An Extension of Zadeh’s Max-Min Composition Operator

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Abstract

We generate the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued $\alpha$-cuts, we get the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$. We prove that the results for the parametric operations are the generalization of Zadeh’s extended algebraic operations.

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1 Introduction

A triangular fuzzy number is the most famous fuzzy number. The membership function of triangular fuzzy number is very simple and consisting of two

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monotone increasing and decreasing lines. We had Zadeh’s max-min compositions for two generalized triangular fuzzy sets and normal fuzzy probability for generalized triangular fuzzy sets. Furthermore, we had exponential fuzzy probability for generalized triangular fuzzy sets\(^\[4\]\). We generated the triangular fuzzy numbers on \(\mathbb{R} \to \mathbb{R}^2\) \([3]\). By defining parametric operations between two regions valued \(\alpha\)-cuts, we get the parametric operations for two triangular fuzzy numbers defined on \(\mathbb{R}^2\). In this paper, we prove that the results for the parametric operations are the generalization of Zadeh’s extended algebraic operations.

2 Preliminaries

Let \(X\) be a set. We define \(\alpha\)-cut and \(\alpha\)-set of the fuzzy set \(A\) with the membership function \(\mu_A(x)\).

**Definition 2.1** An \(\alpha\)-cut of the fuzzy number \(A\) is defined by \(A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}\) if \(\alpha \in (0, 1]\) and \(A_\alpha = \text{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}\) if \(\alpha = 0\). For \(\alpha \in (0, 1]\), the set \(A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}\) is said to be the \(\alpha\)-set of the fuzzy set \(A\), \(A^0\) is the boundary of \(\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}\) and \(A^1 = A_1\).

In the calculations between two fuzzy numbers, the concept of \(\alpha\)-cut is very important. Furthermore, some operations between \(\alpha\)-cuts are very useful and \(\alpha\)-set plays a very important role in a 2-dimensional case.

**Definition 2.2** ([5]) The extended addition \(A(+)B\), extended subtraction \(A(-)B\), extended multiplication \(A(\cdot)B\) and extended division \(A(/)B\) are fuzzy sets with membership functions as follows.

1. \(\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}\), \(x \in A, y \in B\)
2. \(\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}\), \(x \in A, y \in B\)
3. \(\mu_{A(\cdot)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}\), \(x \in A, y \in B\)
4. \(\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}\), \(x \in A, y \in B\)

We proved that for all fuzzy numbers \(A\) and all \(\alpha \in [0, 1]\), there exists a piecewise continuous function \(f_\alpha(t)\) defined on \([0, 1]\) such that \(A_\alpha = \{f_\alpha(t)\mid t \in [0, 1]\}\). If \(A\) is continuous, then the corresponding function \(f_\alpha(t)\) is also continuous. The corresponding function \(f_\alpha(t)\) is said to be the parametric \(\alpha\)-function of \(A\). The parametric \(\alpha\)-function of \(A\) is denoted by \(f_\alpha(t)\) or \(f_A(t)\).
Definition 2.3 ([1]) Let $A$ and $B$ be two continuous fuzzy numbers defined on $\mathbb{R}$ and $f_A(t), f_B(t)$ be the parametric $\alpha$-functions of $A$ and $B$, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their $\alpha$-cuts as follows.

1. parametric addition $A(+)_pB : (A(+)_pB)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0, 1]\}$
2. parametric subtraction $A(-)_pB : (A(-)_pB)_\alpha = \{f_A(t) - f_B(1 - t) \mid t \in [0, 1]\}$
3. parametric multiplication $A(\cdot)_pB : (A(\cdot)_pB)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0, 1]\}$
4. parametric division $A(/)_pB : (A(/)_pB)_\alpha = \{f_A(t)/f_B(1 - t) \mid t \in [0, 1]\}$

Theorem 2.4 ([1]) Let $A$ and $B$ be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have the followings.

1. $A(+)_pB = A(+)B$
2. $A(-)_pB = A(-)B$
3. $A(\cdot)_pB = A(\cdot)B$
4. $A(/)_pB = A(/)B$

3 2-dimensional triangular fuzzy numbers

Definition 3.1 A fuzzy set $A$ with a membership function

$$
\mu_A(x, y) = \begin{cases} 
1 - \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2, \\
0, & \text{otherwise},
\end{cases}
$$

where $a, b > 0$ is called the 2-dimensional triangular fuzzy number and denoted by $(a, x_1, b, y_1)^2$.

