A Present Position-Dependent Conditional Fourier-Feynman Transform and Convolution Product over Continuous Paths

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Abstract

Let $C[0,t]$ denote the function space of all real-valued continuous paths on $[0,t]$. Define $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ by $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$. In the present paper, using a simple formula for the conditional expectation with the conditioning function $X_{n+1}$, we evaluate the $L_p(1 \leq p \leq \infty)$-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions which have the form $f((v_1, x), \cdots, (v_r, x))$ for $x \in C[0,t]$, where $\{v_1, \cdots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and $f \in L_p(\mathbb{R}^r)$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions.
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1 Introduction and preliminaries

Let $C_0[0, t]$ denote the Wiener space, that is, the space of the real-valued continuous functions $x$ on the closed interval $[0, t]$ with $x(0) = 0$. On the space $C_0[0, t]$, the concept of an analytic Fourier-Feynman transform was introduced by Brue [1]. Huffman, Park and Skoug [9] defined a convolution product on $C_0[0, t]$ and then, established various relationships between the analytic Fourier-Feynman transform and the convolution product. Furthermore Chang and Skoug [4] introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product on the Wiener space $C_0[0, t]$. In that paper they also examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were studied by Chang, Cho, Kim, Song and Yoo [3, 7]. In fact Cho and his coauthors [3] introduced the $L_1$-analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and then, established the relationships between them of certain cylinder type functions. Cho [7] extended the relationships between the conditional convolution product and the $L_p(1 \leq p \leq 2)$-analytic conditional Fourier-Feynman transform of the same kind of cylinder type functions. Moreover, on $C[0, t]$, the space of the real-valued continuous paths on $[0, t]$, Kim [12] extended the relationships between the conditional convolution product and the $L_p(1 \leq p \leq \infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick’s Banach algebra $S$ [2]. Cho [5] introduced the $L_1$-analytic conditional Fourier-Feynman transform and the conditional convolution product over $C[0, t]$ and then, established the relationships between them of certain cylinder type functions.

In this paper we further develop the relationships in [3, 5, 7, 12] on the more generalized space $(C[0, t], w_\varphi)$, the analogue of the Wiener space associated with the probability measure $\varphi$ on the Borel class $B(\mathbb{R})$ of $\mathbb{R}$ [10, 14, 15]. For the conditioning function $X_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+2}$ given by $X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0, t]$, we proceed to study the relationships between the conditional convolution product and the analytic conditional Fourier-Feynman transform of the cylinder functions defined on $C[0, t]$. In fact, using a simple formula for the conditional $w_\varphi$-integrals given $X_{n+1}$ [6], we evaluate the $L_p(1 \leq p \leq \infty)$-
measure $\phi$ is called an analogue of the Wiener measure associated with the probability algebra generated by $I$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions. Finally we show that the $L^p$-analytic conditional Fourier-Feynman transform $T_q^{(p)}[((F \ast G)_q|X_{n+1}]$ of the conditional convolution product for the cylinder functions $F$ and $G$, can be expressed by the formula

$$T_q^{(p)}[((F \ast G)_q|X_{n+1}] = \left[ T_q^{(p)}[F|X_{n+1}] \left( \frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\tilde{\xi}_{n+1} + \tilde{\zeta}_{n+1}) \right) \right] \times \left[ T_q^{(p)}[G|X_{n+1}] \left( \frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\tilde{\zeta}_{n+1} - \tilde{\xi}_{n+1}) \right) \right]$$

for a nonzero real $q$, $w\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\tilde{\xi}_{n+1}, \tilde{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X_{n+1}}$ is the probability distribution of $X_{n+1}$ on the Borel class of $\mathbb{R}^{n+2}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the cylinder functions, can be interpreted as the product of the conditional analytic Fourier-Feynman transform of each function.

Throughout this paper let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and the set of complex numbers with positive real parts, respectively.

Now we introduce the concrete form of the probability measure $w\varphi$ on $(C[0,t], \mathcal{B}(C[0,t]))$. For a positive real $t$ let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \leq t$ let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. For $B_j (j = 0, 1, \ldots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure $\varphi$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let

$$m_{\varphi} \left[J_{\vec{t}}^{-1} \left( \prod_{j=0}^{n} B_j \right) \right] = \left[ \prod_{j=1}^{n} \frac{1}{2 \pi (t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^{n} B_j} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d(u_1, \ldots, u_n) d\varphi(u_0).$$

Then $\mathcal{B}(C[0,t])$, the Borel $\sigma$-algebra of $C[0,t]$ coincides with the smallest $\sigma$-algebra generated by $\mathcal{I}$ and there exists a unique probability measure $w\varphi$ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $w\varphi(I) = m_{\varphi}(I)$ for all $I$ in $\mathcal{I}$. This measure $w\varphi$ is called an analogue of the Wiener measure associated with the probability measure $\varphi$ [10, 14, 15, 17].
Let \( \{ e_k : k = 1, 2, \cdots \} \) be a complete orthonormal subset of \( L_2[0, t] \) such that each \( e_k \) is of bounded variation. For \( v \) in \( L_2[0, t] \) and \( x \) in \( C[0, t] \) let \( (v, x) = \lim_{n \to \infty} \sum_{k=1}^{n} \langle v, e_k \rangle \int_{0}^{t} e_k(s)dx(s) \) if the limit exists, where \( \langle \cdot, \cdot \rangle \) denotes the inner product over \( L_2[0, t] \). \( (v, x) \) is called the Paley-Wiener-Zygmund integral of \( v \) according to \( x \). We note that the dot product on the \( r \)-dimensional Euclidean space \( \mathbb{R}^r \) is also denoted by \( \langle \cdot, \cdot \rangle_{\mathbb{R}^r} \).

Applying Theorem 3.5 in [10], we can easily prove the following theorem.

**Theorem 1.1** Let \( \{ h_1, h_2, \cdots, h_r \} \) be an orthonormal subset of \( L_2[0, t] \). For \( i = 1, 2, \cdots, r \), let \( Z_i(x) = (h_i, x) \) on \( C[0, t] \). Then \( Z_1, Z_2, \cdots, Z_r \) are independent and each \( Z_i \) has the standard normal distribution. Moreover, if \( f : \mathbb{R}^r \to \mathbb{R} \) is Borel measurable, then

\[
\int_{C} f(Z_1(x), Z_2(x), \cdots, Z_r(x))dw(x) = \left( \frac{1}{2\pi} \right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(u_1, u_2, \cdots, u_r) \exp \left\{ -\frac{1}{2} \sum_{j=1}^{r} u_j^2 \right\} d(u_1, u_2, \cdots, u_r),
\]

where \( * \) means that if either side exists then both sides exist and they are equal.

Let \( F : C[0, t] \to \mathbb{C} \) be integrable and \( X \) be a random vector on \( C[0, t] \) assuming that the value space of \( X \) is a normed space equipped with the Borel \( \sigma \)-algebra. Then we have the conditional expectation \( E[F|X] \) of \( F \) given \( X \) from a well known probability theory [13]. Furthermore there exists a \( P_X \)-integrable complex-valued function \( \psi \) on the value space of \( X \) such that \( E[F|X](x) = (\psi \circ X)(x) \) for \( w \)-a.e. \( x \in C[0, t] \), where \( P_X \) is the probability distribution of \( X \). The function \( \psi \) is called the conditional \( w \)-integral of \( F \) given \( X \) and it is also denoted by \( E[F|X] \).

