The Ideal Convergence of Difference Strongly of
\( \chi^2 \) in \( p \)-Metric Spaces Defined by Modulus

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Abstract

The aim of this paper is to introduce and study a new concept of the \( \chi^2 \) space via ideal convergence of difference defined by modulus. Some topological properties of the resulting sequence spaces are also examined.

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Keywords: analytic sequence, modulus function, double sequences, \( \chi^2 \) space, n-metric space
We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in [1]. Later on it was investigated by [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19] and many others.

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

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We procure the following sets of double sequences:

$M_u(t) := \{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \},$

$C_p(t) := \{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \},$

$C_{0p}(t) := \{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \},$

$L_u(t) := \{ (x_{mn}) \in w^2 : \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \},$

$C_{0p}(t) := C_p(t) \cap M_u(t) \text{ and } C_{0bp}(t) = C_{0p}(t) \cap M_u(t);$

where $t = (t_{mn})$ be the sequence of positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim’s sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; M_u(t), C_p(t), C_{0p}(t), L_u(t), C_{0bp}(t)$ and $C_{0bp}(t)$ reduce to the sets $M_u, C_p, C_{0p}, L_u, C_{0p}$ and $C_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. [20] have proved that $M_u(t)$ and $C_p(t), C_{0p}(t)$ are complete paranormed spaces of double sequences and obtained the $\alpha-, \beta-, \gamma-$ duals of the spaces $M_u(t)$ and $C_{0p}(t)$. Zeltser (2001) in her phd thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. [22-27] have independently introduced the statistical convergence and Cauchy for double sequences and established the relation between statistical convergent and strongly Cesàro summable double sequences. [28] have defined the spaces $\mathcal{B}_S, \mathcal{B}_S(t), C_{0S}, C_{Sbp}, C_{Sr}$ and $\mathcal{B}V$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $M_u, M_u(t), C_p, C_{0p}, C_p, L_u, C_{0p}$ and $L_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{B}S, \mathcal{B}V, C_{Sbp}$ and the $\beta(\theta)-$ duals of the spaces $C_{Sbp}$ and $C_{Sr}$ of double series. [29] have introduced the Banach space $L_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $L_q$. Subramanian et al. (2010) have studied the space $\chi_{M}^{2}(p, q, u)$ of double sequences and proved some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by [31] as an extension of the definition of strongly Cesàro summable sequences. Connor (1989) further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [33] the four dimensional matrix transformation $(Ar)_{k,l} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k,l}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.
We need the following inequality in the sequel of the paper. For \( a, b \geq 0 \) and \( 0 < p < 1 \), we have

\[
(a + b)^p \leq a^p + b^p
\]

The double series \( \sum_{m,n=1}^\infty x_{mn} \) is called convergent if and only if the double sequence \( (s_{mn}) \) is convergent, where \( s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \) for all \( m, n \in \mathbb{N} \).

A sequence \( x = (x_{mn}) \) is said to be double analytic if \( \sup_{m,n} |x_{mn}|^{1/m+n} < \infty \). The vector space of all double analytic sequences will be denoted by \( \Lambda^2 \). A sequence \( x = (x_{mn}) \) is called double gai sequence if \( ((m+n)! |x_{mn}|)^{1/m+n} \to 0 \) as \( m, n \to \infty \). The double gai sequences will be denoted by \( \chi^2 \).

An FK-space (or a metric space) \( X \) is said to have AK property if \( (\mathcal{I}_{mn}) \) is a Schauder basis for \( X \). Or equivalently \( x[m,n] \to x \).

An FDK-space is a double sequence space endowed with a complete metrizable, locally convex topology under which the coordinate mappings \( x = (x_k) \to (x_{mn})(m, n \in \mathbb{N}) \) are also continuous.

An Orlicz function is a function \( f : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( f(0) = 0, f(x) > 0 \), for \( x > 0 \) and \( f(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( f \) is replaced by \( f(x+y) \leq f(x) + f(y) \), then this function is called modulus function. An modulus function \( f \) is said to satisfy \( \Delta^2 \)-condition for all values \( u \), if there exists \( K > 0 \) such that \( f(2u) \leq Kf(u), u \geq 0 \).

