Existence of Hopf Bifurcation in
a Delay Partial Dependent Predator-Prey Model
with Allelopathic Effect

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Abstract

In this paper, the dynamics of a partial dependent predator-prey model with allelopathic effect and delay is studied. The model has four equilibrium points, i.e., the extinction of prey point, the extinction of predator point, the extinction of both predator and prey point, and the coexistent equilibrium. It is shown that the stability properties of the first three equilibrium points are not affected by the time delay; i.e., the extinction of predator point and the extinction of both predator and prey point are unstable, while the extinction of prey point is conditionally stable. However, the coexistent equilibrium may exhibit a Hopf bifurcation driven by time delay.

Keywords: Partial dependent predator-prey model, allelopathic, Hopf bifurcation

1 Introduction

Since the work of Lotka [6] and Volterra [9], the dynamics of predator-prey interactions have been extensively investigated, including the stability, permanence, periodic solution, bifurcation and chaotic behavior (see e.g. [10-11] and references therein). One of the predator-prey models which is based on the Lotka-Volterra system is a partial dependent predator-prey model proposed by Zhao and Lin [10]
\[
\frac{dx(t)}{dt} = x(t)\left(r_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)\right) \\
\frac{dy(t)}{dt} = y(t)\left(r_2 + a_{21}x(t-\tau) - a_{22}y(t-\tau)\right),
\]  

(1)

where \(x(t)\) and \(y(t)\) are the population densities of prey and predator at time \(t\), respectively; while \(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}, \tau\) are all positive constants. The discrete version of system (1) has been studied by Riva et al. [2], Zhao and Lin [10] and Riva et al. [2] showed that system (1) exhibits a Hopf bifurcation which is controlled by the time delay \(\tau\).

One of important aspects in the dynamics of two-species interaction is allelopathy. Allelopathy can be defined as the direct or indirect effect of one species on another through the production of a chemical released into the environment. The mathematical model that incorporates the allelopathic effects in the dynamics of two species Lotka-Volterra competition system is firstly introduced by Maynard-Smith [7]. Since then, a lot of mathematical models have been proposed to study the influence of allelopathic on the population dynamics of two interacting species, see e.g. [1, 3-4, 8]. Recently, Fitria et al. [5] reconsider the non-delay partial dependent predator prey model (1) and include the allelopathic effects. In the present paper we extend model presented by Fitria et al. [5] by introducing a time delay:

\[
\frac{dx(t)}{dt} = x(t)\left(r_1 - a_{11}x(t-\tau) - a_{12}y(t) - \gamma x(t)y^2(t)\right) \\
\frac{dy(t)}{dt} = y(t)\left(r_2 + a_{21}x(t-\tau) - a_{22}y(t)\right)
\]

(2)

where \(\gamma\) denotes the rate of toxic inhibition for the prey species released by the predator species. The aim of this study is to investigate how the time delay \(\tau\) affects the dynamics of this system. In particular, we will show the existence of Hopf bifurcation which is driven by the time delay \(\tau\).

2 Stability of Equilibria and Existence of Hopf Bifurcation

Before studying the effects of time delay, we first summarize the dynamical properties of system (2) when \(\tau = 0\). Detailed analysis of this case can be seen in Fitria et al. [5]. Without time delay (\(\tau = 0\)), system (2) has four equilibrium points, i.e., the trivial or the extinction of both prey and predator equilibrium point \(E_1(0,0)\), the extinction of prey equilibrium point \(E_2(0, r_2 / a_{22})\), the extinction of predator point \(E_3(r_1 / a_{11}, 0)\) and the coexistent equilibrium point \(E_4(x^*, y^*)\), where \(x^* = (a_{22}y^* - r_2) / a_{21}\) and \(y^*\) is the real positive root of the cubic equation

\[-\gamma a_{22}y^{*3} + \gamma r_2 y^{*2} - (a_{11}a_{22} + a_{12}a_{21})y^{*} + a_{11}r_2 + a_{21}r_1 = 0\]

(3)
According to the Descartes’ rule, equation (3) has at least one positive root. Due to the biological nature, \( E_4 \) exists if \( y^* > r_2 / a_{22} \). It was shown by Fitria et al. [5] that \( E_1(0,0) \) and \( E_3(\eta / a_{11},0) \) are always unstable; while \( E_2(0,r_2 / a_{22}) \) is locally asymptotically stable if \( a_{22}\eta < a_{12}r_2 \), and \( E_4(x^*, y^*) \) is always locally asymptotically stable whenever it exists.