Throughout this paper let \( 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t \) be a partition of \([0, t]\) unless otherwise specified. For any \( x \) in \( C[0, t] \) define the polygonal function \([x] \) of \( x \) by

\[
[x](s) = \sum_{j=1}^{n+1} \chi_{(t_{j-1}, t_j]}(s) \left( \frac{t_j - s}{t_j - t_{j-1}} x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} x(t_j) \right) + \chi_{(t_0]}(s)x(t_0)
\]

for \( s \in [0, t] \), where \( \chi_{(t_{j-1}, t_j]} \) and \( \chi_{(t_0]} \) denote the indicator functions. Similarly, for \( \tilde{\mathbf{x}}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2} \), define the polygonal function \([\tilde{\mathbf{x}}_{n+1}] \) of \( \tilde{\mathbf{x}}_{n+1} \) by (1), where \( x(t_j) \) is replaced by \( \xi_j \) for \( j = 0, 1, \cdots, n + 1 \).

In the following theorem we introduce a simple formula for the conditional \( w \)-integrals on \( C[0, t] \) [6].
Theorem 1.2 Let $F : C[0, t] \to \mathbb{C}$ be integrable and let $X_{n+1} : C[0, t] \to \mathbb{R}^{n+2}$ be given by

$$X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1})).$$

(2)

Then, for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$,

$$E[F|X_{n+1}](\xi_{n+1}) = E[F(x - [x] + [\xi_{n+1}]),

(3)

where the expectation is taken over the variable $x$ and $P_{X_{n+1}}$ is the probability distribution of $X_{n+1}$ on $(\mathbb{R}^{n+2}, B(\mathbb{R}^{n+2}))$.

For a function $F : C[0, t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ and $X_{n+1}^\lambda(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$, where $X_{n+1}$ is given by (2). Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional $w_\varphi$-integral and (3), $E[F^\lambda|X_{n+1}^\lambda](\xi_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\xi_{n+1}])$ for $P_{X_{n+1}}^\lambda$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X_{n+1}}^\lambda$ is the probability distribution of $X_{n+1}^\lambda$ on $(\mathbb{R}^{n+2}, B(\mathbb{R}^{n+2}))$.

Throughout this paper, for $y \in C[0, t]$ let $I_F^\lambda(y, \xi_{n+1}) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi_{n+1}])$ unless otherwise specified, where the expectation is taken over the variable $y$. If $I_F^\lambda(0, \xi_{n+1})$ has the analytic extension $J_F^\lambda(\xi_{n+1})$ on $\mathbb{C}_+$ as a function of $\lambda$, then it is called the analytic conditional Wiener $w_\varphi$-integral of $F$ given $X_{n+1}$ with parameter $\lambda$ and denoted by $E^{anw_\varphi}[F|X_{n+1}]^\lambda(\xi_{n+1}) = J_F^\lambda(\xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a nonzero real $q$, $E^{anw_\varphi}[F|X_{n+1}]^\lambda(\xi_{n+1})$ has the limit as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic conditional Feynman $w_\varphi$-integral of $F$ given $X_{n+1}$ with parameter $\lambda$ and denoted by $E^{anF_\varphi}[F|X_{n+1}]^\lambda(\xi_{n+1}) = \lim_{\lambda \to -iq} E^{anw_\varphi}[F|X_{n+1}]^\lambda(\xi_{n+1})$.

2 The conditional Fourier-Feynman transform

For a given extended real number $p$ with $1 < p \leq \infty$ suppose that $p$ and $p'$ are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let $F_n$ and $F$ be measurable functions such that $\lim_{n \to \infty} J_F^\lambda(x) = F(x)$ and $\lim_{n \to \infty} \int_{C} |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $I_n(w^{p'})(F_n) = F$ and call $F$ the limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by a continuously varying parameter.

We now define the analytic conditional Fourier-Feynman transform of the functions on $C[0, t]$.

Definition 2.1 Let $F$ be defined on $C[0, t]$ and $X_{n+1}$ be given by (2). For $\lambda \in \mathbb{C}_+$ and $w_\varphi$-a.e. $y \in C[0, t]$, let

$$T_\lambda[F|X_{n+1}](y, \xi_{n+1}) = E^{anw_\varphi}[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi_{n+1}])$$

for $\xi_{n+1} \in \mathbb{R}^{n+2}$, where $E^{anw_\varphi}[F|X_{n+1}]^\lambda(\xi_{n+1})$. 


for $P_{X_{n+1}}$-a.e. $\tilde{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a nonzero real $q$ and $w$-$a.e.$ $y \in C[0, t]$, define the $L_1$-analytic conditional Fourier-Feynman transform $T^{(1)}_q[F|X_{n+1}]$ of $F$ given $X_{n+1}$ by the formula

$$T^{(1)}_q[F|X_{n+1}](y, \tilde{\xi}_{n+1}) = E_{anf}^q[F(y + \cdot)|X_{n+1}](\tilde{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\tilde{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \leq \infty$, define the $L_p$-analytic conditional Fourier-Feynman transform $T^{(p)}_q[F|X_{n+1}]$ of $F$ given $X_{n+1}$ by the formula

$$T^{(p)}_q[F|X_{n+1}](\cdot, \tilde{\xi}_{n+1}) = \lim_{\lambda \to -iq} \left( w^p \right) (T_\lambda[F|X_{n+1}](\cdot, \tilde{\xi}_{n+1}))$$

for $P_{X_{n+1}}$-a.e. $\tilde{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$. For each $j = 1, \cdots, n+1$, let $\alpha_j = \frac{1}{\sqrt{t_j - t_{j-1}}} \chi_{(t_{j-1}, t_j]}$ on $[0, t]$. Let $V$ be the subspace of $L_2[0, t]$ generated by $\{\alpha_1, \cdots, \alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of $V$. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0, t]$ to $V$ and $V^\perp$, respectively. Throughout this paper let $\{v_1, v_2, \cdots, v_r\}$ be an orthonormal subset of $L_2[0, t]$ such that $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ is an independent set. Let $\{e_1, \cdots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \cdots, r$, let

$$\mathcal{P}^\perp v_l = \sum_{j=1}^r \alpha_{lj} e_j$$

be the linear combinations of the $e_j$s and let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{r1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1r} & \alpha_{2r} & \cdots & \alpha_{rr} \end{bmatrix}$$

be the transpose of the coefficient matrix of the combinations. We can also regard $A$ as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by

$$T_A \vec{z} = \vec{z} A,$$

where $\vec{z}$ is any row-vector in $\mathbb{R}^r$. We note that $A$ is invertible so that $T_A$ is an isomorphism.

For convenience let $(\vec{v}, x) = ((v_1, x), \cdots, (v_r, x))$ for $x \in C[0, t]$. For $1 \leq p \leq \infty$ let $\mathcal{A}^{(p)}_r$ be the space of the cylinder functions $F_r$ of the form given by

$$F_r(x) = f_r(\vec{v}, x)$$

for $P_{X_{n+1}}$-a.e. $\tilde{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists.
for \( w_{\varphi} \)-a.e. \( x \in C[0, t] \), where \( f_r \in L_p(\mathbb{R}^r) \). We note that, without loss of generality, we can take \( f_r \) to be Borel measurable.

We evaluate the conditional Fourier-Feynman transform of cylinder function in the following theorem.

**Theorem 2.2** Let \( X_{n+1} \) be given by (2). Let \( F_r \in A_r^{(p)} (1 \leq p \leq \infty) \) and \( f_r \) be related by (6). Then for \( \lambda \in \mathbb{C}_+ \), \( w_{\varphi} \)-a.e. \( y \in C[0, t] \) and \( P_{X_{n+1}} \)-a.e. \( \xi_{n+1} \in \mathbb{R}^{n+2} \), \( T_{\lambda}[F_r|X_{n+1}](y, \xi_{n+1}) \) exists and it is given by

\[
T_{\lambda}[F_r|X_{n+1}](y, \xi_{n+1}) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^r} f_r((\vec{v}, y) + (\vec{v}, [\xi_{n+1}]) + T_A \vec{z}) \exp \left\{ -\frac{\lambda}{2} \| \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\vec{z},
\]

where \( T_A \) is given by (5). As a function of \( y \), \( T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1}) \in A_r^{(p)} \).

