \exp\left\{ i\langle \vec{z}, T_{A^T}\vec{u}\rangle_{\mathbb{R}^r} - \frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r} \right\} \\ & d\vec{z}\, d\rho(\vec{u}) \\ = & \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + [\vec{\xi}_{n+1}]), \vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r} \right\} d\rho(\vec{u}), \end{aligned}$$

where the last equality follows from the well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{15}$$

for $a \in \mathbb{C}_+$ and any real b. Let $T_q^{(p)}[\Phi|X_{n+1}](y,\vec{\xi}_{n+1})$ be formally given by (14), where λ is replaced by $-iq$. For $p=1$, the final result follows from the dominated convergence theorem. Now let $1<p\le\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then we have

$$|T_\lambda[\Phi|X_{n+1}](y,\vec{\xi}_n) - T_q^{(p)}[\Phi|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'} \le (2\|\rho\|)^{p'}$$

so that by the dominated convergence theorem

$$\int_C |T_\lambda[\Phi|X_{n+1}](y,\vec{\xi}_{n+1}) - T_q^{(p)}[\Phi|X_{n+1}](y,\vec{\xi}_{n+1})|^{p'} dw_\varphi(y)$$

converges to 0 as λ approaches $-iq$ through $\mathbb{C}_+$. □

Theorem 4.2 *Let $1\le p\le\infty$. Then, under the assumptions as given in Theorem 4.1, we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1},\vec{\zeta}_{n+1}\in\mathbb{R}^{n+2}$*

$$\|T_{\bar\lambda}[T_\lambda[\Phi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](\cdot,\vec{\zeta}_{n+1}) - \phi(\vec{v},\cdot+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])\|_p \to 0$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

Proof. By Theorem 4.1, $T_{\bar\lambda}[T_\lambda[\Phi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$ is well-defined so that we have for $\lambda\in\mathbb{C}_+$

$$\begin{aligned}
&T_{\bar\lambda}[T_\lambda[\Phi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})\\
=\ &\int_{\mathbb{R}^r}\exp\bigg\{i\langle(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r} - \frac{1}{2\bar\lambda}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r}\bigg\}\\
&d\rho(\vec{z})\\
=\ &\int_{\mathbb{R}^r}\exp\bigg\{i\langle(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r} - \frac{\mathrm{Re}\,\lambda}{|\lambda|^2}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r}\bigg\}d\rho(\vec{z})
\end{aligned}$$

which converges to $\phi(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])$ as λ approaches $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. We have

$$\begin{aligned}
&|T_{\bar\lambda}[T_\lambda[\Phi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) - \phi(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}])|\\
=\ &\bigg|\int_{\mathbb{R}^r}\bigg[\exp\bigg\{i\langle(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r} - \frac{\mathrm{Re}\,\lambda}{|\lambda|^2}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r}\bigg\}\\
&-\exp\{i\langle(\vec{v},y+[\vec{\xi}_{n+1}+\vec{\zeta}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r}\}\bigg]d\rho(\vec{z})\bigg|\\
\le\ &\int_{\mathbb{R}^r}\bigg|\exp\bigg\{-\frac{\mathrm{Re}\,\lambda}{|\lambda|^2}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r}\bigg\}-1\bigg|d|\rho|(\vec{z})
\end{aligned}$$

so that the inequality is independent of y, and for $1 \leq p < \infty$

$$\int_C |T_{\bar{\lambda}}[T_\lambda[\Phi|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) - \phi(\vec{v}, y + [\vec{\xi}_{n+1} + \vec{\zeta}_{n+1}])|^p dw_\varphi(y) \leq \left[\int_{\mathbb{R}^r} \left|\exp\left\{-\frac{\operatorname{Re}\lambda}{|\lambda|^2}\|T_{A^T}\vec{z}\|^2_{\mathbb{R}^r}\right\} - 1\right| d|\rho|(\vec{z})\right]^p$$

which converges to 0 as λ approaches $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem which completes the proof. □

Theorem 4.3 *Let ϕ_1, ϕ_2 and ρ_1, ρ_2 be related by (12), respectively, and let $\Phi_1(x) = \phi_1(\vec{v}, x)$ and $\Phi_2(x) = \phi_2(\vec{v}, x)$ for w_φ-a.e. $x \in C[0,t]$. Furthermore let X_{n+1} be given by (2). Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $[(\Phi_1 * \Phi_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})$ exists and it is given by*

$$[(\Phi_1 * \Phi_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) = \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\left\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{z}+\vec{w}\rangle_{\mathbb{R}^r} + \langle(\vec{v},[\vec{\xi}_{n+1}]), \vec{z}-\vec{w}\rangle_{\mathbb{R}^r}] - \frac{1}{4\lambda}\|T_{A^T}(\vec{z}-\vec{w})\|^2_{\mathbb{R}^r}\right\} d\rho_1(\vec{z})d\rho_2(\vec{w}),$$

where T_{A^T} is as given in Theorem 4.1. For nonzero real q, $[(\Phi_1 * \Phi_2)_q|X_{n+1}](y, \vec{\xi}_{n+1})$ is given by the above equation, where λ is replaced by $-iq$. Furthermore, as a function of y, $[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1}) \in \mathcal{A}_r^{(\infty)}$.

