Fixed Point Theorems for Set-valued
Contraction Mappings in Metric Spaces

Seong-Hoon Cho

Department of Mathematics
Hanseo University, Seosan
Chungnam, 356-706, South Korea

Copyright © 2015 Seong-Hoon Cho. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In the paper, new set-valued contraction mappings are introduced and a fixed point theorem for such mappings is established. An example to illustrate main result is given.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point, Contraction mappings, Set-valued mapping, Metric space

1 Introduction and preliminaries

Let \((X, d)\) be a metric space. We denote by \(CB(X)\) the family of nonempty closed and bounded subsets of \((X, d)\). Let \(H(\cdot, \cdot)\) be the Pompeiu-Hausdorff distance on \(CB(X)\), i.e.,

\[
H(A, B) = \max \{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, \quad \text{for } A, B \in CB(X),
\]

where \(d(a, B) = \inf \{d(a, b) : b \in B\}\) is the distance from the point \(a\) to the subset \(B\).

Denote \(\delta(x, A) = \sup \{d(x, y) : y \in A\}\), where \(A \in CB(X)\).

Especially, the authors of [9] obtained a generalization of the result of [13]. They proved the following result.

**Theorem 1.1.** [9] Let \((X,d)\) be a complete metric space. Suppose that a set-valued mapping \(T : X \to CB(X)\) satisfies the following condition:

\[
H(Tx,Ty) \leq km(x,y)
\]  

for all \(x, y \in X\), where \(k \in [0,1)\) and \(m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}\). If \(x \to d(x,Tx)\) is lower semi-continuous, then \(T\) has a fixed point in \(X\). That is, there exists an \(x_* \in X\) such that \(x_* \in Tx_*\).

Let \((X,d)\) be a metric space and \(x_0 \in X\), and let \(T : X \to CB(X)\) be a set-valued mapping. Then, a sequence \(\{x_n\}_{n=0}^\infty\) defined by \(x_n \in Tx_{n-1}, n = 1,2,\cdots\) is called a (Picard) orbit of \(T\) at initial point \(x_0\).

**Theorem 1.2.** [3] Let \((X,d)\) be a complete metric space. If a set-valued mapping \(T : X \to CB(X)\) satisfies the following condition:

there exist two constants \(\theta \in (0,1)\) and \(L \geq 0\) such that

\[
H(Tx,Ty) \leq \theta d(x,y) + Ld(y,Tx)
\]

for all \(x, y \in X\), then \(T\) has a fixed point in \(X\). Moreover, for any \(x_0 \in X\), there exists an orbit \(\{x_n\}\) of \(T\) at the initial point \(x_0\) that converges to a fixed point \(x_*\) of \(T\), for which the following estimates hold:

there exists a constant \(k \in (0,1)\) such that

(i) for \(n = 0,1,2,\cdots\),

\[
d(x_n,x_*) \leq \frac{k^n}{1-k}d(x_0,x_1);
\]

(ii) for \(n = 1,2,3,\cdots\),

\[
d(x_n,x_*) \leq \frac{k}{1-k}d(x_{n-1},x_n).
\]

Recently, the authors of [11] obtained the following result:
Theorem 1.3. [11] Let \((X, d)\) be a complete metric space. If a set-valued mapping \(T : X \to CB(X)\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1} d(x, y)
\]

for all \(x, y \in X\), then \(T\) has a fixed point in \(X\).

In this paper, we introduce new set-valued contraction mappings and establish a new fixed point theorem for such mappings, which is a generalization of the results of [3, 9, 11].

Lemma 1.1. Let \((X, d)\) be a metric space. Suppose that \(A, B \in CB(X)\) and \(c > 0\). If \(H(A, B) < c\) and \(a \in A\), then there exists \(b \in B\) such that \(d(a, b) < c\).