The $\alpha$-cut $A_\alpha$ of a 2-dimensional triangular fuzzy number $A = (a, x_1, b, y_1)^2$ is an interior of ellipse in an $xy$-plane including the boundary

$$
A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2(1-\alpha)^2\}
$$

$$
= \{(x, y) \in \mathbb{R}^2 \mid \left(\frac{x-x_1}{a(1-\alpha)}\right)^2 + \left(\frac{y-y_1}{b(1-\alpha)}\right)^2 \leq 1\}.
$$

Theorem 3.2 ([3]) Let $A$ be a convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$ be the $\alpha$-set of $A$. Then for all $\alpha \in (0, 1)$, there exist piecewise continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$
A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}.
$$
If \( A \) is a continuous convex fuzzy number defined on \( \mathbb{R}^2 \), then the \( \alpha \)-set \( A^\alpha \) is a closed circular convex subset in \( \mathbb{R}^2 \).

**Definition 3.3** ([3]) Let \( A \) and \( B \) be convex fuzzy numbers defined on \( \mathbb{R}^2 \) and
\[
A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \},
\]
\[
B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \}
\]
be the \( \alpha \)-sets of \( A \) and \( B \), respectively. For \( \alpha \in (0, 1) \), we define that the parametric addition \( A(+)_pB \), parametric subtraction \( A(-)_pB \), parametric multiplication \( A(\cdot)_pB \) and parametric division \( A(\bigg/)_pB \) of two fuzzy numbers \( A \) and \( B \) are fuzzy numbers that have their \( \alpha \)-sets as follows.

1. \( A(+)_pB : (A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \} \)
2. \( A(-)_pB : (A(-)_pB)^\alpha = \{(x(t), y(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \}, \) where
\[
\begin{align*}
x_\alpha(t) &= \begin{cases} f_1^\alpha(t) - g_1^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi \\ f_1^\alpha(t) - g_1^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi \end{cases}
\end{align*}
\]
and
\[
\begin{align*}
y_\alpha(t) &= \begin{cases} f_2^\alpha(t) - g_2^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi \\ f_2^\alpha(t) - g_2^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi \end{cases}
\end{align*}
\]
3. \( A(\cdot)_pB : (A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \} \)
4. \( A(\bigg/)_pB : (A(\bigg/)_pB)^\alpha = \{(x(t), y(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi \}, \) where
\[
\begin{align*}
x_\alpha(t) &= \frac{f_1^\alpha(t)}{g_1^\alpha(t + \pi)} \quad (0 \leq t \leq \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t - \pi)} \quad (\pi \leq t \leq 2\pi)
\end{align*}
\]
and
\[
\begin{align*}
y_\alpha(t) &= \frac{f_2^\alpha(t)}{g_2^\alpha(t + \pi)} \quad (0 \leq t \leq \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t - \pi)} \quad (\pi \leq t \leq 2\pi)
\end{align*}
\]

For \( \alpha = 0 \) and \( \alpha = 1 \), \( (A(\cdot)_pB)^0 = \lim_{\alpha \to 0^+} (A(\cdot)_pB)^\alpha \) and \( (A(\cdot)_pB)^1 = \lim_{\alpha \to 1^-} (A(\cdot)_pB)^\alpha \), where \( * = +, -, \cdot, / \).

**Theorem 3.4** ([3]) Let \( A = (a_1, x_1, b_1, y_1)^2 \) and \( B = (a_2, x_2, b_2, y_2)^2 \) be two 2-dimensional triangular fuzzy numbers. Then we have the followings.

1. \( A(+)_pB = \left( a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2 \right)^2 \)
2. \( A(-)_pB = \left( a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2 \right)^2 \)
(3) \((A\cdot p)B\)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}, \text{ where}

\[ x_\alpha(t) = x_1 x_2 + (x_1 a_2 + x_2 a_1)(1 - \alpha) \cos t + a_1 a_2 (1 - \alpha)^2 \cos^2 t \]

and

\[ y_\alpha(t) = y_1 y_2 + (y_1 b_2 + y_2 b_1)(1 - \alpha) \sin t + b_1 b_2 (1 - \alpha)^2 \sin^2 t. \]

(4) \((A/)pB\)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}, \text{ where}

\[ x_\alpha(t) = \frac{x_1 + a_1 (1 - \alpha) \cos t}{x_2 - a_2 (1 - \alpha) \cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1 (1 - \alpha) \sin t}{y_2 - b_2 (1 - \alpha) \sin t}. \]

Thus \((A+)pB\) and \((A-)pB\) become 2-dimensional triangular fuzzy numbers, but \((A\cdot)pB\) and \((A/)pB\) need not to be 2-dimensional triangular fuzzy numbers.

**Theorem 3.5** Parametric operations on \(\mathbb{R}^2\) in Definition 3.3 are the generalization of Zadeh’s extension principles on \(\mathbb{R}\) in Definition 2.2.