In the following we investigate the influence of time delay in system (2). It is clear that the equilibrium points of system (2) with time delay are exactly the same as those of system without delay. The dynamical analysis of system (2) is performed by first considering the transformation \( x(t) = \hat{x} + \varepsilon X(t) \) and \( y(t) = \hat{y} + \varepsilon Y(t) \) where \((\hat{x}, \hat{y})\) is an equilibrium point and \( 0 < \varepsilon << 1 \) to obtain the linear system of delay differential equations

\[
\frac{dU(t)}{dt} = AU(t) + BU(t - \tau)
\]

where \( U(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \), \( A = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \), \( B = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} \), \( v_1 = \eta - a_{11}\hat{x} - a_{12}\hat{y} - 2\gamma \hat{x}\hat{y}^2 \)
\( v_2 = -a_{12}\hat{x} - 2\gamma \hat{x}^2 \hat{y} \), \( v_3 = 0 \), \( v_4 = r_2 + a_{21}\hat{x} - 2a_{22}\hat{y} \), \( w_1 = -a_{11}\hat{x} \), \( w_2 = 0 \), \( w_3 = a_{21}\hat{y} \) and \( w_4 = 0 \). The characteristic equation of system (4) is given by

\[
\left( \lambda - v_1 - w_1 e^{-\lambda \tau} \right)\left( \lambda - v_4 \right) - v_2 w_3 e^{-\lambda \tau} = 0.
\]

At the trivial equilibrium point \( E_1(0,0) \), the characteristic equation (5) is reduced to

\[
(\lambda - \eta)(\lambda - r_2) = 0.
\]

Hence the eigenvalues are \( \eta > 0 \) and \( r_2 > 0 \), showing that \( E_1(0,0) \) is unstable. The characteristic equation at equilibrium \( E_2(0,r_2 / a_{22}) \) is

\[
\left( \lambda - \left( \eta - \frac{a_{12}r_2}{a_{22}} \right) \right)\left( \lambda + r_2 \right) = 0
\]

where its eigenvalues are \( \lambda_1 = \eta - \frac{a_{12}r_2}{a_{22}} \) and \( \lambda_2 = -r_2 < 0 \). It is clear that \( E_2 \) is stable if condition \( a_{22}\eta < a_{12}r_2 \) holds. The characteristic equation at equilibrium \( E_3(\eta / a_{11},0) \) is

\[
\left( \lambda + \eta e^{-\lambda \tau} \left( \lambda - \left( r_2 + \frac{a_{21}\eta}{a_{11}} \right) \right) \right) = 0
\]

which has two real roots, \( \lambda_1 = -\eta e^{-\lambda \tau} < 0 \) and \( \lambda_2 = r_2 + a_{21}\eta / a_{11} > 0 \). Therefore, equilibrium \( E_3(\eta / a_{11},0) \) is unstable. Hence, the stability properties
of \( E_1(0,0), E_2(0,r_2/a_{22}) \) and \( E_3(\eta_1/a_{11},0) \) are exactly the same as those of system (2) without time delay \( \tau \).

At equilibrium point \( E_4(x^*, y^*) \), the characteristic equation of the linearized system (4) is given by

\[
P(\lambda) + Q(\lambda)e^{-\lambda \tau} = 0
\]

where \( P(\lambda) = \lambda^2 + b_1 \lambda + b_2 \) and \( Q(\lambda) = d_1 \lambda + d_2 \) with

\[
b_1 = a_{22}y^* + y^* a_{21}y^* y^*, \quad b_2 = a_{22}y^* y^* y^*, \quad d_1 = a_{11}y^*
\]

\[
d_2 = a_{11}a_{22}x^* y^* + a_{12}a_{21}x^* y^* + 2a_{21}y^* y^* y^2.
\]

Now suppose that \( \lambda = i \omega, \omega > 0 \) is a root for the characteristic equation (6), then we have

\[
-\omega^2 + b_2 + d_1 \omega \sin \omega \tau + d_2 \cos \omega \tau + (b_1 \omega + d_1 \omega \cos \omega \tau - d_2 \sin \omega \tau) = 0
\]

