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Developing a Single Step Hybrid Block Method with Generalized Three Off-step Points for the Direct Solution of Second Order Ordinary Differential Equations

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Abstract

A new one step hybrid block method with generalized three off-step points using interpolation and collocation approach for solving initial value problems of second order ordinary differential equation directly is proposed. In deriving this method, the power series used as basis function to approximate the solution is interpolated at the last two points of the three off-step points while its second derivative is collocated at all points in the selected interval. The method is proven to be zero stable, consistent, convergent and of order five. For the purpose of testing, specific points of the developed method are chosen to solve second order initial value problem directly. The numerical results obtained indicate that the new method outperforms the previous methods in terms of accuracy.

Mathematics Subject Classification: 65L06, 65L05

Keywords: Hybrid method, Block method, Second order differential equation, Power series, Three off-step points

1 Introduction

In this paper, we consider the direct solution to general second order initial value problem(IVPs) of the form

$$y^{''} = f(x, y, y^{'}),\ y(a) = \eta_0,\ y^{'}(a) = \eta_1,\ x \in [a, b]. \tag{1}$$

Many scholars have been developed numerical methods for solving second order IVPs directly. These techniques have been introduced in many literature such as [7] [8], [9], [5] and others. The idea of hybrid method which involves the use off step points was introduced to overcome the zero stability barrier in linear multistep method. This barrier implies that the highest order of zero stability of linear multistep method when steplength k is odd is k +1 and k +2 when k is even[6]. [2] proposed a hybrid block method in which two off-step points were considered in the development the method but the accuracy of the method are low. [9] developed three step with one off-step points. However, this method is also of low accuracy when it was applied to solve (1). Subsequently, one step with one off-step was developed by [1] but the accuracy of the method is still not encouraging.

Hybrid block method depends on the off-step points chosen. To the best of our knowledge, one step hybrid block methods have been developed only to specific point(s). This paper, therefore, attempts to develop one step hybrid block method with generalized three off-step points for solving second initial value problems of second order ordinary differential equations(ODEs) directly.

2 Derivation of the Method

In this section, a hybrid one step block method with generalized three off-step points x_{n+s_1}, x_{n+s_2} and x_{n+s_3} for solving (1) is derived.
Let the approximate solution to equation(1) be the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i, x \in [x_n, x_{n+1}] \tag{2}$$

where $n = 0, 1, 2, ..., N-1$, v denotes of the number of interpolation points which is equal to the order of ODE, m represents the number of collocation points, $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.
The second derivative of (2) is given by

$$\begin{aligned} y^{''}(x) &= f(x, y, y^{'}) \\ &= \sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x - x_n}{h}\right)^{i-2}. \end{aligned} \tag{3}$$

where $v = 2$ and $m = 5$. Interpolating (2) at x_{n+s_2} and x_{n+s_3}, and collocating (3) at all points i.e $x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3}, x_{n+1}$ in the selected interval yields the following equations which can be written in matrix form.

$$\begin{pmatrix} 1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 \\ 1 & s_3 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_1 & \frac{12}{h^2}s_1^2 & \frac{20}{h^2}s_1^3 & \frac{30}{h^2}s_1^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_2 & \frac{12}{h^2}s_2^2 & \frac{20}{h^2}s_2^3 & \frac{30}{h^2}s_2^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2}s_3 & \frac{12}{h^2}s_3^2 & \frac{20}{h^2}s_3^3 & \frac{30}{h^2}s_3^4 \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+s_2} \\ y_{n+s_3} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix} \tag{4}$$

Gaussian elimination method is then applied to find the coefficients of a_i for $i = 0(1)6$. Then, the value of a_i are substituted into equation (2) to give a continuous implicit scheme of the form

$$y(x) = \sum_{i=2}^{3} \alpha_{s_i} y_{n+s_i} + \sum_{i=0}^{1} \beta_{s_i} f_{n+i} + \sum_{i=1}^{3} \beta_{s_i} f_{n+s_i} \tag{5}$$

The first derivative of equation (5) is

$$y^{'}(x) = \sum_{i=2}^{3} \frac{\partial}{\partial x} \alpha_{s_i}(x) y_{n+s_i} + \sum_{i=0}^{1} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} + \sum_{i=1}^{3} \frac{\partial}{\partial x} \beta_{s_i}(x) f_{n+s_i} \tag{6}$$

where

$$\alpha_{s_3} = \frac{(x_n - x + hs_2)}{(h(s_2 - s_3))}$$

$$\alpha_{s_2} = \frac{(x - x_n - hs_3)}{(h(s_2 - s_3))}$$

$$\begin{aligned} \beta_0 = &\frac{(x_n - x + hs_3)(x_n - x + hs_2)}{(60 s_1 s_2 s_3 h^4)} (12x^2 x_n^2 - 3hx^3 + 3hx_n^3 - 8xx_n^3 + 2x^4 \\ &+ 2x_n^4 + 2h^4 s_2^3 - h^4 s_2^4 + 2h^4 s_3^3 - h^4 s_3^4 - 3hs_1 x^3 - hs_2 x^3 - hs_3 x^3 + 3hs_1 x_n^3 \\ &+ hs_3 x_n^3 - 9hxx_n^2 + 9hx^2 x_n - 5h^4 s_1 s_2^2 + 2h^4 s_1 s_2^3 - 5h^4 s_1 s_3^2 - 4h^2 s_1 s_3 x x_n \end{aligned}$$

$$-3h^4s_2^2s_3+h^4s_2s_3^3+h^4s_2^3s_3+5h^2s_1x^2+2h^2s_2x^2+2h^3s_2^2x-5h^3s_1s_2x$$
$$+2h^3s_3^2x-h^3s_3^3x+5h^2s_1x_n^2+2h^2s_2x_n^2-2h^3s_2^2x_n+h^3s_2^3x_n-2h^2s_2s_3xx_n$$
$$+h^3s_3^3x_n+h^4s_2^2s_3^2-h^2s_2^2x^2-h^2s_3^2x^2-h^2s_2^2x_n^2-h^2s_3^2x_n^2+15h^4s_1s_2s_3$$
$$-5h^3s_1s_3x-3h^3s_2s_3x+5h^3s_1s_2x_n+5h^3s_1s_3x_n+3h^3s_2s_3x_n-9hs_1xx_n^2$$
$$-10h^2s_1xx_n-3hs_2xx_n^2+3hs_2x^2x_n-4h^2s_2xx_n-3hs_3xx_n^2+3hs_3x^2x_n$$
$$-3h^4s_1s_2s_3^2-3h^4s_1s_2^2s_3+2h^2s_1s_2x^2+2h^3s_1s_2^2x+2h^2s_1s_3x^2+2h^2s_1s_2x_n^2$$
$$+2h^3s_1s_3^2x+h^3s_2s_3^2x+h^3s_2^2s_3x-2h^3s_1s_2^2x_n+2h^2s_1s_3x_n^2-2h^3s_1s_3^2x_n$$
$$-h^3s_2^2s_3x_n+2h^2s_2^2xx_n+2h^2s_3^2xx_n-3h^3s_1s_2s_3x+3h^3s_1s_2s_3x_n+hs_2x_n^3$$
$$+2h^2s_3x_n^2+2h^2s_3x^2-h^3s_2s_3^2x_n+2h^4s_1s_3^3-3h^4s_2s_3^2-2h^3s_3^2x_n-h^3s_2^3x$$
$$+9hs_1x^2x_n-4h^2s_3xx_n+h^2s_2s_3x_n^2-4h^2s_1s_2xx_n+h^2s_2s_3x^2-8x^3x_n)$$