**Proof.** Consider two 2-dimensional triangular fuzzy numbers \(A = (a_1, x_1, b_1, 0)^2\) and \(B = (a_2, x_2, b_2, 0)^2\). By Theorem 3.4,

1. \((A+)pB = (a_1 + a_2, x_1 + x_2, b_1 + b_2, 0)^2\)
2. \((A-)pB = (a_1 + a_2, x_1 - x_2, b_1 + b_2, 0)^2\)
3. \((A\cdot)pB = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}, \text{ where}

\[ x_\alpha(t) = x_1 x_2 + (x_1 a_2 + x_2 a_1)(1 - \alpha) \cos t + a_1 a_2 (1 - \alpha)^2 \cos^2 t \]

and

\[ y_\alpha(t) = b_1 b_2 (1 - \alpha)^2 \sin^2 t. \]

4. \((A/)pB \} = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}, \text{ where}

\[ x_\alpha(t) = \frac{x_1 + a_1 (1 - \alpha) \cos t}{x_2 - a_2 (1 - \alpha) \cos t} \quad \text{and} \quad y_\alpha(t) = -\frac{b_1}{b_2}. \]

The intersections of these 2-dimensional triangular fuzzy numbers and vertical \(xz\)-plane \((y = 0)\) are as follows.

1. \((A+)pB\) : Note that

\[ \mu_{A(+)pB}(x, y) = 1 - \sqrt{\left(\frac{x - x_1 - x_2}{a_1 + a_2}\right)^2 + \left(\frac{y}{b_1 + b_2}\right)^2}. \]

If \(y = 0\) and \(\mu_{A(+)pB}(x, y) = 0\),

\[ x = x_1 + x_2 \pm (a_1 + a_2). \]
Thus the intersection is the symmetric triangular fuzzy number $C$ on the $xz$-plane with $\mu_C(x_1 + x_2) = 1$ and the zero cut

$$C_0 = [x_1 + x_2 - (a_1 + a_2), x_1 + x_2 + (a_1 + a_2)].$$

(2) $A(-)pB$; Note that

$$\mu_{A(-)pB}(x, y) = 1 - \sqrt{\left(\frac{x - x_1 + x_2}{a_1 + a_2}\right)^2 + \left(\frac{y}{b_1 + b_2}\right)^2}.$$ 

If $y = 0$ and $\mu_{A(-)pB}(x, y) = 0$,

$$x = x_1 - x_2 \pm (a_1 + a_2).$$

Thus the intersection is the symmetric triangular fuzzy number $D$ on the $xz$-plane with $\mu_D(x_1 - x_2) = 1$ and the zero cut

$$D_0 = [x_1 - x_2 - (a_1 + a_2), x_1 - x_2 + (a_1 + a_2)].$$

(3) $A(\cdot)pB$; If $\alpha = 0$,

$$x_0(t) = x_1x_2 + (x_1a_2 + x_2a_1) \cos t + a_1a_2 \cos^2 t.$$ 

Since

$$x_0(0) = x_1x_2 + x_1a_2 + x_2a_1 + a_1a_2 \text{ and } x_0(\pi) = x_1x_2 - (x_1a_2 + x_2a_1) + a_1a_2,$$

the intersection is a fuzzy number $E$ on the $xz$-plane with $\mu_E(x_1x_2) = 1$ and the zero cut

$$E_0 = [x_1x_2 - (x_1a_2 + x_2a_1) + a_1a_2, x_1x_2 + x_1a_2 + x_2a_1 + a_1a_2].$$

(4) $A(/)pB$; If $\alpha = 0$,

$$x_0(t) = \frac{x_1 + a_1 \cos t}{x_2 - a_2 \cos t}.$$ 

Since

$$x_0(0) = \frac{x_1 + a_1}{x_2 - a_2 \text{ and } x_0(\pi) = \frac{x_1 - a_1}{x_2 + a_2},$$

the intersection is a fuzzy number $F$ on the $xz$-plane with $\mu_F(x_1x_2) = 1$ and the zero cut

$$F_0 = \left[\frac{x_1 - a_1}{x_2 + a_2}, \frac{x_1 + a_1}{x_2 - a_2}\right].$$

On the other hand, the intersection of 2-dimensional triangular fuzzy number $A = (a_1, x_1, b_1, 0)^2$ and vertical $xz$-plane ($y = 0$) is the symmetric triangular fuzzy number $G$ on the $xz$-plane with $\mu_G(x_1) = 1$ and the zero cut
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\[ G_0 = [x_1 - a_1, x_1 + a_1]. \]

The intersection of 2-dimensional triangular fuzzy number \( B = (a_2, x_2, b_2, 0)^2 \) and vertical \( xz \)-plane \( (y = 0) \) is the symmetric triangular fuzzy number \( H \) on the \( xz \)-plane with \( \mu_H(x_2) = 1 \) and the zero cut

\[ H_0 = [x_2 - a_2, x_2 + a_2]. \]

For two triangular fuzzy numbers \( G \) and \( H \), the following result for Zadeh’s extension principle is well known.

\[ G(+)H = C, \ G(-)H = D, \ G(\cdot)H = E \text{ and } G(/)H = F \]

The proof is complete.

References


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