**Proof.** For \( \lambda > 0 \), \( w_{\varphi} \)-a.e. \( y \in C[0, t] \) and \( P_{X_{n+1}} \)-a.e. \( \xi_{n+1} \in \mathbb{R}^{n+2} \)

\[
I_{F_r}^\lambda(y, \xi_{n+1}) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^r} f_r((\vec{v}, y) + (\vec{v}, [\xi_{n+1}]) + T_A \vec{z}) \exp \left\{ -\frac{\lambda}{2} \| \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\vec{z}
\]

by Lemma 2.1 of [5]. We note that if \( 1 \leq p < \infty \), then by the change of variable theorem

\[
\int_{\mathbb{R}^r} |f_r((\vec{v}, y) + (\vec{v}, [\xi_{n+1}]) + T_A \vec{z})|^p d\vec{z} = |\det(A^{-1})| \| f_r \|_{p}^p < \infty,
\]

where \( A^{-1} \) is the inverse of the matrix \( A \) given by (5). Now, by Morera’s theorem with aids of Hölder’s inequality and the dominated convergence theorem, we have (7) for \( \lambda \in \mathbb{C}_+ \). To prove the remainder of the theorem, for \( \lambda \in \mathbb{C}_+ \) and for \( \vec{u} \in \mathbb{R}^r \), let

\[
\Psi(\lambda, \vec{u}) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \exp \left\{ -\frac{\lambda}{2} \| \vec{u} \|_{\mathbb{R}^r}^2 \right\}.
\]

By the change of variable theorem

\[
\gamma(\lambda, \vec{u}, \xi_{n+1}) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^r} f_r((\vec{u}, y) + (\vec{u}, [\xi_{n+1}]) + T_A \vec{z}) \exp \left\{ -\frac{\lambda}{2} \| \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\vec{z}
\]

\[
= \int_{\mathbb{R}^r} f_r((T_A \vec{z} - T_A \vec{z}) \Psi(\lambda, \vec{u})) d\vec{z}
\]

\[
= (f_r(T_A \cdot) \Psi(\lambda, \vec{u}))(\vec{z} + T_A^{-1}).
\]

By (8), \( f_r(T_A \cdot) \in L_p(\mathbb{R}^r) \) and \( \Psi(\lambda, \vec{u}) \in L_1(\mathbb{R}^r) \) so that \( f_r(T_A \cdot) \Psi(\lambda, \vec{u}) \in L_p(\mathbb{R}^r) \) by Young’s inequality [8, p.232]. Now \( \gamma(\lambda, \vec{u}, \xi_{n+1}) = (f_r(T_A \cdot) \Psi(\lambda, \vec{u}))(\vec{z} + T_A^{-1}) \in L_p(\mathbb{R}^r) \) by the change of variable theorem which completes the proof. \( \square \)
Theorem 2.3 Let $X_{n+1}$ be given by (2). Let $F_r \in \mathcal{A}_r^{(p)}(1 \leq p \leq 2)$ and $f_r$ be related by (6). Then for a nonzero real $q$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$, $T_q^{(p)}[F_r|X_{n+1}](y,\xi_{n+1})$ exists and it is given by (7), where $\lambda$ is replaced by $-iq$. Furthermore, as a function of $y$, $T_q^{(p)}[F_r|X_{n+1}](\cdot,\xi_{n+1}) \in \mathcal{A}_r^{(p')}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 < p \leq 2$ and $p' = \infty$ if $p = 1$.

Proof. When $p = 1$, the results follow from Theorem 2.2 of [5]. Suppose that $1 < p \leq 2$. For either $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, for $u \in \mathbb{R}^r$ and $\xi_{n+1} \in \mathbb{R}^{n+2}$, let $\Psi(\lambda, u)$ and $\gamma(\lambda, u, \xi_{n+1})$ be given by (9) and (10), respectively. By (8), $f_r(T_{A'}A)$ belongs to $L_p(\mathbb{R}^r)$ so that $f_r(T_{A'}A)\ast \Psi(\lambda, \cdot) \in L_p(\mathbb{R}^r)$ by Lemma 1.1 of [11]. Now $\gamma(\lambda, \cdot, \xi_{n+1}) = (f_r(T_{A'}A)\ast \Psi(\lambda, \cdot))(((\bar{v}, [\bar{\xi}_{n+1}]) + A)A^{-1})$ belongs to $L_p(\mathbb{R}^r)$ by the change of variable theorem. Let $T_q^{(p)}[F_r|X_{n+1}](y,\xi_{n+1}) = \gamma(-iq, (\bar{v}, y), \xi_{n+1})$ formally for $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$. Then

$$
\int_C |T_\lambda[F_r|X_{n+1}](y,\xi_{n+1}) - T_q^{(p)}[F_r|X_{n+1}](y,\xi_{n+1})|^{p'}dw_\varphi(y)
$$

$$
= \int_C \left| \gamma(\lambda, (\bar{v}, y), \xi_{n+1}) - \gamma(-iq, (\bar{v}, y), \xi_{n+1}) \right|^{p'}dw_\varphi(y)
$$

$$
= \left( \frac{1}{2\pi} \right)^{\frac{\lambda}{2}} \int_{\mathbb{R}^r} \left| \gamma(\lambda, u, \xi_{n+1}) - \gamma(-iq, \bar{u}, \xi_{n+1}) \right|^{p'} \exp \left\{ -\frac{1}{2} \| u \|_{\mathbb{R}^r}^2 \right\} \| u \|_{\mathbb{R}^r}^{p'} \| u \|_{\mathbb{R}^r} \right\} \| u \|_{\mathbb{R}^r}^{p'} \right\} \| u \|_{\mathbb{R}^r} \right\} \| u \|_{\mathbb{R}^r} \right\} \| u \|_{\mathbb{R}^r} \right\} \| u \|_{\mathbb{R}^r} \right\}
$$

by Theorems 1.1 and 2.2. Let $z = ((\bar{v}, [\bar{\xi}_{n+1}]) + \bar{u})A^{-1}$. Then, by the change of variable theorem,

$$
\int_C |T_\lambda[F_r|X_{n+1}](y,\xi_{n+1}) - T_q^{(p)}[F_r|X_{n+1}](y,\xi_{n+1})|^{p'}dw_\varphi(y)
$$

$$
\leq |\det(A)| \int_{\mathbb{R}^r} |(f_r(T_{A'}A)\ast \Psi(\lambda, \cdot))(z) - (f_r(T_{A'}A)\ast \Psi(-iq, \cdot))(z)|^{p'}dz
$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by Lemma 1.2 of [11]. Now the proof is completed. \hfill \Box

Theorem 2.4 Let $X_{n+1}$ be given by (2). Let $F_r \in \mathcal{A}_r^{(p)}(1 \leq p \leq \infty)$ and $f_r$ be related by (6). For $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1}, \xi_{n+1} \in \mathbb{R}^{n+2}$, let $F_r(y, \xi_{n+1}, \xi_{n+1}) = f_r((\bar{v}, y) + (\bar{v}, [\bar{\xi}_{n+1} + \xi_{n+1} + \xi_{n+1}]))$. Then, for a nonzero real $q$,

$$
\int_C |T_\lambda[T_\lambda[F_r|X_{n+1}](\cdot,\xi_{n+1})|X_{n+1}](y,\xi_{n+1}) - F_r(y,\xi_{n+1},\xi_{n+1})|^{p'}dw_\varphi(y) \to 0
$$
for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$

$$T_X[T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \zeta_{n+1}) \to F_{r1}(y, \xi_{n+1}, \zeta_{n+1})$$

as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$. 