$$\beta_{s_1}=\frac{-(x_n-x+hs_3)(x_n-x+hs_2)}{60h^4s_1(s_1-1)(s_1-s_3)(s_1-s_2)}(h^4s_2^4-h^4s_2^3s_3-2h^4s_2^3-h^4s_2^2s_3^2$$
$$-h^4s_2s_3^3+3h^4s_2s_3^2+h^4s_3^4-2h^4s_3^3+h^3s_2^3x-h^3s_2^3x_n-h^3s_2^2s_3x+h^3s_2^2s_3x_n$$
$$+2h^3s_2^2x_n-h^3s_2s_3^2x+h^3s_2s_3^2x_n+3h^3s_2s_3x-3h^3s_2s_3x_n+h^3s_3^3x-h^3s_3^3x_n$$
$$+2h^3s_3^2x_n+h^2s_2^2x^2-2h^2s_2^2xx_n+h^2s_2^2x_n^2-h^2s_2s_3x^2+2h^2s_2s_3xx_n-3hx_n^3$$
$$+4h^2s_2xx_n-2h^2s_2x_n^2+h^2s_3^2x^2-2h^2s_3^2xx_n+h^2s_3^2x_n^2-2h^2s_3x^2-hs_3x_n^3$$
$$-2h^2s_3x_n^2+hs_2x^3-3hs_2x^2x_n+3hs_2xx_n^2-hs_2x_n^3+hs_3x^3-2x^4+8x^3x_n$$
$$-3hs_3x^2x_n+4h^2s_3xx_n+3hs_3xx_n^2-h^2s_2s_3x_n^2-2h^3s_3^2x-2h^3s_2^2x+8xx_n^3$$
$$+3h^4s_2^2s_3-2h^2s_2x^2+3hx^3-9hx^2x_n+9hxx_n^2-12x^2x_n^2-2x_n^4)$$

$$\beta_{s_2}=\frac{(x_n-x+hs_3)(x_n-x+hs_2)}{60h^4s_2(s_2-1)(s_2-s_3)(s_1-s_2)}(3hx^3-12x^2x_n^2+8xx_n^3+8x^3x_n$$
$$-2h^4s_2s_3^2+3h^4s_2^3-2h^4s_2^4-2h^4s_3^3+3hs_1x^3-2hs_2x^3-3hs_1x_n^3-2h^4s_1s_3^3$$
$$-2h^2s_2s_3xx_n-2x_n^4+9hxx_n^2-9hx^2x_n-5h^4s_1s_2^2+3h^4s_1s_2^3+5h^4s_1s_3^2$$
$$+h^4s_2s_3^3+h^4s_2^3s_3-5h^2s_1x^2+3h^2s_2x^2+3h^3s_2^2x-2h^3s_2^3x+5h^3s_1s_2x_n$$
$$-5h^2s_1x_n^2+3h^2s_2x_n^2-3h^3s_2^2x_n+2h^3s_2^3x_n-2h^2s_3x_n^2+2h^3s_3^2x_n-h^3s_3^3x_n$$
$$-2h^2s_2^2x^2-2h^2s_2^2x_n^2+h^2s_3^2x_n^2+5h^4s_1s_2s_3-5h^3s_1s_2x+5h^3s_1s_3x-3hx_n^3$$
$$-5h^3s_1s_3x_n+2h^3s_2s_3x_n+9hs_1xx_n^2-9hs_1x^2x_n+10h^2s_1xx_n-6hs_2xx_n^2$$
$$-6h^2s_2xx_n+3hs_3xx_n^2-3hs_3x^2x_n+4h^2s_3xx_n-2h^4s_1s_2s_3^2-2h^4s_1s_2^2s_3$$
$$+3h^3s_1s_2^2x-2h^2s_1s_3x^2-2h^3s_1s_3^2x+h^2s_2s_3x^2+h^3s_2s_3^2x+h^3s_2^2s_3x-2x^4$$
$$-3h^3s_1s_2^2x_n-2h^2s_1s_3x_n^2+2h^3s_1s_3^2x_n+h^2s_2s_3x_n^2-h^3s_2s_3^2x_n-h^3s_2^2s_3x_n$$
$$+4h^2s_2^2xx_n-hs_3x_n^3-2h^2s_3^2xx_n-2h^3s_1s_2s_3x+2h^3s_1s_2s_3x_n+3h^2s_1s_2x_n^2$$
$$-6h^2s_1s_2xx_n-2h^3s_2s_3x-2h^4s_2^2s_3+3h^2s_1s_2x^2+6hs_2x^2x_n+4h^2s_1s_3xx_n$$
$$-2h^2s_3x^2+h^4s_2^2s_3^2+2hs_2x_n^3+h^2s_3^2x^2+h^3s_3^3x-2h^3s_3^2x+hs_3x^3+h^4s_3^4)$$

$$\beta_{s_3}=\frac{(x_n-x+hs_3)(x_n-x+hs_2)}{(60h^4s_3(s_3-1)(s_2-s_3)(s_1-s_3))}(12x^2x_n^2-3hx^3+3hx_n^3-8xx_n^3$$