2 Fixed point theorems

Let \((X, d)\) be a metric space, and \(T : X \to CB(X)\) be a set-valued mapping, and let \(x_0 \in X\). Then, \(X\) is called \(T\)-orbitally complete if any Cauchy subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) converges to some point in \(X\), where \(\{x_n\}\) is a Picard orbit of \(T\) at initial point \(x_0 \in X\).

Note that if \(T : X \to X\) is a single-valued mapping, then \(X\) is \(T\)-orbitally complete if any Cauchy subsequence \(\{x_{n(k)}\}\) of the sequence \(\{x_n\}\) defined by \(x_{n+1} = Tx_n\), \(n = 0, 1, 2, \cdots, x_0 \in X\) converges to some point in \(X\).

Also, note that completeness implies \(T\)-orbitally completeness, for a set (or single)-valued mapping \(T\).

Theorem 2.1. Let \((X, d)\) be a metric space, and let \(T : X \to CB(X)\) be a set-valued mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1} m(x, y) + Ld(y, Tx)
\]

for all \(x, y \in X\), where \(L \geq 0\), then \(T\) has a fixed point in \(X\). Moreover, for any \(x_0 \in X\), there exists an orbit \(\{x_n\}\) of \(T\) at the initial point \(x_0\) that converges to a fixed point \(x_*\) of \(T\), for which (1.3) and (1.4) hold.

Proof. Let \(x_0 \in X\), and let \(x_1 \in Tx_0\).

If \(x_0 = x_1\), then \(x_0 \in Tx_0\), and the proof is finished.

Assume that \(x_0 \neq x_1\).
From (2.1) we have
\[
\begin{align*}
d(x_1, Tx_1) \\ \leq H(Tx_0, Tx_1) \\ \leq \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{\delta(x_0, Tx_0) + d(x_1, Tx_1) + 1} m(x_0, x_1) + L d(x_1, Tx_0) \\ \leq \frac{d(x_0, Tx_1) + d(x_1, Tx_1) + 1}{\delta(x_0, Tx_0) + d(x_1, Tx_1) + 1} m(x_0, x_1) \\ \leq \frac{d(x_0, x_1) + d(x_1, Tx_1)}{\delta(x_0, Tx_0) + d(x_1, Tx_1) + 1} m(x_0, x_1) \\ = \beta_0 m(x_0, x_1), \quad \text{where } \beta_0 = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{\delta(x_0, Tx_0) + d(x_1, Tx_1) + 1}.
\end{align*}
\]

We deduce that
\[
m(x_0, x_1) \\ = \max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{1}{2} \{d(x_0, Tx_1) + d(x_1, Tx_0)\}\} \\ \leq \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1), \frac{1}{2} \{d(x_0, x_1) + d(x_1, Tx_1)\}\} \\ = \max\{d(x_0, x_1), d(x_1, Tx_1)\}.
\]

From (2.2) and (2.3) we obtain
\[
d(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \beta_0 \max\{d(x_0, x_1), d(x_1, Tx_1)\}
\]
which implies
\[
d(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \beta_0 d(x_0, x_1), \quad \text{because } \beta_0 < 1.
\]

Let \( r \in (\beta_0, 1) \) be fixed.
Then, \( H(Tx_0, Tx_1) < rd(x_0, x_1) \). By Lemma 1.1, there exists \( x_2 \in Tx_1 \) such that
\[
d(x_1, x_2) < rd(x_1, x_0).
\]
If \( x_1 = x_2 \), then \( T \) has a fixed point, and the proof is finished.
Assume that \( x_1 \neq x_2 \).

Let
\[
\alpha_0 = \frac{d(x_0, x_1) + d(x_1, x_2)}{d(x_0, x_1) + d(x_1, x_2) + 1}.
\]
Then, \( \beta_0 \leq \alpha_0 \).