**Proof.** Note that $T_X[T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \zeta_{n+1})$ is well-defined by Theorem 2.2. For $\lambda \in \mathbb{C}_+$, $w_\varphi$-a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2}$,

$$T_X[T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \zeta_{n+1})$$

$$= \left( \frac{|\lambda|^2}{4\pi \text{Re} \lambda} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f_r(T_A((\bar{\nu}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y))A^{-1} - i\bar{\eta})$$

$$\times \exp \left\{ - \frac{|\lambda|^2}{4\text{Re} \lambda}\|\bar{\eta}\|^2_{\mathbb{R}^n} \right\} d\bar{\eta}$$

$$= \left( \frac{|\lambda|^2}{2\text{Re} \lambda} \right)^{\frac{n}{2}} \left( f_r(T_{A'}) \ast \Psi \left( 1, \left( \frac{|\lambda|^2}{2\text{Re} \lambda} \right)^{\frac{1}{2}} \right) \right) \left( (\bar{\nu}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y) \right)A^{-1}$$

by Theorem 2.3 of [5], where $\Psi$ is given by (9). We have $\int_{\mathbb{R}^n} \Psi(1, \bar{u})d\bar{u} = 1$ and hence $\Psi(1, \cdot) \in L_1(\mathbb{R}^n)$, and $\sup \{|\Psi(1, \bar{z})| : \|\bar{z}\|_{\mathbb{R}^n} \geq \|\bar{u}\|_{\mathbb{R}^n} \} = \Psi(1, \bar{u})$ is an $L_1$-function of $\bar{u}$. Furthermore $f_r(T_{A'})$ is in $L_1(\mathbb{R}^n)$ by (8). Let $\epsilon = \left( \frac{2\text{Re} \lambda}{|\lambda|^2} \right)^{\frac{1}{2}} > 0$.

By Theorem 1.1 and Theorem 1.18 of [16]

$$\int_C |T_X[T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \zeta_{n+1}) - F_{r1}(y, \xi_{n+1}, \zeta_{n+1})|^p dw_\varphi(y)$$

$$= \int_C \epsilon^{-\tau} \left( f_r(T_{A'}) \ast \Psi \left( 1, \left( \frac{|\lambda|^2}{2\text{Re} \lambda} \right)^{\frac{1}{2}} \right) \right) \left( (\bar{v}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y) \right)A^{-1} - f_r(T_A((\bar{v}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y))A^{-1}) \right|^p dw_\varphi(y)$$

$$\leq |\det(A)| \int_{\mathbb{R}^n} \left| \epsilon^{-\tau} \left( f_r(T_{A'}) \ast \Psi \left( 1, \left( \frac{|\lambda|^2}{2\text{Re} \lambda} \right)^{\frac{1}{2}} \right) \right) \right|^p d\bar{u}$$

which converges to 0 if $1 \leq p < \infty$, and by Theorem 1.25 of [16]

$$T_X[T_{\lambda}[F_r|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \zeta_{n+1})$$

$$= \epsilon^{-\tau} \left( f_r(T_{A'}) \ast \Psi \left( 1, \left( \frac{|\lambda|^2}{2\text{Re} \lambda} \right)^{\frac{1}{2}} \right) \right) \left( (\bar{v}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y) \right)A^{-1}$$

which converges to

$$f_r(T_A((\bar{v}, [\xi_{n+1} + \zeta_{n+1}]) + (\bar{v}, y))A^{-1}) = F_{r1}(y, \xi_{n+1}, \zeta_{n+1})$$

if $1 \leq p \leq \infty$, as $\lambda$ approaches $-iq$ through $\mathbb{C}_+$. \[\square\]
3 The conditional convolution product

In this section we derive the conditional convolution product of cylinder functions and investigate various relationships between the conditional Fourier-Feynman transform and convolution product. To do this we need the following definition.

**Definition 3.1** Let \(X_{n+1}\) be given by (2), and \(F\) and \(G\) be defined on \(C[0,t]\). Define the conditional convolution product \([ (F * G)_\lambda |X_{n+1} ]\) of \(F\) and \(G\) given \(X_{n+1}\) by the formula, for \(w_\varphi\)-a.e. \(y \in C[0,t]\)

\[
[(F * G)_\lambda |X_{n+1}](y, \xi_{n+1}) = \left\{
\begin{array}{ll}
E^\Lambda_{w_\varphi} \left[ F \left( \frac{y + \cdot}{\sqrt{2}} \right) G \left( \frac{y - \cdot}{\sqrt{2}} \right) \right]_{X_{n+1}} (\xi_{n+1}), & \lambda \in \mathbb{C}_+; \\
E^\Lambda_{w_\varphi} \left[ F \left( \frac{y + \cdot}{\sqrt{2}} \right) G \left( \frac{y - \cdot}{\sqrt{2}} \right) \right]_{X_{n+1}} (\xi_{n+1}), & \lambda = -iq; \ q \in \mathbb{R} - \{0\}
\end{array}\right.
\]

if they exist for \(P_{X_{n+1}}\)-a.e. \(\xi_{n+1} \in \mathbb{R}^{n+2}\). If \(\lambda = -iq\), we can replace \([ (F * G)_\lambda |X_{n+1} ]\) by \([ (F * G)_{iq} |X_{n+1} ]\).

**Theorem 3.2** Let \(F_r \in \mathcal{A}_r^{(p_1)}, G_r \in \mathcal{A}_r^{(p_2)}\) and \(f_r, g_r\) be related by (6), respectively, where \(1 \leq p_1, p_2 \leq \infty\). Furthermore let \(\frac{1}{p_1} + \frac{1}{p'_1} = 1\), \(\frac{1}{p_2} + \frac{1}{p'_2} = 1\) and \(X_{n+1}\) be given by (2). Then for \(\lambda \in \mathbb{C}_+, w_\varphi\)-a.e. \(y \in C[0,t]\) and \(P_{X_{n+1}}\)-a.e. \(\xi_{n+1} \in \mathbb{R}^{n+2}\), \([ (F_r * G_r)_\lambda |X_{n+1} ](y, \xi_{n+1})\) exists and it is given by

\[
[(F_r * G_r)_\lambda |X_{n+1}](y, \xi_{n+1}) = \int_{\mathbb{R}^r} f_r \left( \frac{1}{\sqrt{2}} \left((\vec{v}, y + [\vec{\xi}_{n+1}]) + T_A \vec{z} \right) \right) \times g_r \left( \frac{1}{\sqrt{2}} \left((\vec{v}, y - [\vec{\xi}_{n+1}]) - T_A \vec{z} \right) \right) \Psi(\lambda, \vec{z}) d\vec{z},
\]

where \(T_A\) and \(\Psi\) are given by (5) and (9), respectively. Furthermore, as functions of \(y\), \([ (F_r * G_r)_\lambda |X_{n+1} ](y, \xi_{n+1}) \in \mathcal{A}_r^{(1)}\) if either \(p_2 \leq p'_1\) or \(p_1 \leq p'_2\), \([ (F_r * G_r)_\lambda |X_{n+1} ](y, \xi_{n+1}) \in \mathcal{A}_r^{(p_2)}\) if \(p_2 \geq p'_1\) and \([ (F_r * G_r)_\lambda |X_{n+1} ](y, \xi_{n+1}) \in \mathcal{A}_r^{(p_1)}\) if \(p_1 \geq p'_2\).