The real and imaginary parts of this equation are respectively given by

\[
-\omega^2 + b_2 + d_1 \omega \sin \omega \tau + d_2 \cos \omega \tau = 0
\]

\[
b_1 \omega + d_1 \omega \cos \omega \tau - d_2 \sin \omega \tau = 0
\]

or equivalently

\[
-\omega^2 + b_1 = -d_1 \omega \sin \omega \tau - d_2 \cos \omega \tau
\]

\[
b_1 \omega = -d_1 \omega \cos \omega \tau + d_2 \sin \omega \tau.
\]

Squaring both sides gives

\[
\omega^4 - 2b_1 \omega^2 + b_2^2 = \omega^2 \left( \cos \omega \tau \right)^2 + 2d_1 \omega \sin \omega \tau \cos \omega \tau + d_1^2 \omega^2 \left( \sin \omega \tau \right)^2
\]

\[
b_1^2 \omega^2 = \omega^2 \left( \sin \omega \tau \right)^2 - 2d_1 \omega \sin \omega \tau \cos \omega \tau + d_1^2 \omega^2 \left( \cos \omega \tau \right)^2.
\]

Adding both equations and regrouping by powers of \( \omega \), we obtain the following fourth order polynomial

\[
\omega^4 + p \omega^2 + q = 0
\]

where \( p = b_1^2 - d_1^2 - 2b_2 \) and \( q = b_2^2 - d_2^2 \). Substituting \( m = \omega^2 \) into equation (7) leads to the following quadratic equation

\[
K(m) = m^2 + pm + q.
\]

\( \omega \) is real if and only if \( m > 0 \) and hence we directly have the following lemma.

**Lemma 1**

Suppose \( \tau \geq 0 \). If equation (8) has no positive roots then the equilibrium point \( E_4(x^*, y^*) \) is asymptotically stable.

If we define \( D = p^2 - 4q \), then the following cases emerge.

**Case 1.** Equation (8) has no positive root if at least one of the following is satisfied
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\[(S_1) \quad D < 0,\]
\[(S_2) \quad D > 0, \quad p \geq 0, \quad q > 0,\]
\[(S_3) \quad D = 0, \quad p \geq 0,\]

Case 2. If \( D > 0, \quad p < 0, \quad q > 0 \) holds, then equation (8) has two positive roots

Case 3. Equation (8) has one positive root if one of the following is satisfied

\[(R_1) \quad q < 0,\]
\[(R_2) \quad q = 0, \quad p < 0,\]
\[(R_3) \quad D = 0, \quad p < 0.\]

A Hopf bifurcation occurs when equation (8) has at least one real positive root \( m \) such that the real part of complex conjugate pair of eigenvalues of system (4) crosses zero as \( \tau \) varies. We assume that condition \( R_1 \) holds, i.e. \( q < 0 \). We will find the values of \( \tau \) such that system (4) has a pair of purely imaginary eigenvalues \( \lambda = \pm i\alpha_0 \). It can be shown that equation (6) has a pair of purely imaginary roots if \( \tau = \tau_k \) where

\[
\tau_k = \frac{1}{\alpha_0} \cos^{-1}\left(\frac{d_2\left(-\omega_0^2 + b_2\right) + b_1 d_1 \alpha_0^2}{-d_1^2 \omega_0^2 - d_2^2}\right) + \frac{2k\pi}{\alpha_0}, \quad k = 0, 1, 2, 3, \ldots \tag{9}
\]

Let \( \lambda(\tau) = \phi(\tau) + i\omega(\tau) \) be a root of equation (6) satisfying \( \phi(\tau_k) = 0, \omega(\tau_k) = \alpha_0 \).

Substituting \( \lambda(\tau) \) into equation (6) gives the following equation

\[
\alpha^2 - \omega^2 + b_2 \alpha + b_2 + e^{-\alpha\tau} a_1 \cos \omega\tau + e^{-\alpha\tau} d_1 \omega \sin \omega\tau + e^{-\alpha\tau} d_2 \cos \omega\tau +
\left(2\alpha\omega + b_1 \omega - e^{-\alpha\tau} a_1 \sin \omega\tau + e^{-\alpha\tau} d_1 \omega \cos \omega\tau - e^{-\alpha\tau} d_2 \sin \omega\tau\right)i = 0. \tag{10}
\]