Let \( k \) be fixed such that \( \max\{\alpha_0, r\} < k < 1 \).
Then, from (2.4) we have
\[
d(x_1, x_2) < kd(x_0, x_1).
\]

(2.4)
Again, from (2.1) we have

\[
\begin{align*}
    d(x_2, Tx_2) \\
    \leq & H(Tx_1, Tx_2) \\
    \leq & \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{\delta(x_1, Tx_1) + d(x_2, Tx_2) + 1} m(x_1, x_2) + Ld(x_2, Tx_1) \\
    = & \frac{d(x_1, x_2) + d(x_2, Tx_2)}{\delta(x_1, Tx_1) + d(x_2, Tx_2) + 1} m(x_1, x_2) \\
    \leq & \frac{d(x_1, x_2) + d(x_2, Tx_2)}{\delta(x_1, Tx_1) + d(x_2, Tx_2) + 1} m(x_1, x_2) \\
    = & \beta_1 m(x_1, x_2), \text{ where } \beta_1 = \frac{d(x_1, x_2) + d(x_2, Tx_2)}{\delta(x_1, Tx_1) + d(x_2, Tx_2) + 1}.
\end{align*}
\]

(2.6)

We have

\[
\begin{align*}
    m(x_1, x_2) \\
    = & \max\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{1}{2}\{d(x_1, Tx_2) + d(x_2, Tx_1)\}\} \\
    \leq & \max\{d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), \frac{1}{2}\{d(x_1, x_2) + d(x_2, Tx_2)\}\} \\
    = & \max\{d(x_1, x_2), d(x_2, Tx_2)\}.
\end{align*}
\]

(2.7)

From (2.6) and (2.7) we have

\[
H(Tx_1, Tx_2) \leq \beta_1 d(x_1, x_2).
\]

(2.8)

We deduce that \( \beta_1 \leq \alpha_0 \). In fact, we obtain

\[
\beta_1 - \alpha_0 = \frac{d(x_2, Tx_2) - d(x_0, x_1)}{[\delta(x_1, Tx_1) + d(x_2, Tx_2) + 1][d(x_0, x_1) + d(x_1, x_2) + 1]} < 0,
\]

because \( d(x_2, Tx_2) \leq d(x_2, x_1) < d(x_0, x_1) \).

Thus,

\[
H(Tx_1, Tx_2) \leq \alpha_0 d(x_1, x_2) < kd(x_1, x_2).
\]

By applying Lemma 1.1 with \( x_2 \in Tx_1 \), we can choose \( x_3 \in Tx_2 \) such that

\[
d(x_2, x_3) < kd(x_1, x_2).
\]

If \( x_2 = x_3 \), then the proof is finished.

Assume that \( x_2 \neq x_3 \).

From (2.5) we have

\[
d(x_2, x_3) < k^2 d(x_0, x_1).
\]

(2.9)
Let 
\[ \alpha_1 = \frac{d(x_1, x_2) + d(x_2, x_3)}{d(x_1, x_2) + d(x_2, x_3) + 1}. \]

Then, it is easy to see that \( \alpha_1 \leq \alpha_0 \).

Let 
\[ \beta_2 = \frac{d(x_2, x_3) + d(x_3, Tx_3)}{\delta(x_2, Tx_2) + d(x_3, Tx_3) + 1}. \]

Then, we deduce \( \beta_2 \leq \alpha_1 \).

From (2.1) we have
\[ d(x_3, Tx_3) \leq H(Tx_2, Tx_3) \leq \beta_2 \max\{d(x_2, x_3), d(x_3, Tx_3)\} \]

which implies
\[ H(Tx_2, Tx_3) \leq \beta_2 d(x_2, x_3). \]

Thus we have
\[ H(Tx_2, Tx_3) \leq \beta_2 d(x_2, x_3) \leq \alpha_1 d(x_2, x_3) \leq \alpha_0 d(x_2, x_3) < kd(x_2, x_3). \]

By applying Lemma 1.1 with \( x_3 \in Tx_2 \), we can choose \( x_4 \in Tx_3 \) such that
\[ d(x_3, x_4) < kd(x_2, x_3). \]

Assume that \( x_3 \neq x_4 \).