$$+6h^2s_1s_3xx_n + 2h^4s_2^3 - h^4s_2^4 - 3h^4s_3^3 + 2h^4s_3^4 - 3hs_1x^3 + 2hs_3x^3 + 3hs_1x_n^3$$
$$-2hs_3x_n^3 - 9hxx_n^2 + 9hx^2x_n - 5h^4s_1s_2^2 + 2h^4s_1s_2^3 + 5h^4s_1s_3^2 - 4h^2s_1s_2xx_n$$
$$-h^4s_2s_3^3 - h^4s_2^3s_3 + 5h^2s_1x^2 + 2h^2s_2x^2 + 2h^3s_2^2x + 2h^4s_1s_2^2s_3 + 5h^3s_1s_3x$$
$$+5h^2s_1x_n^2 + 2h^2s_2x_n^2 - 2h^3s_2^2x_n + h^3s_2^3x_n - 3h^2s_3x_n^2 + 3h^3s_3^2x_n - 2h^3s_3^3x_n$$
$$-h^2s_2^2x^2 + 2h^2s_3^2x^2 - h^2s_2^2x_n^2 + 2h^2s_3^2x_n^2 - 5h^4s_1s_2s_3 - 5h^3s_1s_2x - h^4s_2^2s_3^2$$
$$+5h^3s_1s_2x_n - 5h^3s_1s_3x_n - 2h^3s_2s_3x_n - 9hs_1xx_n^2 + 9hs_1x^2x_n - 10h^2s_1xx_n$$
$$+3hs_2x^2x_n - 4h^2s_2xx_n + 6hs_3xx_n^2 - 6hs_3x^2x_n + 6h^2s_3xx_n + 2h^4s_1s_2s_3^2$$
$$+2h^2s_1s_2x^2 + 2h^3s_1s_2^2x - 3h^2s_1s_3x^2 - 3h^3s_1s_3^2x - h^2s_2s_3x^2 + 2h^3s_2s_3x$$
$$+2h^2s_1s_2x_n^2 - 2h^3s_1s_2^2x_n - 3h^2s_1s_3x_n^2 + 3h^3s_1s_3^2x_n - h^2s_2s_3x_n^2 - 3h^4s_1s_3^3$$
$$+h^3s_2^2s_3x_n - 2h^3s_1s_2s_3x_n + 2h^2s_2^2xx_n - 4h^2s_3^2xx_n + 2h^3s_3^3x - 3hs_2xx_n^2$$
$$+2h^4s_2s_3^2 - h^3s_2^3x - 3h^2s_3x^2 - 3h^3s_3^2x + 2h^4s_2^2s_3 + hs_2x_n^3 + 2x^4 - h^3s_2s_3^2x$$
$$-8x^3x_n + h^3s_2^2s_3x + h^3s_2s_3^2x_n + 2h^3s_1s_2s_3x + 2h^2s_2s_3xx_n - hs_2x^3 + 2x_n^4)$$

$$\beta_1 = -\frac{(x_n - x + hs_3)(x_n - x + hs_2)}{(60h^4(s_3-1)(s_2-1)(s_1-1))}(2s_1h^4s_2^3 + h^4s_2^2s_3^2 - 3s_1h^4s_2^2s_3$$
$$-h^3s_2^2s_3x_n + h^4s_2^3s_3 + h^4s_2s_3^3 - 3s_1h^4s_2s_3^2 - h^4s_3^4 + 2s_1h^4s_3^3 - h^3s_2^3x$$
$$+h^3s_2^3x_n + h^3s_2^2s_3x - h^4s_2^4 + 2s_1h^3s_2^2x - 2s_1h^3s_2^2x_n + h^3s_2s_3^2x - h^3s_2s_3^2x_n$$
$$-3s_1h^3s_2s_3x + 3s_1h^3s_2s_3x_n - h^3s_3^3x + h^3s_3^3x_n + 2s_1h^3s_3^2x - 2s_1h^3s_3^2x_n$$
$$+h^2s_2s_3x_n^2 + 2s_1h^2s_2x^2 - 4s_1h^2s_2xx_n + 2s_1h^2s_2x_n^2 - h^2s_3^2x^2 + 2h^2s_3^2xx_n$$
$$-8xx_n^3 + 2s_1h^2s_3x^2 - 4s_1h^2s_3xx_n + 2s_1h^2s_3x_n^2 - hs_2x^3 + 3hs_2x^2x_n$$
$$+3hs_3x^2x_n - 3hs_3xx_n^2 + hs_3x_n^3 - 3s_1hx^3 + 9s_1hx^2x_n - 9s_1hxx_n^2$$
$$+3s_1hx_n^3 - 8x^3x_n + hs_2x_n^3 + 2x^4 - h^2s_3^2x_n^2 - 3hs_2xx_n^2 - 2h^2s_2s_3xx_n$$
$$+2x_n^4 + 12x^2x_n^2 - h^2s_2^2x^2 + 2h^2s_2^2xx_n - h^2s_2^2x_n^2 + h^2s_2s_3x^2 - hs_3x^3)$$

Evaluating (5) at the non-interpolating point i.e x_n, x_{n+s_1}, x_{n+1} and (6) at all points produces the discrete schemes and its derivative. Combining the discrete schemes and its derivative at x_n gives the equation in a matrix form as follows

$$A^{[3]_2}Y_m^{[3]_2} = B_1^{[3]_2}R_1^{[3]_2} + B_2^{[3]_2}R_2^{[3]_2} + h^2\left[D^{[3]_2}R_3^{[3]_2} + E^{[3]_2}R_4^{[3]_2}\right] \qquad (7)$$

where

$$A^{[3]_2} = \begin{pmatrix} 0 & \frac{s_3}{s_2-s_3} & \frac{s_2}{s_2-s_3} & 0 \\ 1 & \frac{-(s_1-s_3)}{(s_2-s_3)} & \frac{(s_1-s_2)}{(s_2-s_3)} & 0 \\ 0 & \frac{(s_3-1)}{(s_2-s_3)} & \frac{-(s2-1)}{(s2-s3)} & 1 \\ 0 & \frac{-1}{(h(s2-s3))} & \frac{1}{(h(s2-s3))} & 0 \end{pmatrix},\ Y_m^{[3]_2} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix}$$

$$B_1^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ R_1^{[3]_2} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},\ B_2^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$R_2^{[3]_2} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},\ D^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_2} \\ 0 & 0 & 0 & D_{24}^{[3]_2} \\ 0 & 0 & 0 & D_{34}^{[3]_2} \\ 0 & 0 & 0 & D_{44}^{[3]_2} \end{pmatrix},\ R_3^{[3]_3} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

$$E^{[3]_2} = \begin{pmatrix} E_{11}^{[3]_2} & E_{12}^{[3]} & E_{13}^{[3]_2} & E_{14}^{[3]_2} \\ E_{21}^{[3]_2} & E_{22}^{[3]_2} & E_{23}^{[3]_2} & E_{24}^{[3]_2} \\ E_{31}^{[3]_2} & E_{32}^{[3]_2} & E_{33}^{[3]_2} & E_{34}^{[3]_2} \\ E_{41}^{[3]_2} & E_{42}^{[3]_2} & E_{43}^{[3]_2} & E_{44}^{[3]_2} \end{pmatrix} \text{and } R^{[4]_3} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}$$