Proof. By Theorem 3.2, we have for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}
&[(\Phi_1 * \Phi_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) \\
= & \int_{\mathbb{R}^r} \phi_1\left(\frac{1}{\sqrt{2}}[(\vec{v}, y + [\vec{\xi}_{n+1}]) + T_A\vec{z}]\right)\phi_2\left(\frac{1}{\sqrt{2}}[(\vec{v}, y - [\vec{\xi}_{n+1}]) - T_A\vec{z}]\right) \\
&\times \Psi(\lambda,\vec{z})d\vec{z} \\
= & \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\left\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r} + \langle(\vec{v},[\vec{\xi}_{n+1}]),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}]\right\} \\
&\times \int_{\mathbb{R}^r} \exp\left\{\frac{i}{\sqrt{2}}\langle T_A\vec{z}, \vec{u}-\vec{w}\rangle_{\mathbb{R}^r} - \frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\right\} d\vec{z}d\rho_1(\vec{u})d\rho_2(\vec{w}) \\
= & \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\left\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r} + \langle(\vec{v},[\vec{\xi}_{n+1}]),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}] - \frac{1}{4\lambda}\|T_{A^T}(\vec{u} \right. \\
&\left. -\vec{w})\|^2_{\mathbb{R}^r}\right\} d\rho_1(\vec{u})d\rho_2(\vec{w}),
\end{aligned}$$

where the last equality follows from (15). By the dominated convergence theorem, we have the theorem. □

Now we have the final theorem of our work.

Theorem 4.4 *Let X_{n+1} be given by (2), q be nonzero real and $1 \le p \le \infty$. Furthermore let Φ_1 and Φ_2 be as given in Theorem 4.3. Then we have for w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned} & T_q^{(p)}[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ = & \left[T_q^{(p)}[\Phi_1|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}+\vec{\xi}_{n+1})\right)\right] \\ & \times\left[T_q^{(p)}[\Phi_2|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1}-\vec{\xi}_{n+1})\right)\right]. \end{aligned}$$

Proof. By Theorems 2.2 and 4.3 we have for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} & T_\lambda[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ = & \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\bigg\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y+[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r}+\langle(\vec{v},y+[\vec{\zeta}_{n+1} \\ & -\vec{\xi}_{n+1}]),\vec{w}\rangle_{\mathbb{R}^r}]+\frac{1}{4qi}\|T_{A^T}(\vec{z}-\vec{w})\|^2_{\mathbb{R}^r}+\frac{i}{\sqrt{2}}\langle\vec{u},T_{A^T}(\vec{z}+\vec{w})\rangle_{\mathbb{R}^r}-\frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\bigg\} \\ & d\vec{u}\,d\rho_1(\vec{z})d\rho_2(\vec{w}) \\ = & \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\bigg\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y+[\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}]),\vec{z}\rangle_{\mathbb{R}^r}+\langle(\vec{v},y+[\vec{\zeta}_{n+1}-\vec{\xi}_{n+1}]), \\ & \vec{w}\rangle_{\mathbb{R}^r}]+\frac{1}{4qi}\|T_{A^T}(\vec{z}-\vec{w})\|^2_{\mathbb{R}^r}-\frac{1}{4\lambda}\|T_{A^T}(\vec{z}+\vec{w})\|^2_{\mathbb{R}^r}\bigg\}d\rho_1(\vec{z})d\rho_2(\vec{w}), \end{aligned}$$

where the last equality follows from (15). Let $T_q^{(p)}[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|$ $X_{n+1}]$ $(y,\vec{\zeta}_{n+1})$ be given by the right hand side of the last equality, where λ is replaced by $-iq$. The existence of $T_q^{(1)}[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$ follows from the dominated convergence theorem. Now let $1 < p \le \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then we have by the dominated convergence theorem

$$\begin{aligned} & \int_C |T_\lambda[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1}) \\ & -T_q^{(p)}[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})|^{p'}dw_\varphi(y) \\ \le & \bigg[\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\bigg|\exp\bigg\{-\frac{1}{4\lambda}\|T_{A^T}(\vec{z}+\vec{w})\|^2_{\mathbb{R}^r}\bigg\}-\exp\bigg\{\frac{1}{4qi}\|T_{A^T}(\vec{z}+\vec{w})\|^2_{\mathbb{R}^r}\bigg\}\bigg| \\ & d|\rho_1|(\vec{z})d|\rho_2|(\vec{w})\bigg]^{p'} \longrightarrow 0 \end{aligned}$$

as λ approaches $-iq$ through $\mathbb{C}_+$ which shows the existence of $T_q^{(p)}[[(\Phi_1 * \Phi_2)_q|X_{n+1}](\cdot,\vec{\xi}_{n+1})|X_{n+1}](y,\vec{\zeta}_{n+1})$. Now the equality in the theorem follows from Theorems 3.4, 4.1 and 4.3. □

Remark 4.5 *Without using Theorem 3.4, we can prove Theorem 4.4 with aids of Theorems 4.1 and 4.3.*

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