From (2.9) we have
\[ d(x_3, x_4) < k^3d(x_0, x_1). \]

Continuing this process, we obtain a sequence \( \{x_n\} \subset X \) such that for all \( n = 1, 2, \ldots \),
\[ x_n \in Tx_{n-1}, \quad x_{n-1} \neq x_n \]
and
\[ d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \tag{2.10} \]

and
\[ d(x_n, x_{n+1}) \leq k^n d(x_0, x_1). \tag{2.11} \]

Note that \( \{x_n\} \) is an orbit of \( T \) at the initial point \( x_0 \).

For \( m > n \), we obtain
\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_m, x_m) \]
\[ \leq (k^n + k^{n+1} + \cdots + k^{m-1})d(x_0, x_1) \]
\[ \leq \frac{k^n}{1-k} d(x_0, x_1). \tag{2.12} \]
Thus, \( \{x_n\} \) is a Cauchy sequence in \( X \).

By the \( T \)-orbitally completeness of \( X \), there exists \( x_* \in X \) such that
\[
\lim_{n \to \infty} x_n = x_*.
\]

From (2.1) we have
\[
\begin{align*}
    d(x_{n+1}, Tx_*) & \leq H(Tx_n, Tx_*) \\
    & \leq \frac{d(x_n, Tx_*) + d(x_*, Tx_n)}{d(x_*, Tx_*) + 1} m(x_n, x_*) + Ld(x_*, Tx_n) \\
    & \leq \frac{d(x_n, Tx_*) + d(x_*, x_{n+1})}{d(x_*, Tx_*) + 1} m(x_n, x_*) + Ld(x_*, x_{n+1}).
\end{align*}
\]

We deduce
\[
\begin{align*}
m(x_n, x_*) &= \max\{d(x_n, x_*), d(x_n, Tx_n), d(x_*, Tx_n), \frac{1}{2}\{d(x_n, Tx_*) + d(x_*, Tx_n)\}\} \\
& \leq \max\{d(x_n, x_*), d(x_n, x_{n+1}), d(x_*, Tx_*) + \frac{1}{2}\{d(x_n, Tx_*) + d(x_*, x_{n+1})\}\}
\end{align*}
\]

Thus, we have
\[
\lim_{n \to \infty} m(x_n, x_*) \leq d(x_*, Tx_*).
\]

By letting \( n \to \infty \) in the above inequality (2.13) and by using (2.14), we have
\[
d(x_*, Tx_*) \leq \frac{d(x_*, Tx_*)}{d(x_*, Tx_*) + 1} d(x_*, Tx_*)
\]
which implies \( d(x_*, Tx_*) = 0 \). Thus, \( x_* \in Tx_* \), and so \( T \) has a fixed point.

We now show that (1.3) and (1.4) are satisfied.

From (2.12) we have
\[
\begin{align*}
d(x_n, x_*) & \leq d(x_n, x_m) + d(x_m, x_*) \\
& \leq \frac{k^n}{1-k} d(x_0, x_1) + d(x_m, x_*), \text{ where } m > n.
\end{align*}
\]

Letting \( m \to \infty \) in above inequality, we have (1.3).

From (2.10) we have
\[
\begin{align*}
d(x_n, x_{n+p}) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\
& \leq (k + k^2 + \cdots + k^p)d(x_{n-1}, x_n) \\
& = \frac{k(1 - k^p)}{1-k} d(x_{n-1}, x_n).
\end{align*}
\]

Letting \( p \to \infty \) in above inequality, we have (1.4).
By Theorem 2.1, we have the following corollaries.

**Corollary 2.2.** Let \((X, d)\) be a metric space, and let \(T : X \to CB(X)\) be a set-valued mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} n(x, y) + Ld(y, Tx)
\]

for all \(x, y \in X\), where \(L \geq 0\) and \(n(x, y) = \max\{d(x, y), \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}\), then \(T\) has a fixed point in \(X\).