**Proof.** Using the same method as used in the proof of Theorem 2.4 of [5], for \(\lambda > 0\), \(w_\varphi\)-a.e. \(y \in C[0,t]\) and \(P_{X_{n+1}}\)-a.e. \(\xi_{n+1} \in \mathbb{R}^{n+2}\),

\[
[(F_r * G_r)_\lambda |X_{n+1}](y, \xi_{n+1}) = \int_{\mathbb{R}^r} f_r \left( \frac{1}{\sqrt{2}} \left((\vec{v}, y + [\vec{\xi}_{n+1}]) + T_A \vec{z} \right) \right) \times g_r \left( \frac{1}{\sqrt{2}} \left((\vec{v}, y - [\vec{\xi}_{n+1}]) - T_A \vec{z} \right) \right) \Psi(\lambda, \vec{z}) d\vec{z}
\]

by Lemma 2.1 of [5]. Now, for \( \lambda \in \mathbb{C}_+ \) and \( \vec{u} \in \mathbb{R}^r \), let

\[
\gamma_1(\vec{u}) = \int_{\mathbb{R}^r} f_r \left( \frac{1}{\sqrt{2}} (\vec{u} + (\vec{v}, [\xi_{n+1}]) + T_A \vec{z}) \right)
\times g_r \left( \frac{1}{\sqrt{2}} (\vec{u} - (\vec{v}, [\xi_{n+1}]) - T_A \vec{z}) \right) \Psi(\lambda, \vec{z}) d\vec{z}
\]

formally, and suppose that \( p_2 \leq p'_1 \). Let \( \vec{p} = \frac{1}{\sqrt{2}} (\vec{u} + T_A \vec{z}) \) and \( \vec{q} = \frac{1}{\sqrt{2}} (\vec{u} - T_A \vec{z}) \). Then \( \vec{u} = \frac{1}{\sqrt{2}} (\vec{p} + \vec{q}) \) and \( \vec{z} = \frac{1}{\sqrt{2}} (\vec{p} - \vec{q}) A^{-1} \). By the change of variable theorem,

\[
\int_{\mathbb{R}^r} |\gamma_1(\vec{u})| d\vec{u} \leq |\det(A^{-1})| \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f_r \left( \vec{p} + \frac{1}{\sqrt{2}} (\vec{v}, [\xi_{n+1}]) \right) \times g_r \left( \vec{q} - \frac{1}{\sqrt{2}} (\vec{v}, [\xi_{n+1}]) \right) \Psi \left( \lambda, \frac{1}{\sqrt{2}} (\vec{p} - \vec{q}) A^{-1} \right) d\vec{q} d\vec{p},
\]

where \( f_{r_1}(\vec{p}) = f_r(\vec{p} + \frac{1}{\sqrt{2}} (\vec{v}, [\xi_{n+1}])) \), \( g_{r_1}(\vec{p}) = g_r(\vec{p} - \frac{1}{\sqrt{2}} (\vec{v}, [\xi_{n+1}])) \) and \( \Psi_1(\vec{p}) = \Psi(\lambda, \frac{1}{\sqrt{2}} \vec{p} A^{-1}) = (\frac{\lambda}{2\pi})^\frac{n}{2} \exp\{ -\frac{1}{4} \| \vec{p} A^{-1} \|^2 \} \). Now let \( \frac{1}{p_2} + \frac{1}{q} = \frac{1}{p'_1} + 1 \). Since \( p_2 \leq p'_1 \), we have \( \frac{1}{q} = 1 + \frac{1}{p'_1} - \frac{1}{p_2} = \frac{p'_1 p_2 - p_2}{p'_1 p_2} \) and \( 0 \leq p'_1 (p_2 - 1) + p_2 = p'_1 p_2 + p_2 - p'_1 \leq p'_1 p_2 \) if \( 1 \leq p_1 \leq \infty \) and \( 1 \leq p_2 < \infty \). Hence \( 1 \leq q \leq \infty \) for \( 1 \leq p_1, p_2 \leq \infty \). On the other hand, \( \Psi_1 \in L_q(\mathbb{R}^r) \). Now by the general form of Young’s inequality [8, Theorem 8.9] and Hölder’s inequality

\[
\int_{\mathbb{R}^r} |\gamma_1(\vec{u})| d\vec{u} \leq |\det(A^{-1})| \| f_{r_1} \|_{p_1} \| (g_{r_1} * |\Psi_1|) \|_{p'_1} \leq |\det(A^{-1})| \| f_{r_1} \|_{p_1} \| g_{r_1} \|_{p_2} \| \Psi_1 \|_{q} < \infty
\]

which shows that \( \gamma_1 \in L_1(\mathbb{R}^r) \) and hence \( (F_r * G_r)_\lambda |X_{n+1}\rangle \left( \cdot, \xi_{n+1} \right) \in \mathcal{A}_r^{(1)} \). Similarly \( (F_r * G_r)_\lambda |X_{n+1}\rangle \left( \cdot, \xi_{n+1} \right) \in \mathcal{A}_r^{(1)} \) if \( p_1 \leq p'_1 \). Suppose that \( 1 < p'_1 \leq p_2 < \infty \). Then by Hölder’s inequality and the change of variable theorem

\[
\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2} d\vec{u} \leq \| \det(A^{-1}) \|^{2\frac{p_2}{p'_1}} \| f_{r_1} \|_{p_1}^{p_2} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \left( \Psi(\lambda, \vec{z}) g_r \left( \frac{1}{\sqrt{2}} T_A ((\vec{u} - (\vec{v}, [\xi_{n+1}])) A^{-1} - \vec{z}) \right) \right) \frac{p_2}{p'_1} d\vec{z} d\vec{u} - (\vec{v}, [\xi_{n+1}])) A^{-1} - \vec{z} \right) \right)^{p_2} d\vec{z} d\vec{u}
\]

\[
= \| \det(A^{-1}) \|^{2\frac{p_2}{p'_1}} \| f_{r_1} \|_{p_1}^{p_2} \int_{\mathbb{R}^r} \left( \left| \Psi(\lambda, \cdot) \right|^{p'_1} \left| g_r \left( \frac{1}{\sqrt{2}} T_A \right) \right|^{p'_1} \right) ((\vec{u} - (\vec{v}, [\xi_{n+1}])) A^{-1})^{p_2} d\vec{u}.
\]
Let \( \vec{z} = (\vec{u} - (\vec{v}, [\xi_{n+1}]))A^{-1} \). By Young’s inequality and the change of variable theorem

\[
\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2}d\vec{u} \leq \|\det(A)||\det(A^{-1})|2^\frac{p_2}{p_1}\|f_r\|_{p_2}^{p_2}\|\Psi(\lambda, \cdot)|_{p_1}^{p_2} \times \left\|g_r\left(\frac{1}{\sqrt{2}}T_A^\sigma\right)\right\|_{p_2}^{p_2} \\
= \|\det(A)||\det(A^{-1})|2^\frac{p_2}{p_1}\|f_r\|_{p_2}^{p_2}\|\Psi(\lambda, \cdot)|_{p_1}^{p_2}\|g_r\|_{p_2}^{p_2} < \infty.
\]

Suppose that \( 1 = p_1' \leq p_2 < \infty \). Then \( p_1 = \infty \). Hence by the change of variable theorem and Young’s inequality

\[
\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2}d\vec{u} \leq \|f_r\|_{\infty}^{p_2}\int_{\mathbb{R}^r} \left[\left|\Psi(\lambda, \cdot)\right|g_r\left(\frac{1}{\sqrt{2}}T_A^\sigma\right)\right]^{p_2}((\vec{u} - (\vec{v}, [\xi_{n+1}]))A^{-1})d\vec{u} \\
\leq \|\det(A)||f_r\|_{\infty}^{p_2}\|\Psi(\lambda, \cdot)|_{p_1}^{p_2}\left\|g_r\left(\frac{1}{\sqrt{2}}T_A^\sigma\right)\right\|_{p_2}^{p_2} \\
= 2^\frac{p_2}{p_1}||f_r\|_{\infty}^{p_2}\|\psi(\lambda, \cdot)|_{p_1}^{p_2}\|g_r\|_{p_2}^{p_2} < \infty.
\]