The non-zero elements of $D^{[3]_2}$ and $E^{[3]_2}$ are given by

$$\begin{aligned}
D_{14}^{[3]_2} &= \frac{1}{(60s_1)}(s_2^2s_3^2 - 5s_1s_2^2 + 2s_1s_2^3 - 5s_1s_3^2 + 2s_1s_3^3 - 3s_2s_3^2 - 3s_2^2s_3 + s_2s_3^3 \\
&\quad + s_2^3s_3 + 2s_2^3 - s_2^4 + 2s_3^3 - s_3^4 - 3s_1s_2s_3^2 - 3s_1s_2^2s_3 + 15s_1s_2s_3) \\
D_{24}^{[3]_2} &= -\frac{1}{(60s_1s_2s_3)}(s_1 - s_2)(s_1 - s_3)(s_1^4 - s_1^3s_2 - s_1^3s_3 - 2s_1^3 - s_1^2s_2^2 + s_3^4 \\
&\quad + 3s_1^2s_2 - s_1^2s_3^2 + 3s_1^2s_3 - s_1s_2^3 + 2s_1s_2^2s_3 + 3s_1s_2^2 + 2s_1s_2s_3^2 + 2s_1^2s_2s_3 \\
&\quad - 2s_2^3 - s_1s_3^3 + 3s_1s_3^2 + s_2^4 - s_2^3s_3 - s_2^2s_3^2 + 3s_2^2s_3 - s_2s_3^3 + 3s_2s_3^2 \\
&\quad - 2s_3^3 - 12s_1s_2s_3) \\
D_{34}^{[3]_2} &= \frac{1}{(60s_1s_2s_3)}(s_2 - 1)(s_3 - 1)(2s_1 + s_2 + s_3 + s_2^2s_3^2 - 3s_1s_2 - 3s_1s_3 \\
&\quad - 3s_1s_2^2 + 2s_1s_2^3 - 3s_1s_3^2 + 2s_1s_3^3 - 2s_2s_3^2 - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + s_2^2 \\
&\quad + s_2^3 - 2s_2s_3 - s_2^4 + s_3^2 + s_3^3 - s_3^4 - 3s_1s_2s_3^2 - 3s_1s_2^2s_3 + 12s_1s_2s_3 - 1) \\
D_{44}^{[3]_2} &= -\frac{1}{h(60s_1s_2s_3)}(s_2^2s_3^3 - 3s_2^2s_3^2 + s_2^3s_3^2 - 5s_1s_2^3 + 2s_1s_2^4 - 5s_1s_3^3 + 2s_1s_3^4 \\
&\quad - 3s_2s_3^3 - 3s_2^3s_3 + s_2s_3^4 + s_2^4s_3 + 2s_2^4 + 2s_3^4 - s_3^5 + 15s_1s_2s_3^2 - 3s_1s_2s_3^3 \\
&\quad + 15s_1s_2^2s_3 - 3s_1s_2^3s_3 - 3s_1s_2^2s_3^2 - s_2^5) \\
E_{11}^{[3]_2} &= \frac{s_2s_3}{60s_1(s_1 - s_2)(s_1 - s_3)(s_1 - 1)}(s_2^3s_3 - s_2^4 + 2s_2^3 + s_2^2s_3^2 - 3s_2^2s_3 + s_2s_3^3 \\
&\quad - s_3^4 - 3s_2s_3^2 + 2s_3^3) \\
E_{12}^{[3]} &= \frac{s_3}{60(s_1 - s_2)(s_2 - s_3)(s_2 - 1)}(s_2^2s_3^2 - 5s_1s_2^2 + 3s_1s_2^3 + 5s_1s_3^2 - 2s_1s_3^3 \\
&\quad - 2s_3^3 - 2s_2s_3^2 - 2s_2^2s_3 + s_2s_3^3 + s_2^3s_3 + 3s_2^3 + s_3^4 - 2s_1s_2s_3^2 - 2s_1s_2^2s_3 \\
&\quad + 5s_1s_2s_3 - 2s_2^4) \\
E_{13}^{[3]_2} &= \frac{-s_2}{60(s_1 - s_3)(s_2 - s_3)(s_3 - 1)}(s_2^2s_3^2 + 5s_1s_2^2 - 2s_1s_2^3 - 5s_1s_3^2 + 3s_2s_3^3 \\
&\quad + s_2s_3^3 + s_2^3s_3 - 2s_2^3 + s_2^4 + 3s_3^3 - 2s_3^4 - 2s_1s_2s_3^2 - 2s_1s_2^2s_3 + 5s_1s_2s_3 \\
&\quad - 2s_2s_3^2 - 2s_2^2s_3) \\
E_{14}^{[3]_2} &= -\frac{s_2s_3}{(60s_1 - 60)(s_2 - 1)(s_3 - 1)}(-s_2^4 + s_2^3s_3 + 2s_1s_2^3 + s_2^2s_3^2 - 3s_1s_2^2s_3
\end{aligned}$$

$$+s_2s_3^3-3s_1s_2s_3^2-s_3^4+2s_1s_3^3)$$

$$E_{21}^{[3]_2} = \frac{1}{(60s_1(s_1-1))}(2s_1^4-s_1^3s_2-s_1^3s_3-3s_1^3-s_1^2s_2^2+s_1^2s_2s_3+2s_1^2s_2-s_2^4$$
$$+2s_1^2s_3-s_1s_2^3+s_1s_2^2s_3+2s_1s_2^2+s_1s_2s_3^2-3s_1s_2s_3-s_1s_3^3+2s_1s_3^2$$
$$+s_2s_3^3-s_1^2s_3^2+s_2^3s_3+2s_2^3+s_2^2s_3^2-3s_2^2s_3-3s_2s_3^2-s_3^4+2s_3^3)$$

$$E_{22}^{[3]_2} = \frac{-(s_1-s_3)}{(60s_2(s_2-s_3)(s_2-1))}(2s_1^2s_2-s_1^4-s_1^3s_2+s_1^3s_3+2s_1^3-s_1^2s_2^2+s_1^2s_3^2$$
$$+s_1^2s_2s_3-3s_1^2s_3-s_1s_2^3+s_1s_2^2s_3+2s_1s_2^2+s_1s_2s_3^2-3s_1s_2s_3+2s_2s_3^2$$
$$+s_1s_3^3-3s_1s_3^2+2s_2^4-s_2^3s_3-3s_2^3-s_2^2s_3^2+2s_2^2s_3-s_2s_3^3+2s_3^3-s_3^4)$$

$$E_{23}^{[3]_2} = \frac{(s_1-s_2)}{(60s_3(s_2-s_3)(s_3-1))}(2s_1^3-s_1^4+s_1^3s_2-s_1^3s_3+s_1^2s_2^2+s_1^2s_2s_3+2s_2^3$$
$$-s_1^2s_3^2+2s_1^2s_3+s_1s_2^3+s_1s_2^2s_3-3s_1s_2^2+s_1s_2s_3^2-3s_1s_2s_3-s_2^4+2s_3^4$$
$$-s_1s_3^3-3s_1^2s_2+2s_1s_3^2-s_2^3s_3-s_2^2s_3^2+2s_2^2s_3-s_2s_3^3+2s_2s_3^2-3s_3^3)$$