**Remark 1:** An Modulus function satisfies the inequality \( f(\lambda x) \leq \lambda f(x) \) for all \( \lambda \) with \( 0 < \lambda < 1 \).

1.1. **Lemma.** Let \( f \) be an modulus function which satisfies \( \Delta^2 \)-condition and let \( 0 < \delta < 1 \). Then for each \( t \geq \delta \), we have \( f(t) \leq K \delta^{-1} f(2) \) for some constant \( K > 0 \).

Let \( M \) and \( \Phi \) be mutually complementary modulus functions. Then, we have

(i) For all \( u, y \geq 0 \),

\[
uy \leq M(u) + \Phi(y), \quad (Young's\ inequality)[34]
\]

(ii) For all \( u \geq 0 \),

\[
\eta(u) = M(u) + \Phi(\eta(u)).
\]
(iii) For all \( u \geq 0 \), and \( 0 < \lambda < 1 \),
\[
M(\lambda u) \leq \lambda M(u).
\]

[35] used the idea of Orlicz function to construct Orlicz sequence space
\[
\ell_M = \{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \},
\]
The space \( \ell_M \) with the norm
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \((1 \leq p < \infty)\),
the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{mn}) \) of modulus function is called a Musielak-modulus function. A sequence
\( g = (g_{mn}) \) defined by
\[
g_{mn}(v) = \sup \{ |v| - f_{mn}(u) : u \geq 0 \}, m, n = 1, 2, \ldots
\]
is called the complementary function of a Musielak-modulus function \( f \). For a given Musielak modulus function \( f \),
the Musielak-modulus sequence space \( t_f \) is defined by
\[
t_f = \left\{ x \in w^3 : M_f(|x_{mnk}|)^{1/m+n+k} \to 0, \text{as } m, n, k \to \infty \right\},
\]
where \( M_f \) is a convex modular defined by
\[
M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.
\]
We consider \( t_f \) equipped with the Luxemburg metric space, (i.e)
Let \( (X_i, d_i) \), \( i \in I \) be a family of metric spaces such that each two elements of the family are disjoint. Denote \( X : \bigcup_{i \in I} X_i \). If we define
\[
d(x, y) = \begin{cases} 
  d_i(x, y), & \text{if } x, y \in X_i \\
  +\infty, & \text{if } x \in X_i, y \in X_j, i \neq j
\end{cases}
\]
then the pair \( (X, d) \) is a Luxemburg metric space. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows
\[
Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \},
\]
for \( Z = c, c_0 \) and \( \ell_\infty \), where \( \Delta x_k = x_k - x_{k+1} \) for all \( k \in \mathbb{N} \).
Here \( c, c_0 \) and \( \ell_\infty \) denote the classes of convergent.null and bounded scalar valued single sequences respectively. The difference sequence space \( bv_p \) of the classical space \( \ell_p \) is introduced and studied in the case \( 1 \leq p \leq \infty \) by Başar and Altay and in the case \( 0 < p < 1 \). The spaces \( c(\Delta), c_0(\Delta), \ell_\infty(\Delta) \) and \( bv_p \) are Banach spaces normed by
\[
\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty) .
\]
Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by
\[
Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \},
\]
where \( Z = \Lambda^2, \chi^2 \) and \( \Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1} \) for all \( m, n \in \mathbb{N} \). The generalized difference double notion has the following representation: 
\[ \Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{m+1n+1} - \Delta^{m-1} x_{m+1n+1} + \Delta^{m-1} x_{m+1n+1} + \Delta^{m-1} x_{m+1n+1} + \Delta^{m-1} x_{m+1n+1} \]
and also this generalized difference double notion has the following binomial representation:
\[ \Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}. \]

2. Definitions and Preliminaries

Let \( \Delta^m X \) be a non empty set. A non-void class \( I \subseteq 2^{\Delta^m X} \) (power set of \( \Delta^m X \)) is called an ideal if \( I \) is additive (i.e \( A, B \in I \Rightarrow A \cup B \in I \)) and hereditary (i.e \( A \in I \) and \( B \subseteq A \Rightarrow B \in I \)). A non-empty family of sets \( F \subseteq 2^{\Delta^m X} \) is said to be a filter on \( \Delta^m X \) if \( \phi \notin F; A, B \in F \Rightarrow A \cap B \in F \) and \( A \in F, A \subseteq B \Rightarrow B \in F \). For each ideal \( I \) there is a filter \( F(I) \) given by \( F(I) = \{ K \subseteq N : N \setminus K \in I \} \). A non-trivial ideal \( I \subset 2^{\Delta^m X} \) is called admissible if and only if \( \{ \{ x \} : x \in \Delta^m X \} \subset I \).