Let \( 1 < p_1' \leq p_2 = \infty \). Then we have by the change of variable theorem and Hölder’s inequality

\[
|\gamma_1(\vec{u})| \leq \|g_r\|_{\infty}\int_{\mathbb{R}^r} \left|f_r\left(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{v}, [\xi_{n+1}]) + T_A\vec{z}]\right)\Psi(\lambda, \vec{z})\right|d\vec{z} \\
\leq \|g_r\|_{\infty}\|\det(A^{-1})|2^\frac{1}{p_1}\|f_r\|_{p_1}\|\Psi(\lambda, \cdot)|_{p_1'}
\]

for \( \vec{u} \in \mathbb{R}^r \). Let \( 1 = p_1' \leq p_2 = \infty \). Then \( p_1 = \infty \) and we have

\[
|\gamma_1(\vec{u})| \leq \|f_r\|_{\infty}\|g_r\|_{\infty}\|\Psi(\lambda, \cdot)|_{1}
\]

for \( \vec{u} \in \mathbb{R}^r \). Consequently we have \( \gamma_1 \in L_{p_2}(\mathbb{R}^r) \) for \( 1 \leq p \leq \infty \) and hence \([F_r * G_r)_\lambda|X_{n+1}|(\cdot, \xi_{n+1}) \in A_r^{(p_2)}\). Similarly we can prove \([F_r * G_r)_\lambda|X_{n+1}|(\cdot, \xi_{n+1}) \in A_r^{(p_1)}\) if \( p_1 \geq p_2' \). Note that the existence of \([F_r * G_r)_\lambda|X_{n+1}| \) follows from Morera’s theorem with an aid of the dominated convergence theorem.

**Theorem 3.3** Let \( X_{n+1} \) be given by (2) and \( q \) be a nonzero real number. Then for \( \lambda \in \mathbb{C}_+ \) or \( \lambda = -iq \), and \( \xi_{n+1} \in \mathbb{R}^{n+2} \), we have the following:

1. If \( F_r, G_r \in A_r^{(1)} \), then \([F_r * G_r)_\lambda|X_{n+1}|(\cdot, \xi_{n+1}) \in A_r^{(1)}\),

2. If \( F_r, G_r \in A_r^{(2)} \), then \([F_r * G_r)_\lambda|X_{n+1}|(\cdot, \xi_{n+1}) \in A_r^{(\infty)}\),
3. if \( F_r \in A_r(1) \) and \( G_r \in A_r(2) \), then \([(F_r * G_r)_\lambda |X_{n+1}](\cdot, \tilde{\xi}_{n+1}) \in A_r(2)\),

4. if \( F_r \in A_r(1) \) and \( G_r \in A_r(1) \cap A_r(2) \), then \([(F_r * G_r)_\lambda |X_{n+1}](\cdot, \tilde{\xi}_{n+1}) \in A_r(1) \cap A_r(2) \), and

5. if \( F_r \in A_r(1) \) and \( G_r \in A_r(\infty) \), then \([(F_r * G_r)_\lambda |X_{n+1}](\cdot, \tilde{\xi}_{n+1}) \in A_r(\infty)\).

**Proof.** Let \( F_r, G_r \) and \( f_r, g_r \) be related by (6), respectively. Furthermore, for \( \lambda \in \mathbb{C}_+ \) or \( \lambda = -i\bar{q} \), let \( \gamma_1 \) be given by (11).

1. The result follows from Theorem 2.4 of [5].

2. We have for \( \bar{u} \in \mathbb{R}^r \)

\[
|\gamma_1(\bar{u})|^2 \leq \|\Psi(\lambda, \cdot)\|_{\infty}^2 \left[ \int_{\mathbb{R}^r} \left| f_r \left( \frac{1}{\sqrt{2}}[\bar{u} + (\bar{v}, [\tilde{\xi}_{n+1}]) + T_A \bar{z}] \right) \right|^2 d\bar{z} \right] \\
\times \left[ \int_{\mathbb{R}^r} \left| g_r \left( \frac{1}{\sqrt{2}}[\bar{u} - (\bar{v}, [\tilde{\xi}_{n+1}]) - T_A \bar{z}] \right) \right|^2 d\bar{z} \right] \\
= 2^r |\det(A^{-1})|^2 \|\Psi(\lambda, \cdot)\|_{\infty}^2 \|f_r\|_2^2 \|g_r\|_2^2 < \infty
\]

by Hölder’s inequality and the change of variable theorem. By the dominated convergence theorem, \([(F_r * G_r)_q |X_{n+1}] \) exists and (2) follows.

3. We have

\[
\int_{\mathbb{R}^r} |\gamma_1(\bar{u})|^2 d\bar{u} \\
\leq \|\Psi(\lambda, \cdot)\|_{\infty}^2 \int_{\mathbb{R}^r} \left[ \int_{\mathbb{R}^r} \left| f_r \left( \frac{1}{\sqrt{2}}[\bar{u} + (\bar{v}, [\tilde{\xi}_{n+1}]) + T_A \bar{z}] \right) \right| g_r \left( \frac{1}{\sqrt{2}}[\bar{u} - (\bar{v}, [\tilde{\xi}_{n+1}]) - T_A \bar{z}] \right) d\bar{z} \right] \int_{\mathbb{R}^r} \left| f_r \left( \frac{1}{\sqrt{2}}[\bar{u} + (\bar{v}, [\tilde{\xi}_{n+1}]) + T_A \bar{w}] \right) \right| g_r \left( \frac{1}{\sqrt{2}}[\bar{u} - (\bar{v}, [\tilde{\xi}_{n+1}]) - T_A \bar{w}] \right) d\bar{u} d\bar{w}.
\]

Let \( \alpha = \frac{1}{\sqrt{2}}(\bar{u} + T_A \bar{z}) \) and \( \beta = \frac{1}{\sqrt{2}}(\bar{u} + T_A \bar{w}) \). Then we have by the change of variable theorem and Hölder’s inequality

\[
\int_{\mathbb{R}^r} |\gamma_1(\bar{u})|^2 d\bar{u} \\
\leq 2^r |\det(A^{-1})|^2 \|\Psi(\lambda, \cdot)\|_{\infty}^2 \int_{\mathbb{R}^r} \left| f_r \left( \alpha + \frac{1}{\sqrt{2}}(\bar{v}, [\tilde{\xi}_{n+1}]) \right) \right| \int_{\mathbb{R}^r} \left| f_r \left( \beta \right) \right| \left| g_r \left( \sqrt{2}\bar{u} - \frac{1}{\sqrt{2}}(\bar{v}, [\tilde{\xi}_{n+1}]) - \alpha \right) \right| \left| g_r \left( \sqrt{2}\bar{u} - \frac{1}{\sqrt{2}}(\bar{v}, [\tilde{\xi}_{n+1}]) - \beta \right) \right| d\bar{u} d\bar{v} d\alpha d\beta \\
\leq 2^r |\det(A^{-1})|^2 \|\Psi(\lambda, \cdot)\|_{\infty}^2 \|f_r\|_2^2 \|g_r\|_2^2 < \infty.
\]
4. The result follows from (1) and (3).

5. It follows immediately from $F_r \in \mathcal{A}_r^{(1)}$ and the dominated convergence theorem.

Now the proof is completed. \qed

Now applying the same method as used in the proof of Theorem 4.1 of [5], we have the following theorem from Theorems 2.2 and 3.2.