**Corollary 2.3.** Let \((X, d)\) be a metric space, and let \(T : X \to CB(X)\) be a set-valued mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} d(x, y) + Ld(y, Tx)
\]

for all \(x, y \in X\), where \(L \geq 0\), then \(T\) has a fixed point in \(X\).

**Corollary 2.4.** Let \((X, d)\) be a metric space, and let \(T : X \to CB(X)\) be a set-valued mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} m(x, y)
\]

for all \(x, y \in X\), then \(T\) has a fixed point in \(X\).

**Corollary 2.5.** Let \((X, d)\) be a metric space, and let \(T : X \to CB(X)\) be a set-valued mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
H(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1} m(x, y)
\]

for all \(x, y \in X\), then \(T\) has a fixed point in \(X\).

**Remark 2.1.** Corollary 2.5 is a generalization of Theorem 5 of [11]. If we have \(X\) is complete and \(m(x, y) = d(x, y)\), Corollary 2.5 becomes Theorem 5 of [11].
Corollary 2.6. Let \((X, d)\) be a metric space, and let \(T : X \to X\) be a mapping. Suppose that \(X\) is \(T\)-orbitally complete. If \(T\) satisfies the following condition:

\[
d(Tx, Ty) \\
\leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \{d(x, Ty) + d(y, Tx)\}\} \\
+ Ld(y, Tx)
\]

for all \(x, y \in X\), where \(L \geq 0\), then

(a) \(T\) has a fixed point in \(X\);

(b) \(d(x_*, y_*) \geq \frac{1-L}{2}\), whenever \(x_*\) and \(y_*\) are two distinct fixed point of \(T\);

(c) for each \(x_0 \in X\), the Picard iteration \(\{x_n\}\) given by \(x_{n+1} = Tx_n\), \(n = 0, 1, 2, \cdots\), converges to a fixed point \(x_*\) of \(T\), and the following estimates hold:

there exists a constant \(k \in (0, 1)\) such that

(i) for \(n = 0, 1, 2, \cdots\),

\[
d(x_n, x_*) \leq \frac{k^n}{1-k}d(x_0, x_1);
\]

(ii) for \(n = 1, 2, 3, \cdots\),

\[
d(x_n, x_*) \leq \frac{k}{1-k}d(x_{n-1}, x_n).
\]

Proof. By taking single valued mapping in Theorem 2.1, we obtain (a) and (c).

To prove (b), let \(x_*\) and \(y_*\) be two distinct fixed points of \(T\).

We have

\[
d(x_*, y_*) = d(Tx_*, Ty_*) \\
\leq \frac{d(x_*, Ty_*) + d(y_*, Tx_*)}{d(x_*, Tx_*) + d(y_*, Ty_*) + 1} \max\{d(x_*, y_*), d(x_*, Tx_*), d(y_*, Ty_*), \frac{1}{2} \{d(x_*, Ty_*) + d(y_*, Tx_*)\}\} \\
+ Ld(y_*, Tx_*) \\
= 2d(x_*, y_*)d(x_*, y_*) + Ld(x_*, y_*).
\]

Hence, \(d(x_*, y_*) \geq \frac{1-L}{2}\). \qed
Corollary 2.7. Let $(X, d)$ be a metric space, and let $T : X \to X$ be a mapping. Suppose that $X$ is $T$-orbitally complete. If $T$ satisfies the following condition:

$$
d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max \{d(x, y), \frac{1}{2} \{d(x, Tx) + d(y, Ty)\}, \frac{1}{2} \{d(x, Ty) + d(y, Tx)\}\}
$$

for all $x, y \in X$, where $L \geq 0$, then

(a) $T$ has a fixed point in $X$;
(b) $d(x_*, y_*) \geq \frac{1-L}{2}$, whenever $x_*$ and $y_*$ are two distinct fixed points of $T$;
(c) $\{T^n x\}$ converges to a fixed point, for all $x \in X$.