A double sequence space \( E \) is said to be solid or normal if \( (\alpha_{mn} \Delta^m x_{mn}) \in E \), whenever \( (\Delta^m x_{mn}) \in E \) and for all double sequences \( \alpha = (\alpha_{mn}) \) of scalars with \( |\alpha_{mn}| \leq 1 \) for all \( m, n \in \mathbb{N} \).

Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( w \), where \( n \leq w \). A real valued function \( d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \) on \( X \) satisfying the following four conditions:

(i) \( \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0 \) if and only if \( d_1(x_1, 0), \ldots, d_n(x_n, 0) \) are linearly dependent,

(ii) \( \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \) is invariant under permutation,

(iii) \( \|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \), \( \alpha \in \mathbb{R} \)

(iv) \( d_p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p} \) for \( 1 \leq p < \infty \); (or)

(v) \( d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup \{ d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n) \} \),

for \( x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y \) is called the \( p \)-product metric of the Cartesian product of \( n \)-vector of the norms of the \( n \)-sub spaces.

A trivial example of \( p \)-product metric of \( n \)-metric space is the \( p \)-norm space is \( X = \mathbb{R} \) equipped with the following Euclidean metric in the product space is the \( p \)-norm:

\[ \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_E = \sup \{ |\det(d_{mn}(x_{mn}, 0))| \} = \left\{ \begin{array}{cccc} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \cdots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \cdots & d_{2n}(x_{2n}, 0) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \cdots & d_{nn}(x_{nn}, 0) \end{array} \right\} \]
where \( x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots n \).

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( p \)- metric. Any complete \( p \)-metric space is said to be \( p \)-Banach metric space.
3. Main Results

In this section we introduce the notion of different types of $I$-convergent double sequences. This generalizes and unifies different notions of convergence for $\chi^2$. We shall denote the ideal of $2^{N \times N}$ by $I_2$.

Let $I_2$ be an ideal of $2^{N \times N}$, $f$ be an modulus function, $\eta = (\eta_{mn})$ be a double analytic sequence of strictly positive real numbers and $(\Delta^m X, \|(d_1(x_1,0), \ldots, d_n(x_n,0))\|_p)$ be an $p$-product of $n$ metric spaces is the $p$ norm of the $n$-vector of the norms of the $n$ subspaces. Further $\chi^2 (p - \Delta^m X)$ denotes $\Delta^m X-$valued sequence space. Now, we define the following sequence spaces:

$$\chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta = x = (\Delta^m x_{mn}) \in \chi^2 (p - \Delta^m X) : \forall \epsilon > 0,$$

$$\{ (r, s) \in N \times N : \frac{1}{r^2} \sum_{m=1}^r \sum_{n=1}^s f \| (\Delta^m x_{mn})^{1/m+n}, d_1(x_1,0), \ldots, d_n(x_n-1,0) \|_p \} \eta_{mn} \geq \epsilon \} \in I_2,$$ for every $d_1(x_1,0), \ldots, d_n(x_n-1,0) \in \Delta^m X$.

$$\Lambda_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta = x = (x_{mn}) \in \Lambda^2 (p - \Delta^m X) : \exists K > 0,$$

$$\{ (r, s) \in N \times N : \frac{1}{r^2} \sum_{m=1}^r \sum_{n=1}^s f \| (\Delta^m x_{mn})^{1/m+n}, d_1(x_1,0), \ldots, d_n(x_n-1,0) \|_p \} \eta_{mn} \geq K \} \in I_2,$$ for every $d_1(x_1,0), \ldots, d_n(x_n-1,0) \in \Delta^m X$.

$$\Lambda_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta = x = (x_{mn}) \in \Lambda^2 (p - \Delta^m X) : \exists K > 0,$$

$$\{ (r, s) \in N \times N : \frac{1}{r^2} \sum_{m=1}^r \sum_{n=1}^s f \| (\Delta^m x_{mn})^{1/m+n}, d_1(x_1,0), \ldots, d_n(x_n-1,0) \|_p \} \eta_{mn} \leq K \} \text{, for}
$$

every $d_1(x_1,0), \ldots, d_n(x_n-1,0) \in \Delta^m X$.