$$E_{24}^{[3]_2} = \frac{-(s_1-s_2)(s_1-s_3)}{(60(s_1-1)(s_2-1)(s_3-1))}(s_1^2s_3^2-s_1^4+s_1^3s_2+s_1^3s_3+s_1^2s_2^2+s_2^3s_3$$
$$-2s_1^2s_2s_3+s_1s_2^3-2s_1s_2^2s_3-2s_1s_2s_3^2+s_1s_3^3-s_2^4+s_2^2s_3^2+s_2s_3^3-s_3^4)$$

$$E_{31}^{[3]_2} = \frac{(s_2-1)(s_3-1)}{(60s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(-s_2^4+s_2^3s_3+s_2^3+s_2^2s_3^2-2s_2^2s_3+s_2^2$$
$$+s_2s_3^3-2s_2s_3^2-2s_2s_3+s_2-s_3^4+s_3^3+s_3^2+s_3-1)$$

$$E_{32}^{[3]_2} = \frac{-(s_3-1)}{(60s_2(s_1-s_2)(s_2-s_3)}(2s_1-s_2+s_3-s_2^2s_3^2+2s_1s_2-3s_1s_3+s_2s_3$$
$$-3s_1s_2^3-3s_1s_3^2+2s_1s_3^3+s_2s_3^2+s_2^2s_3-s_2s_3^3-s_2^3s_3-s_2^2-s_2^3+2s_2^4$$
$$+s_3^2+s_3^3+2s_1s_2^2+2s_1s_2s_3^2+2s_1s_2^2s_3-s_3^4-3s_1s_2s_3-1)$$

$$E_{33}^{[3]_2} = \frac{(s_2-1)}{(60s_3(s_1-s_3)(s_2-s_3)}(2s_1+s_2-s_3-s_2^2s_3^2-3s_1s_2+2s_1s_3+s_2s_3$$
$$+2s_1s_2^3+2s_1s_3^2-3s_1s_3^3+s_2s_3^2+s_2^2s_3-s_2s_3^3-s_2^3s_3+s_2^2+s_2^3-s_2^4$$
$$-s_3^2-s_3^3-3s_1s_2^2+2s_3^4+2s_1s_2s_3^2+2s_1s_2^2s_3-3s_1s_2s_3-1)$$

$$E_{34}^{[3]_2} = \frac{-1}{(60s_1-60)}(s_2^2s_3^2-s_2-s_3-3s_1+2s_1s_2+2s_1s_3+s_2s_3+2s_1s_2^2$$
$$+2s_1s_3^2+2s_1s_3^3+s_2s_3^2+s_2^2s_3+s_2s_3^3+s_2^3s_3-s_2^2-s_2^3-s_2^4-s_3^2$$
$$+2s_1s_2^3-3s_1s_2s_3^2-3s_1s_2^2s_3-3s_1s_2s_3-s_3^3-s_3^4+2)$$

$$E_{41}^{[3]_2} = \frac{-1}{(60hs_1(s_1-s_2)(s_1-s_3)(s_1-1))}(-s_2^5+s_2^4s_3+2s_2^4+s_2^3s_3^2-3s_2^3s_3$$
$$+s_2^2s_3^3-3s_2^2s_3^2+s_2s_3^4-3s_2s_3^3-s_3^5+2s_3^4)$$

$$E_{42}^{[3]_2} = \frac{1}{(60hs_2(s_1-s_2)(s_2-s_3)(s_2-1))}(2s_2^2s_3^2-s_2^2s_3^3-s_2^3s_3^2+5s_1s_2^33s_1s_2^4$$
$$+2s_1s_3^4+2s_2s_3^3+2s_2^3s_3-s_2s_3^4-s_2^4s_3-3s_2^4+2s_3^4-5s_1s_2s_3^2+2s_2^5$$
$$-5s_1s_2^2s_3-5s_1s_3^3+2s_1s_2s_3^3+2s_1s_2^3s_3+2s_1s_2^2s_3^2-s_3^5)$$

$$E_{43}^{[3]_2} = \frac{-1}{(60hs_3(s_1-s_3)(s_2-s_3)(s_3-1))}(2s_2^2s_3^2 - s_2^2s_3^3 - s_2^3s_3^2 - 5s_1s_2^3 + 2s_2^4 \\ -3s_1s_3^4 + 2s_2s_3^3 + 2s_2^3s_3 - s_2s_3^4 - s_2^4s_3 - 3s_3^4 + 2s_3^5 - 5s_1s_2s_3^2 + 2s_1s_2^4 \\ +5s_1s_3^3 - 5s_1s_2^2s_3 + 2s_1s_2s_3^3 + 2s_1s_2^3s_3 + 2s_1s_2^2s_3^2 - s_2^5)$$

$$E_{44}^{[3]_2} = \frac{1}{(h(60s_1-60)(s_2-1)(s_3-1))}(-s_2^5 + s_2^4s_3 + 2s_1s_2^4 + s_2^3s_3^2 - 3s_1s_2^3s_3 \\ +s_2^2s_3^3 - 3s_1s_2^2s_3^2 + s_2s_3^4 - 3s_1s_2s_3^3 - s_3^5 + 2s_1s_3^4)$$

Multiplying equation (7) by the inverse of $A^{[3]_2}$ gives the following one step with generalized three off-step points block method.

$$I^{[2]_3}Y_m^{[3]_2} = \bar{B}_1^{[3]_2}R_1^{[3]_2} + \bar{B}_2^{[3]_2}R_2^{[3]_2} + h^2\bar{D}^{[3]_2}R_3^{[3]_2} + h^2\bar{E}^{[3]_2}R_4^{[3]_2} \qquad (8)$$

where

$$I^{[2]_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_1^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_2^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix}$$

$$\bar{D}^{[3]_2} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_2} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_2} \end{pmatrix}, \bar{E}^{[3]_2} = \begin{pmatrix} \bar{E}_{11}^{[3]_2} & \bar{E}_{12}^{[3]} & \bar{E}_{13}^{[3]_2} & \bar{E}_{14}^{[3]_2} \\ \bar{E}_{21}^{[3]_2} & \bar{E}_{22}^{[3]_2} & \bar{E}_{23}^{[3]_2} & \bar{E}_{24}^{[3]_2} \\ \bar{E}_{31}^{[3]_2} & \bar{E}_{32}^{[3]_2} & \bar{E}_{33}^{[3]_2} & \bar{E}_{34}^{[3]_2} \\ \bar{E}_{41}^{[3]_2} & \bar{E}_{42}^{[3]_2} & \bar{E}_{43}^{[3]_2} & \bar{E}_{44}^{[3]_2} \end{pmatrix}$$

with

$$\bar{D}_{14}^{[3]_2} = \frac{-s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3)}{(60s_2s_3)}$$

$$\bar{D}_{24}^{[3]_2} = \frac{s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3)}{(60s_1s_3)}$$