Corollary 2.8. Let $(X, d)$ be a metric space, and let $T : X \to X$ be a mapping. Suppose that $X$ is $T$-orbitally complete. If $T$ satisfies the following condition:

$$
d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y) + L d(y, Tx)
$$

for all $x, y \in X$, where $L \geq 0$, then

(a) $T$ has a fixed point in $X$;
(b) $d(x_*, y_*) \geq \frac{1-L}{2}$, whenever $x_*$ and $y_*$ are two distinct fixed points of $T$;
(c) $\{T^n x\}$ converges to a fixed point, for all $x \in X$.

Remark 2.2. Corollary 2.8 is a generalization of Theorem 1 of [11]. If we have $X$ is complete and $L = 0$, then Corollary 2.8 reduce to Theorem 1 of [11].

The following example illustrates Theorem 2.1.

Example 2.1. Let $X = \{0, \frac{1}{2}, 1\}$, and let $d : X \times X \to [0, \infty)$ be defined by

$$
d(0, \frac{1}{2}) = 2, d(1, \frac{1}{2}) = \frac{5}{2}, d(0, 1) = 3,
$$

$$
d(x, x) = 0 \text{ for all } x \in X, \text{ and } d(a, b) = d(b, a) \text{ for all } a, b \in X.
$$

Then $(X, d)$ is a metric space.

We define a set-valued mapping $T : X \to CB(X)$ by

$$
T x = \begin{cases} 
\{0\} & (x = 0), \\
\{\frac{1}{2}, 1\} & (x = \frac{1}{2}), \\
\{0, 1\} & (x = 1).
\end{cases}
$$
Then, $X$ is $T$-orbitally complete.

Let $L = 0$.

We now show that condition (2.1) is satisfied.

We consider four cases.

Case 1. Let $x = y$. Then we have $H(Tx, Ty) = 0$. Hence condition (2.1) is satisfied.

Case 2. Let $x = 0$ and $y = \frac{1}{2}$. Then we have

$$H(Tx, Ty) = 3 < 8 = \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} d(x, y).$$

Hence, (2.1) is satisfied.

Case 3. Let $x = 0$ and $y = 1$. Then we have

$$H(Tx, Ty) = 3 < 9 = \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} d(x, y).$$

Hence, (2.1) is satisfied.

Case 4. Let $x = \frac{1}{2}$ and $y = 1$. Then we have

$$H(Tx, Ty) = 2 < 5 = \frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + d(y, Ty) + 1} d(x, y).$$

Hence, (2.1) is satisfied.

Thus, $T$ satisfies all conditions of Theorem 2.1 and $T$ has a fixed point.

Note that condition (1.1) of Theorem 1.1 is not satisfied. In fact, if condition (1.1) is satisfied, then we have, for $x = 0$ and $y = \frac{1}{2}$,

$$3 = H(Tx, Ty) \leq km(x, y) = 2k, \text{ where } 0 \leq k < 1.$$

Thus we obtain $k \geq \frac{3}{2}$, which is a contradiction. Thus, condition (1.1) of Theorem 1.1 is not satisfied.

Condition (1.2) of Theorem 1.2 is also not satisfied. If condition (1.2) holds, then we obtain, for $x = \frac{1}{2}$ and $y = 1$,

$$2 = H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) = \frac{1}{2} \theta, \text{ where } 0 < \theta < 1.$$

Hence, $\theta \geq 4$, which is a contradiction. Thus, condition (1.2) of Theorem 1.2 is not satisfied.

Also, note that condition (1.5) of Theorem 1.3 is not satisfied. In fact, for $x = 0$ and $y = \frac{1}{2}$, we have

$$\frac{d(x, Ty) + d(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1} d(x, y) = \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9} < 3 = H(Tx, Ty).$$

Hence, condition (1.5) of Theorem 1.3 is not satisfied.
Therefore, Theorem 2.1 is a generalization of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Acknowledgments. This research was supported by Hanseo University.

References


**Received: August 4, 2015; Published: September 12, 2015**