**Theorem 3.4** Let $F_r, G_r \in \cup_{1 \leq p \leq \infty} \mathcal{A}_r^{(p)}$ and let $X_{n+1}$ be given by (2). Then for $\lambda \in \mathbb{C}_+, w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2}$, we have

$$T_\lambda [(F_r * G_r)_\lambda |X_{n+1}](\cdot, \xi_{n+1}) |X_{n+1}](y, \zeta_{n+1})$$

$$= \left[ T_\lambda [F_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} + \xi_{n+1})) \right]$$

$$\times \left[ T_\lambda [G_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} - \xi_{n+1})) \right].$$

We have the following relationships between the conditional Fourier-Feynman transform and the convolution products from Theorems 2.3, 3.3, 3.4 and Theorem 4.1 of [5].

**Theorem 3.5** Let $X_{n+1}$ be given by (2) and $q$ be nonzero real. Then we have the followings;

1. if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then we have for $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2}$

$$T_q^{(1)} [(F_r * G_r)_q |X_{n+1}](\cdot, \xi_{n+1}) |X_{n+1}](y, \zeta_{n+1})$$

$$= \left[ T_q^{(1)} [F_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} + \xi_{n+1})) \right]$$

$$\times \left[ T_q^{(1)} [G_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} - \xi_{n+1})) \right],$$

2. if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(2)}$, then we have for $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_{n+1}}$-a.e. $\xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2}$

$$T_q^{(2)} [(F_r * G_r)_q |X_{n+1}](\cdot, \xi_{n+1}) |X_{n+1}](y, \zeta_{n+1})$$

$$= \left[ T_q^{(1)} [F_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} + \xi_{n+1})) \right]$$

$$\times \left[ T_q^{(2)} [G_r |X_{n+1}](\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} - \xi_{n+1})) \right].$$
4 Evaluation formulas for bounded cylinder functions

Let \( \hat{M}(\mathbb{R}^r) \) be the set of all functions \( \phi \) on \( \mathbb{R}^r \) defined by

\[
\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{z} \rangle_{\mathbb{R}^r}\}d\rho(\vec{z}), \tag{12}
\]

where \( \rho \) is a complex Borel measure of bounded variation over \( \mathbb{R}^r \). For \( w_\varphi \)-a.e. \( x \in C[0, t] \), let \( \Phi \) be given by

\[
\Phi(x) = \phi(\vec{v}, x) \tag{13}
\]

where \( \phi \) is given by (12).

Now we have the following theorem.

**Theorem 4.1** Let \( 1 \leq p \leq \infty \). Let \( X_{n+1} \) and \( \Phi \) be given by (2) and (13), respectively. Then for \( \lambda \in \mathbb{C}_+ \), \( w_\varphi \)-a.e. \( y \in C[0, t] \) and \( P_{X_{n+1}} \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \), \( T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) \) exists and it is given by

\[
T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) = \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + \vec{\xi}_{n+1}), \vec{z} \rangle_{\mathbb{R}^r} - \frac{\lambda}{2\lambda} \| T_{A^T} \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\rho(\vec{z}), \tag{14}
\]

where \( A^T \) is the transpose of \( A \) and \( T_{A^T} = \vec{z}A^T \) for \( \vec{z} \in \mathbb{R}^r \). For nonzero real \( q \), \( w_\varphi \)-a.e. \( y \in C[0, t] \) and \( P_{X_{n+1}} \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \), \( T_q^{(p)}[\Phi|X_{n+1}](y, \vec{\xi}_{n+1}) \) exists and it is given by (14), where \( \lambda \) is replaced by \(-iq\). Furthermore, as functions of \( y \), \( T_q^{(p)}[\Phi|X_{n+1}](\cdot, \vec{\xi}_{n+1}) \in A_{r^{(\infty)}} \).

**Proof.** For \( \lambda \in \mathbb{C}_+ \), \( w_\varphi \)-a.e. \( y \in C[0, t] \) and \( P_{X_{n+1}} \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \) we have by Theorem 2.2

\[
T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1})
= \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^r} \phi((\vec{v}, y + [\vec{\xi}_{n+1}] + T_A \vec{z}) \exp\left\{-\frac{\lambda}{2} \| \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\vec{z}
= \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + [\vec{\xi}_{n+1}], \vec{u} \rangle_{\mathbb{R}^r} \right\} \int_{\mathbb{R}^r} \exp\left\{ i\langle \vec{z}, T_{A^T} \vec{u} \rangle_{\mathbb{R}^r} - \frac{\lambda}{2} \| \vec{z} \|_{\mathbb{R}^r}^2 \right\} d\vec{z} d\rho(\vec{u})
= \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + [\vec{\xi}_{n+1}], \vec{u} \rangle_{\mathbb{R}^r} - \frac{1}{2\lambda} \| T_{A^T} \vec{u} \|_{\mathbb{R}^r}^2 \right\} d\rho(\vec{u}),
\]

where the last equality follows from the well-known integration formula

\[
\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left( \frac{\pi}{a} \right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a} \right\} \tag{15}
\]
for $a \in \mathbb{C}_+$ and any real $b$. Let $T^{(p)}_q[\Phi|X_{n+1}](y, \xi_{n+1})$ be formally given by (14), where $\lambda$ is replaced by $-iq$. For $p = 1$, the final result follows from the dominated convergence theorem. Now let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have

$$|T_\lambda[\Phi|X_{n+1}](y, \xi_n) - T^{(p)}_q[\Phi|X_{n+1}](y, \xi_{n+1})|^p' \leq (2\|\rho\|)^{p'}$$

so that by the dominated convergence theorem

$$\int_C |T_\lambda[\Phi|X_{n+1}](y, \xi_{n+1}) - T^{(p)}_q[\Phi|X_{n+1}](y, \xi_{n+1})|^p' d\varphi(y)$$

converges to 0 as $\lambda$ approaches $-iq$ through $\mathbb{C}_+$.

\begin{proof}
By Theorem 4.1, $T_\lambda[T_\lambda[\Phi|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \xi_{n+1})$ is well-defined so that we have for $\lambda \in \mathbb{C}_+$

$$T_\lambda[T_\lambda[\Phi|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \xi_{n+1}) = \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}], \vec{z} \rangle_{\mathbb{R}^r} - \frac{1}{2\lambda} \|T_{A^r} \vec{z}\|^2_{\mathbb{R}^r} - \frac{1}{2\lambda} \|T_{A^r} \vec{z}\|^2_{\mathbb{R}^r} \right\} d\rho(\vec{z})$$

$$= \int_{\mathbb{R}^r} \exp\left\{ i\langle (\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}], \vec{z} \rangle_{\mathbb{R}^r} - \frac{\text{Re}\lambda}{|\lambda|^2} \|T_{A^r} \vec{z}\|^2_{\mathbb{R}^r} \right\} d\rho(\vec{z})$$

which converges to $\phi(\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}]$ as $\lambda$ approaches $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. We have

$$|T_\lambda[T_\lambda[\Phi|X_{n+1}](\cdot, \xi_{n+1})|X_{n+1}](y, \xi_{n+1}) - \phi(\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}])|$$

$$= \int_{\mathbb{R}^r} \left[ \exp\left\{ i\langle (\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}], \vec{z} \rangle_{\mathbb{R}^r} - \frac{\text{Re}\lambda}{|\lambda|^2} \|T_{A^r} \vec{z}\|^2_{\mathbb{R}^r} \right\} - \exp\left\{ i\langle (\vec{v}, y + [\xi_{n+1} + \vec{\xi}_{n+1}], \vec{z} \rangle_{\mathbb{R}^r} \right\} \right] d\rho(\vec{z})$$

$$\leq \int_{\mathbb{R}^r} \left| \exp\left\{ -\frac{\text{Re}\lambda}{|\lambda|^2} \|T_{A^r} \vec{z}\|^2_{\mathbb{R}^r} \right\} - 1 \right| d|\rho|(\vec{z})$$
so that the inequality is independent of \( y \), and for \( 1 \leq p < \infty \)
\[
\int_C |T_XT_\lambda[\Phi|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}(y, \vec{\xi}_{n+1}) - \phi(\vec{v}), \quad y + [\vec{\xi}_{n+1} + \vec{\xi}_{n+1}]| \, dw(x) \leq \left[ \int_{\mathbb{R}^r} \left\{ -\frac{\text{Re} \lambda}{|\lambda|^2} \|T_{A^r}\vec{z}\|_{\mathbb{R}^r}^2 \right\}^{\frac{1}{2}} - 1 \right] \, |d\rho| \cdot (\vec{z}) \right]^{\frac{p}{2}}
\]
which converges to 0 as \( \lambda \) approaches \(-iq\) through \( C_+ \) by the dominated convergence theorem which completes the proof. □