$$\bar{D}_{34}^{[3]_2} = \frac{-s_3^2(5s_1s_3 - 20s_1s_2 + 5s_2s_3 - 2s_1s_3^2 - 2s_2s_3^2 - 2s_3^2 + s_3^3 + 5s_1s_2s_3)}{(60s_1s_2)}$$

$$\bar{D}_{44}^{[3]_2} = \frac{(2s_1 + 2s_2 + 2s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 20s_1s_2s_3 - 1)}{(60s_1s_2s_3)}$$

$$\bar{E}_{11}^{[3]_2} = \frac{s_1^2(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))}$$

$$\bar{E}_{12}^{[3]_2} = \frac{s_1^4(5s_3 - 2s_1 - 2s_1s_3 + s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$

$$\bar{E}_{13}^{[3]_2} = \frac{-s_1^4(5s_2 - 2s_1 - 2s_1s_2 + s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$

$$\bar{E}_{14}^{[3]_2} = \frac{s_1^4(5s_2s_3 - 2s_1s_3 - 2s_1s_2 + s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}$$

$$\bar{E}_{21}^{[3]_2} = \frac{-s_2^4(5s_3 - 2s_2 - 2s_2s_3 + s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}_{22}^{[3]_2} = \frac{s_2^2(10s_1s_3 - 5s_1s_2 - 5s_2s_3 + 3s_1s_2^2 + 3s_2^2s_3 + 3s_2^2 - 2s_2^3 - 5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}_{23}^{[3]_2} = \frac{s_2^4(2s_2 - 5s_1 + 2s_1s_2 - s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}_{24}^{[3]_2} = \frac{-s_2^4(2s_1s_2 - 5s_1s_3 + 2s_2s_3 - s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}$$
$$\bar{E}_{31}^{[3]_2} = \frac{h^2s_3^4(2s_3 - 5s_2 + 2s_2s_3 - s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}_{32}^{[3]_2} = \frac{-s_3^4(2s_3 - 5s_1 + 2s_1s_3 - s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}_{33}^{[3]_2} = \frac{s_3^2(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}_{34}^{[3]_2} = \frac{s_3^4(5s_1s_2 - 2s_1s_3 - 2s_2s_3 + s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))}$$
$$\bar{E}_{41}^{[3]_2} = \frac{-(5s_2s_3 - 2s_3 - 2s_2 + 1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}$$
$$\bar{E}_{42}^{[3]_2} = \frac{(5s_1s_3 - 2s_3 - 2s_1 + 1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}$$
$$\bar{E}_{43}^{[3]_2} = \frac{-(5s_1s_2 - 2s_2 - 2s_1 + 1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}$$
$$\bar{E}_{44}^{[3]_2} = \frac{(3s_1 + 3s_2 + 3s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 10s_1s_2s_3 - 2)}{(60(s_3-1)(s_2-1)(s_1-1))}$$

Equation (8) can also be written as

$$\begin{aligned} y_{n+s_1} &= y_n + hs_1y_n^{'} \\ &+ \frac{-h^2s_1^2(5s_1s_2 + 5s_1s_3 - 20s_2s_3 - 2s_1^2s_2 - 2s_1^2s_3 - 2s_1^2 + s_1^3 + 5s_1s_2s_3)}{(60s_2s_3)} f_n \\ &+ \frac{h^2s_1^2(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\ &+ \frac{h^2s_1^4(5s_3 - 2s_1 - 2s_1s_3 + s_1^2)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)} f_{n+s_2} - \frac{h^2s_1^4(5s_2 - 2s_1 - 2s_1s_2 + s_1^2)}{60s_3(s_3-1)(s_2-s_3)(s_1-s_3)} f_{n+s_3} \\ &+ \frac{h^2s_1^4(5s_2s_3 - 2s_1s_3 - 2s_1s_2 + s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \end{aligned}$$

$$\begin{aligned} y_{n+s_2} &= y_n + hs_2y_n^{'} \\ &+ \frac{h^2s_2^2(20s_1s_3 - 5s_1s_2 - 5s_2s_3 + 2s_1s_2^2 + 2s_2^2s_3 + 2s_2^2 - s_2^3 - 5s_1s_2s_3)}{(60s_1s_3)} f_n \end{aligned}$$

$$-\frac{h^2s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}f_{n+s_1}+\frac{h^2s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{60s_3(s_3-1)(s_2-s_3)(s_1-s_3)}f_{n+s_3}$$
$$+\frac{h^2s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2}$$
$$-\frac{h^2s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}f_{n+1} \qquad (9)$$
$$y_{n+s_3}=y_n+hs_3y_n^{'}$$
$$+\frac{-h^2s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)}f_n$$
$$+\frac{h^2s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2)}f_{n+s_1}+\frac{h^2s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{60(s_2-1)(s_1-1)(s_3-1)}f_{n+1}$$
$$+\frac{-h^2s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2}$$
$$+\frac{h^2s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}$$
$$y_{n+1}=y_n+hy_n^{'}$$
$$+\frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)}f_n$$
$$+\frac{-h^2(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1}$$
$$+\frac{h^2(5s_1s_3-2s_3-2s_1+1)}{60s_2(s_2-1)(s_2-s_3)(s_1-s_2)}f_{n+s_2}$$
$$+\frac{-h^2(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}$$
$$+\frac{h^2(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}$$

The first derivative of (8)is given by

$$y_{n+s_1}^{'}=y_n^{'}$$
$$-\frac{hs_1(10s_1s_2+10s_1s_3-30s_2s_3-5s_1^2s_2-5s_1^2s_3-5s_1^2+3s_1^3+10s_1s_2s_3)}{60s_2s_3}f_n$$
$$+\frac{hs_1(20s_1s_2+20s_1s_3-30s_2s_3-15s_1^2s_2-15s_1^2s_3-15s_1^2+12s_1^3+20s_1s_2s_3)}{60(s_1-1)(s_1-s_3)(s_1-s_2)}f_{n+s_1}$$
$$+\frac{hs_1^3(10s_3-5s_1-5s_1s_3+3s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2}$$
$$+\frac{-hs_1^3(10s_2-5s_1-5s_1s_2+3s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}$$
$$+\frac{hs_1^3(10s_2s_3-5s_1s_3-5s_1s_2+3s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))}f_{n+1}$$