If $\eta = \eta_{mn} = 1$ for all $m, n \in \mathbb{N}$ we obtain

$$\chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta = \chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p],$$

$$\Lambda_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta = \Lambda_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p],$$

The following well-known inequality will be used in this study: $0 \leq \min f_{mn} \eta_{mn} = H_0 \leq \eta_{mn} \leq \sup_{mn} = H < \infty, D = \max (1, 2^H - 1)$, then

$$|x_{mn} + y_{mn}|^{\eta_{mn}} \leq D \{ |x_{mn}|^{\eta_{mn}} + |y_{mn}|^{\eta_{mn}} \}$$

for all $m, n \in \mathbb{N}$ and $x_{mn}, y_{mn} \in \mathbb{C}$. Also $|x_{mn}|^{\eta_{mn}/m+n} \leq \max \left(1, |x_{mn}|^{H/m+n} \right)$ for all $x_{mn} \in \mathbb{C}$.

3.1. Theorem. The classes of sequences $\chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta_{mn}$,

$\Lambda_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta_{mn}$ are linear spaces over the complex field $\mathbb{C}$

Proof: Now we establish the result for the case $\chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta_{mn}$ and the others can be proved similarly. Let $x, y \in \chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta_{mn}$ and $\alpha, \beta \in \mathbb{C}$.

Then

$$\{ (r, s) \in N \times N : \frac{1}{r^2} \sum_{m=1}^r \sum_{n=1}^s \left[ f \| (\Delta^m x_{mn})^{1/m+n}, d_1(x_1,0), \ldots, d_n(x_n-1,0) \|_p \} \eta_{mn} \geq \frac{\epsilon}{2} \} \in I_2$$

and

$$\{ (r, s) \in N \times N : \frac{1}{r^2} \sum_{m=1}^r \sum_{n=1}^s \left[ f \| (\Delta^m y_{mn})^{1/m+n}, d_1(x_1,0), \ldots, d_n(x_n-1,0) \|_p \} \eta_{mn} \geq \frac{\epsilon}{2} \} \in I_2$$

for all $x, y \in \chi_{\Delta_f}^2 [\| (d_1(x_1,0), \ldots, d_n(x_n,0))\|_p]^\eta_{mn}$ and $\alpha, \beta \in \mathbb{C}$.
I_2.

Since \( \| (d_1(x_1,0),\ldots,d_n(x_n,0)) \|_p \) be an \( p \)-product of \( n \) metric spaces is the \( p \)-norm of the \( n \)-vector of the norms of the \( n \) subspaces and \( f \) is an modulus function, the following inequality holds:

\[
\begin{align*}
\frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} & \left[ f \left( \| (\Delta^m x_{mn} + \Delta^m y_{mn}) \|^{1/m+n} \right), d_1(x_1,0), \ldots, d_n(x_n-1,0) \right] \|_p \right] \left[ \eta_{mn}^p \right] \leq \\
\frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} & \left[ f \left( \| (\Delta^m x_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \right] \|_p \right] \left[ \eta_{mn}^n \right] + \\
\frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} & \left[ f \left( \| (\Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \right] \|_p \right] \left[ \eta_{mn}^n \right] + \\
\frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} & \left[ f \left( \| (\Delta^m x_{mn}) \|^{1/m+n} + \| (\Delta^m y_{mn}) \|^{1/m+n} \right), d_1(x_1,0), \ldots, d_n(x_n-1,0) \right] \|_p \right] \left[ \eta_{mn}^n \right] .
\end{align*}
\]

From the above inequality we get

\[
\{(r,s) \in N \times N : \frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn} + \Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \geq \epsilon \} 
\subset \{(r,s) \in N \times N : \frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \geq \frac{\epsilon}{2} \}
\subset \{ (r,s) \in N \times N : \frac{1}{r^s} \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \geq \frac{\epsilon}{2} \} \in I_2.
\]

This completes the proof.