**Theorem 4.3** Let \( \phi_1, \phi_2 \) and \( \rho_1, \rho_2 \) be related by (12), respectively, and let \( \Phi_1(x) = \phi_1(\vec{v}, x) \) and \( \Phi_2(x) = \phi_2(\vec{v}, x) \) for \( w_\varphi \)-a.e. \( x \in C[0, t] \). Furthermore, let \( X_{n+1} \) be given by (2). Then for \( \lambda \in C_+, w_\varphi \)-a.e. \( y \in C[0, t] \) and \( P_x \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \), \( [\Phi_1 \Phi_2](\cdot, \vec{\xi}_{n+1}) \) exists and it is given by

\[
[(\Phi_1 \Phi_2)(\cdot, \vec{\xi}_{n+1})] = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp \left\{ \frac{i}{2\sqrt{2}} \left[ \langle \vec{v}, \vec{z} + \vec{w} \rangle_{\mathbb{R}^r} + \langle \vec{v}, \vec{\xi}_{n+1} \rangle \right] \right\} d\rho_1(\vec{z}) d\rho_2(\vec{w}),
\]

where \( T_{A^r} \) is as given in Theorem 4.1. For nonzero real \( q \), \( [\Phi_1 \Phi_2](\cdot, \vec{\xi}_{n+1}) \) is given by the above equation, where \( \lambda \) is replaced by \(-iq\). Furthermore, as a function of \( y \), \( [\Phi_1 \Phi_2](\cdot, \vec{\xi}_{n+1}) \) \( \in \mathcal{A}(\infty) \).

**Proof.** By Theorem 3.2, we have for \( \lambda \in C_+, w_\varphi \)-a.e. \( y \in C[0, t] \) and \( P_x \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \)

\[
[\Phi_1 \Phi_2](\cdot, \vec{\xi}_{n+1})
= \int_{\mathbb{R}^r} \phi_1 \left( \frac{1}{\sqrt{2}} \left[ \langle \vec{v}, \vec{z} + \vec{\xi}_{n+1} \rangle + T_A\vec{z} \right] \right) \phi_2 \left( \frac{1}{\sqrt{2}} \left[ \langle \vec{v}, \vec{z} - \vec{\xi}_{n+1} \rangle - T_A\vec{z} \right] \right)
\]

\[
\times \Psi(\lambda, \vec{z}) \, d\vec{z}
\]

\[
= \left( \frac{\lambda}{2\pi} \right)^{-\frac{n}{2}} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp \left\{ \frac{i}{2} \left[ \langle \vec{v}, \vec{z} + \vec{w} \rangle_{\mathbb{R}^r} + \langle \vec{v}, \vec{\xi}_{n+1} \rangle , \vec{u} - \vec{w} \rangle_{\mathbb{R}^r} \right] \right\}
\times \int_{\mathbb{R}^r} \exp \left\{ \frac{i}{\sqrt{2}} \left( T_A\vec{z} \right) \, \vec{u} - \vec{w} \right\}_{\mathbb{R}^r} \, - \frac{\lambda}{2} \|\vec{z}\|_{\mathbb{R}^r}^2 \right\} \, d\vec{z} d\rho_1(\vec{u}) d\rho_2(\vec{w})
\]

\[
= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp \left\{ -\frac{i}{\sqrt{2}} \left[ \langle \vec{v}, \vec{z} + \vec{w} \rangle_{\mathbb{R}^r} + \langle \vec{v}, \vec{\xi}_{n+1} \rangle , \vec{u} - \vec{w} \rangle_{\mathbb{R}^r} \right] - \frac{1}{4\lambda} \|T_A\vec{u} - \vec{w}\|_{\mathbb{R}^r}^2 \right\} \, d\rho_1(\vec{u}) d\rho_2(\vec{w}),
\]

where the last equality follows from (15). By the dominated convergence theorem, we have the theorem. □

Now we have the final theorem of our work.
Theorem 4.4 Let \( X_{n+1} \) be given by (2), \( q \) be nonzero real and \( 1 \leq p \leq \infty \). Furthermore let \( \Phi_1 \) and \( \Phi_2 \) be as given in Theorem 4.3. Then we have for \( w_\varphi \)-a.e. \( y \in C[0,t] \) and \( P_{X_{n+1}} \)-a.e. \( \xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2} \)

\[
T_q^{(p)}[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) = \left[ T_q^{(p)}[\Phi_1 | X_{n+1} | \left( \frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} + \xi_{n+1}) \right) \right]
\times \left[ T_q^{(p)}[\Phi_2 | X_{n+1} | \left( \frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\zeta_{n+1} - \xi_{n+1}) \right) \right]
\]

Proof. By Theorems 2.2 and 4.3 we have for \( \lambda \in \mathbb{C}_+ \), \( w_\varphi \)-a.e. \( y \in C[0,t] \) and \( P_{X_{n+1}} \)-a.e. \( \xi_{n+1}, \zeta_{n+1} \in \mathbb{R}^{n+2} \)

\[
T_\lambda[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) = \left( \frac{\lambda}{2\pi} \right)^\frac{1}{2} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (\vec{v}, y + [\zeta_{n+1}, \xi_{n+1}]), \vec{z} \right]_{\mathbb{R}^r} + \left( (\vec{v}, y + [\zeta_{n+1}, \xi_{n+1}]), -\vec{z} \right]_{\mathbb{R}^r} \right\} \frac{1}{4q^i} \| T_{A^r}(\vec{z} - \vec{w}) \|_{\mathbb{R}^r}^2 - \frac{1}{4\lambda} \| T_{A^r}(\vec{z} + \vec{w}) \|_{\mathbb{R}^r}^2 \right\} d\rho_1(\vec{z}) d\rho_2(\vec{w})
\]

where the last equality follows from (15). Let \( T_q^{(p)}[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) \) be given by the right hand side of the last equality, where \( \lambda \) is replaced by \(-iq\). The existence of \( T_q^{(p)}[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) \) follows from the dominated convergence theorem. Now let \( 1 < p \leq \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then we have by the dominated convergence theorem

\[
\int_C \left| T_\lambda[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) \right| - T_q^{(p)}[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1})] d\rho_1(\vec{z}) \rightarrow 0
\]

as \( \lambda \) approaches \(-iq\) through \( \mathbb{C}_+ \) which shows the existence of \( T_q^{(p)}[\langle \Phi_1 * \Phi_2 \rangle q | X_{n+1} | (\cdot, \xi_{n+1}) | X_{n+1} | (y, \zeta_{n+1}) \). Now the equality in the theorem follows from Theorems 3.4, 4.1 and 4.3. \( \square \)
Remark 4.5 Without using Theorem 3.4, we can prove Theorem 4.4 with aids of Theorems 4.1 and 4.3.

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