$$
\begin{aligned}
y'_{n+s_2} &= y'_n \\
&+ \frac{s_2(30s_1s_3 - 10s_1s_2 - 10s_2s_3 + 5s_1s_2^2 + 5s_2^2s_3 + 5s_2^2 - 3s_2^3 - 10s_1s_2s_3)}{(60s_1s_3)} f_n \\
&+ \frac{-s_2^3(10s_3 - 5s_2 - 5s_2s_3 + 3s_2^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{s_2(30s_1s_3 - 20s_1s_2 - 20s_2s_3 + 15s_1s_2^2 + 15s_2^2s_3 + 15s_2^2 - 12s_2^3 - 20s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{s_2^3(5s_2 - 10s_1 + 5s_1s_2 - 3s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \\
&+ \frac{-s_2^3(5s_1s_2 - 10s_1s_3 + 5s_2s_3 - 3s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))}
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+s_3} &= y'_n \qquad (10) \\
&- \frac{s_3(10s_1s_3 - 30s_1s_2 + 10s_2s_3 - 5s_1s_3^2 - 5s_2s_3^2 - 5s_3^2 + 3s_3^3 + 10s_1s_2s_3)}{(60s_1s_2)} f_n \\
&+ \frac{s_3^3(5s_3 - 10s_2 + 5s_2s_3 - 3s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} - \frac{s_3^3(5s_3 - 10s_1 + 5s_1s_3 - 3s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
&+ \frac{s_3(20s_1s_3 - 30s_1s_2 + 20s_2s_3 - 15s_1s_3^2 - 15s_2s_3^2 - 15s_3^2 + 12s_3^3 + 20s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{s_3^3(10s_1s_2 - 5s_1s_3 - 5s_2s_3 + 3s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))} f_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{(5s_1 + 5s_2 + 5s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 30s_1s_2s_3 - 3)}{(60s_1s_2s_3)} f_n \\
&- \frac{(10s_2s_3 - 5s_3 - 5s_2 + 3)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
&+ \frac{(10s_1s_3 - 5s_3 - 5s_1 + 3)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} - \frac{(10s_1s_2 - 5s_2 - 5s_1 + 3)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
&+ \frac{(15s_1 + 15s_2 + 15s_3 - 20s_1s_2 - 20s_1s_3 - 20s_2s_3 + 30s_1s_2s_3 - 12)}{(60(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned}
$$

3 Properties of the Method

3.1 Order of Method

The linear difference operator L associated with (8) is defined as

$$L[y(x);h] = Y_m^{[3]_2} - \bar{B}_1^{[3]_2} R_1^{[3]_2} - \bar{B}_2^{[3]_2} R_2^{[3]_3} - h^2 \left[\bar{D}^{[3]_2} R_3^{[3]_2} + \bar{E}^{[3]_2} R_4^{[3]_2} \right] \quad (11)$$

where $y(x)$ is an arbitrary test function continuously differentiable on [a,b]. The elements of Y_m and $R_3^{[2]_2}$ are expanded in Taylors series respectively and their terms are collected in powers of h to give

$$\begin{bmatrix}
\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_1)^j h^j}{j!} y_n^j - y_n - s_1 h y_n^{'} \\
+\frac{h^2 s_1^2(5s_1s_2+5s_1s_3-20s_2s_3-2s_1^2s_2-2s_1^2s_3-2s_1^2+s_1^3+5s_1s_2s_3)}{(60s_2s_3)} y_n^{''} \\
-\frac{h^2 s_1^4(5s_2s_3-2s_1s_3-2s_1s_2+s_1^2)}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_1^4(5s_3-2s_1-2s_1s_3+s_1^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{s_2^j h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_1^2(5s_1s_2+5s_1s_3-10s_2s_3-3s_1^2s_2-3s_1^2s_3-3s_1^2+2s_1^3+5s_1s_2s_3)}{(60(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+2}}{j!} y_n^{j+2} \\
+\frac{s_1^4(5s_2-2s_1-2s_1s_2+s_1^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+2}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{(s_2)^j h^j}{j!} y_n^j - y_n - s_2 h y_n^{'} \\
-\frac{h^2 s_2^2(20s_1s_3-5s_1s_2-5s_2s_3+2s_1s_2^2+2s_2^2s_3+2s_2^2-s_2^3-5s_1s_2s_3)}{(60s_1s_3)} y_n^{''} \\
+\frac{s_2^4(2s_1s_2-5s_1s_3+2s_2s_3-s_2^2)}{(60(s_3-1)(s_1-1)(s_2-1))} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_2^2(10s_1s_3-5s_1s_2-5s_2s_3+3s_1s_2^2+3s_2^2s_3+3s_2^2-2s_2^3-5s_1s_2s_3)}{(60(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+2}}{j!} y_n^{j+2} \\
+\frac{s_2^4(5s_3-2s_2-2s_2s_3+s_2^2)}{60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_2^4(2s_2-5s_1+2s_1s_2-s_2^2)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+2}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{(s_3)^j h^j}{j!} y_n^j - y_n - s_3 h y_n^{'} \\
+\frac{h^2 s_3^2(5s_1s_3-20s_1s_2+5s_2s_3-2s_1s_3^2-2s_2s_3^2-2s_3^2+s_3^3+5s_1s_2s_3)}{(60s_1s_2)} y_n^{''} \\
-\frac{s_3^4(5s_1s_2-2s_1s_3-2s_2s_3+s_3^2)}{(60(s_2-1)(s_1-1)(s_3-1))} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
+\frac{s_3^4(2s_3-5s_1+2s_1s_3-s_3^2)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_3^4(2s_3-5s_2+2s_2s_3-s_3^2)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+2}}{j!} y_n^{j+2} \\
-\frac{s_3^2(5s_1s_3-10s_1s_2+5s_2s_3-3s_1s_3^2-3s_2s_3^2-3s_3^2+2s_3^3+5s_1s_2s_3)}{(60(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+2}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - h y_n^{'} \\
-\frac{h^2(2s_1+2s_2+2s_3-5s_1s_2-5s_1s_3-5s_2s_3+20s_1s_2s_3-1)}{(60s_1s_2s_3)} y_n^{''} \\
-\frac{(3s_1+3s_2+3s_3-5s_1s_2-5s_1s_3-5s_2s_3+10s_1s_2s_3-2)}{(60(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
-\frac{(5s_1s_3-2s_3-2s_1+1)}{(60s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+2}}{j!} y_n^{j+2} \\
+\frac{(5s_2s_3-2s_3-2s_2+1)}{(60s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+2}}{j!} y_n^{j+2} \\
+\frac{(5s_1s_2-2s_2-2s_1+1)}{(60s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+2}}{j!} y_n^{j+2}
\end{array}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j gives the order of the method to be