From equation (7) and equation (10), we can show that

\[
\left. \frac{d\alpha(\tau)}{d\tau} \right|_{\tau = \tau_0} = \left. \frac{2\alpha_0^4 + p\alpha_0^2}{K_1^2 + K_2^2} \right|_{\tau = \tau_0} = \left. \frac{\alpha_0^4 - q}{K_1^2 + K_2^2} \right|_{\tau = \tau_0} \tag{11}
\]

where \( K_1 = d_1 \sin \omega_0 \tau_0 + d_1 \omega_0 \tau_0 \cos \omega_0 \tau_0 - d_2 \tau_0 \sin \omega_0 \tau_0 - \omega_0 \) and \( K_2 = d_1 \cos \omega_0 \tau_0 - d_1 \omega_0 \tau_0 \sin \omega_0 \tau_0 - d_2 \tau_0 \cos \omega_0 \tau_0 - b_1 \). Since \( q < 0 \), we have that \( \left. \frac{d\alpha(\tau)}{d\tau} \right|_{\tau = \tau_0} > 0 \). Therefore, the transversality condition is satisfied. Hence, we have the following results.
Theorem 1
If \( q < 0 \) then the equilibrium \( E_4(x^*, y^*) \) is asymptotically stable for \( \tau \in (0, \tau_0) \) and unstable when \( \tau > \tau_0 \). In other words, system (2) undergoes a Hopf bifurcation at \( E_4(x^*, y^*) \) when \( \tau = \tau_0 \).

The results in Theorem 1 show the existence of Hopf bifurcation in system (2) which is driven by the time delay \( \tau \).

3 Numerical simulations

To illustrate the results of previous analysis, we perform some simulations by solving system (2) numerically using different values of parameters and time delay. Here, we take the following parameters: \( \eta = 1 \), \( r_2 = 1 \), \( a_{11} = 0.07 \), \( a_{12} = 0.15 \), \( \gamma = 0.008 \), \( a_{21} = 0.05 \) and \( a_{22} = 0.8 \). In this case, system (2) has a coexistent equilibrium \( E_4(7.8426, 1.7402) \). Since \( p = 1.6728 \) and \( q = -0.9615 \), Lemma 1 states that equation (8) has one positive root, i.e., \( m = 0.4524 \). This means that the characteristic equation (6) has a pair of purely imaginary root \( \lambda = \pm i\omega_0 \), where \( \omega_0 = 0.6726 \). Hence, according to Theorem 1 and equation (9), system (2) undergoes a Hopf bifurcation at \( E_4 \) when \( \tau = \tau_0 \approx 2.5939 \). In other words, the coexistent equilibrium \( E_4 \) is asymptotically stable when \( \tau \in [0, \tau_0) \). If the time delay \( \tau \) passes through the critical value \( \tau_0 \), then the coexistent equilibrium loses its stability and the system goes into oscillations. This behavior can be observed from

![Figure 1. Phase portrait of system (2) with \( \tau = 2.5 \). \( E_4 \) is asymptotically stable equilibrium.](image)
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Figure 2. Phase portrait of system (2) with $\tau = 2.7$ and (a) initial value: (6.0, 1.75); (b) initial value: (1,1). $E_4$ is unstable equilibrium and there exists a stable limit cycle.

Numerical solutions shown in Figure 1 and Figure 2. It is seen in Figure 1 that the solution is convergent to $E_4$ when we take $\tau = 2.5 < \tau_0$. For the time delay $\tau = 2.7 > \tau_0$, the solution is periodic which oscillates about $E_4$, see Figure 2. The solution is convergent to a limit cycle. Furthermore, Figure 2(a) and Figure 2(b) show that the limit cycle is stable; meaning that the Hopf bifurcation is supercritical.

4 Conclusion

The dynamics of a partial dependent predator-prey model with allelopathic effect and delay have been investigated. The effect of time delay on system (2) has been discussed. From the stability analysis of all points, it is found that the time delay only affects the stability of the coexistent equilibrium point. Here the time delay is responsible for the stability switch of the coexistent equilibrium point, and a Hopf bifurcation occurs as the time delay increases to a certain threshold. Such analytical results have been confirmed by numerical results. Furthermore, our numerical simulations show that the Hopf bifurcation is supercritical.

References

http://dx.doi.org/10.1016/j.jmaa.2010.01.024

http://dx.doi.org/10.12988/ams.2013.37421

http://dx.doi.org/10.1016/j.nahs.2008.04.001

http://dx.doi.org/10.1016/0304-3800(94)00134-0

http://dx.doi.org/10.1063/1.4914433


http://dx.doi.org/10.1016/j.ecolmodelling.2004.08.021


http://dx.doi.org/10.1016/j.chaos.2009.02.025

http://dx.doi.org/10.1155/2014/104156

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