$[5, 5, 5, 5]^T$ with vector of error constant

$$\bar{C}_7 = \begin{bmatrix} \frac{-s_1^3}{7200}(5s_1s_2 + 5s_1s_3 - 10s_2s_3 - 3s_1^2s_2 - 3s_1^2s_3 - 3s_1^2 + 2s_1^3 + 5s_1s_2s_3) \\ \frac{s_2^3}{7200}(10s_1s_3 - 5s_1s_2 - 5s_2s_3 + 3s_1s_2^2 + 3s_2^2s_3 + 3s_2^2 - 2s_2^3 - 5s_1s_2s_3) \\ \frac{-s_3^3}{7200}(5s_1s_3 - 10s_1s_2 + 5s_2s_3 - 3s_1s_3^2 - 3s_2s_3^2 - 3s_3^2 + 2s_3^3 + 5s_1s_2s_3) \\ \frac{1}{7200}(3s_1 + 3s_2 + 3s_3 - 5s_1s_2 - 5s_1s_3 - 5s_2s_3 + 10s_1s_2s_3 - 2) \end{bmatrix}$$

for all $s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{-5s_1s_3+3s_1^2s_3+3s_1^2-2s_1^3}{5s_1-10s_3-3s_1^2+5s_1s_3}\} \cup \{s_1 = \frac{5s_3-3s_2^2s_3-3s_2^2+s_2^3}{10s_3-5s_2+3s_2^2-5s_2s_3}\}$ $\cup\{s_1 = \frac{-5s_1s_3+3s_2s_3^2+3s_3^2-2s_3^3}{5s_3-10s_2-3s_3^2+5s_2s_3}\} \cup \{s_1 = \frac{-3s_2-3s_3+5s_2s_3+2}{3-5s_2-5s_3+10s_2s_3}\}$.

3.2 Zero stability

Definition 3.1 *The hybrid block method (8) is said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1$, and if $|r_z| = 1$ then the multiplicity of r_z must not greater than two.*

In order to fined the zero-stability of (8), we only consider the first characteristic polynomial of the method according to Definition(3.1) as follows

$$\Pi(r) = |r\, I^{[2]_3} - \bar{B}_1^{[3]_3}| = \left| r \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(r-1)$$

which implies $r = 0, 0, 0, 1$. Hence, our method is zero stable since $|r_z| \leq 1$, and the multiplicity when $|r_z| = 1$ is one for all $s_1, s_2, s_3 \in (0, 1)$.

3.3 Consistency

Definition 3.2 *The one step hybrid block method (8) is said to be consistent if the order of the method is greater than or equal to one i.e. $P \geq 1$*

By above definition the block method (8) is consistent because its order is five.

3.4 Convergence

Theorem 3.3 (Henrici, 1962) *Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent*

Since the method (8) is consistent and zero stable, it implies the method is convergent for all off step points.

3.5 Region of absolute stability

In this article, the locus method was adopted to determine the function of stability region. The method (8) is said to be absolutely stable if for a given h all roots of the characteristic polynomial $\pi(z,h) = \rho(z) - h^2\sigma(z)$, satisfies $|z_t| < 1$. The test equation $y = -\lambda^2 y$ is substituted in (8) where $\bar{h} = -\lambda^2 h^2$ and $\lambda = \frac{df}{dy}$. Substituting $r = \cos\theta - i\sin\theta$ and considering real part yields

$$\begin{aligned}\bar{h}(\theta,h) &= 43200(\cos\theta - 1)/(s_1s_2s_3(4s_1 + 4s_2 + 4s_3 \\ &- 3s_1s_2 - 3s_1s_3 - 3s_2s_3 + 2s_1s_2s_3 + s_1s_2s_3\cos\theta - 5)) \qquad (12)\end{aligned}$$

3.5.1 Numerical Examples

In order to find the accuracy of our methods, the following second order ODEs are tasted. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$.

Exact solution: $y(x) = 1 + \frac{1}{2}ln\left(\frac{2+x}{2-x}\right)$ with $h = 0.05$

Table 1: Comparison of the new method with Jator(2007) for solving Problem 1

x	Exact solution	Computed solution in our method with three off-step points $s_1 = \frac{3}{10}$, $s_2 = \frac{5}{10}$, $s_3 = \frac{7}{10}$	Error in our method, $P = 5$	Errors in Jator(2007), $P = 6$
0.1	1.0500417292784914	1.0500417292784476	$4.374279e^{-14}$	$7.1629e^{-12}$
0.2	1.1003353477310753	1.1003353477308178	$2.577938e^{-13}$	$1.5091e^{-11}$
0.3	1.1511404359364665	1.1511404359351332	$1.333600e^{-12}$	$4.5286e^{-11}$
0.4	1.2027325540540816	1.2027325540493166	$4.765521e^{-12}$	$1.0808e^{-10}$
0.5	1.2554128118829946	1.2554128118695882	$1.340705e^{-11}$	$1.7818e^{-10}$
0.6	1.3095196042031119	1.3095196041705097	$3.260214e^{-11}$	$4.4434e^{-10}$
0.7	1.3654437542713971	1.3654437541989344	$7.246204e^{-11}$	$7.4446e^{-10}$
0.8	1.4236489301936035	1.4236489300410109	$1.525911e^{-10}$	$1.5009e^{-9}$
0.9	1.4847002785940546	1.4847002782817593	$3.122926e^{-10}$	$3.7579e^{-9}$
1.0	1.5493061443340586	1.5493061437002651	$6.337899e^{-10}$	$4.7410e^{-9}$

Problem 2: $y'' + (\frac{6}{x})y' + (\frac{4}{x^2})y = 0$, $y(1) = 1$, $y'(1) = 1$.

Exact solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$ with $h = \frac{1}{320}$.

Table 2: Comparison of the new method with Badmus (2014) for solving Problem 2.

x	Exact solution	Computed solution in our method with one off-step points $s1 = \frac{3}{10}$, $s2 = \frac{5}{10}$, $s3 = \frac{7}{10}$	Error in our method, $P = 5$	Badmus (2014), $P = 8$
1.003125	1.0030765258576961	1.0030765258576961	0.000000	$1.645e^{-7}$
1.006250	1.0060575030835164	1.0060575030837122	$1.9584e^{-13}$	$6.603e^{-7}$
1.009375	1.0089449950888376	1.0089449950894158	$5.7820e^{-13}$	$4.4141e^{-6}$
1.012500	1.0117410181679887	1.0117410181691262	$1.1375e^{-12}$	$1.2993e^{-5}$
1.018750	1.0144475426864139	1.0144475426882791	$1.8651e^{-12}$	$1.6377e^{-5}$
1.028125	1.0170664942356729	1.0170664942384249	$2.7520e^{-12}$	$2.8296e^{-5}$
1.021875	1.0195997547562881	1.0195997547600784	$3.7903e^{-12}$	$5.0516e^{-5}$
1.025000	1.0220491636294322	1.0220491636344042	$4.9720e^{-12}$	$3.8609e^{-5}$
1.028125	1.0244165187384029	1.0244165187446930	$6.2900e^{-12}$	$7.4909e^{-5}$
1.031250	1.0267035775008062	1.0267035775085429	$7.7367e-^{12}$	$1.4588e^{-5}$

4 Conclusion

A new single step hybrid block method with generalized three off-step points for the direct solution of second order ordinary differential equation has been developed successfully. The developed method is zero-stable, consistent and also convergent. When solving the same problems, the numerical results confirm that the new method produces better accuracy if compared to the existing methods.

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