3.2. **Theorem.** The class of sequence \( \chi_{2f}^{2f} \left[ \| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p \right] \) is a paranormed space with respect to the paranorm defined by

\[
g_{rs}(x) = \inf \left\{ \left( \sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn} + \Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \right)^{1/\pi} \leq 1 \right\},
\]

for every \( d_1(x_1,0), \ldots, d_n(x_n-1,0) \in X \).

**Proof:** \( g_{rs}(\theta) = 0 \) and \( g_{rs}(-x) = g_{rs}(x) \) are easy to prove, so we omit them. Let us take \( x,y \in \chi_{2f}^{2f} \left[ \| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p \right] \eta_{mn}^n \). Let

\[
g_{rs}(x) = \inf \left\{ \sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \leq 1, \forall x \in X \right\}
\]

and

\[
g_{rs}(y) = \inf \left\{ \sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \leq 1, \forall x \in X \right\}.
\]

Then we have

\[
\sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn} + \Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \leq
\]

\[
\sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] +
\]

\[
\sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] .
\]

Thus

\[
\sup_{r,s} \left( \sum_{m=1}^{r^s} \sum_{n=1}^{s} f \left( \| (\Delta^m x_{mn} + \Delta^m y_{mn}) \|^{1/m+n} d_1(x_1,0), \ldots, d_n(x_n-1,0) \right) \|_p \right] \left[ \eta_{mn}^n \right] \leq 1
\]

and \( g_{rs}(x + y) = g_{rs}(x) + g_{rs}(y) \).

Now, let \( \lambda_{mn}^{u} \rightarrow \lambda \), where \( \lambda_{mn}^{u}, \lambda \in \mathbb{C} \) and \( g_{rs} (\Delta^m x_{mn} - \Delta^m x_{mn}) \rightarrow 0 \) as \( u \rightarrow \infty \). We have to prove that \( g_{rs} \left( \lambda_{mn} \Delta^m x_{mn} - \lambda \Delta^m x_{mn} \right) \rightarrow 0 \) as \( u \rightarrow \infty \). Let

\[
g_{rs}(x^u) =
\]
\[ \left\{ \sup_{rs} \frac{1}{r^s} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[ f \left( \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right) \right]^{\eta_{mn}} \leq 1, \forall x \in X \right\} \\

\text{and}

\[ g_{rs} \left( x^u - x \right) = \left\{ \sup_{rs} \frac{1}{r^s} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[ f \left( \left( \| \Delta^m x_{mn} - \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right) \right]^{\eta_{mn}} \leq 1, \right\} \text{for all} \ x \in X. \]

We observe that

\[ f \left( \left( \| \frac{\lambda u}{\Delta^m x_{mn} - \lambda \Delta^m x_{mn}} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right) \leq \frac{1}{r^s} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[ f \left( \left( \| \frac{\lambda u}{\Delta^m x_{mn} - \lambda \Delta^m x_{mn}} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right) \right]^{\eta_{mn}} \]

From this inequality, it follows that

\[ \left[ f \left( \left( \| \frac{\lambda u}{\Delta^m x_{mn} - \lambda \Delta^m x_{mn}} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right) \right]^{\eta_{mn}} \leq 1 \]

and consequently

\[ g_{rs} \left( \lambda u \Delta^m x_{mn} - \lambda \Delta^m x_{mn} \right) \leq \left( \| \frac{\lambda u}{\Delta^m x_{mn} - \lambda \Delta^m x_{mn}} \|^{1/m+n} \right)^{\eta_{mn}} \inf_{g_{rs} \left( \Delta^m x_{mn} \right)} \left( 1, \ldots, d_n(x_{n-1}, 0) \right) \]

Hence by our assumption the right hand side tends to zero as \( u, m \) and \( n \to \infty \). This completes the proof.

3.3. Theorem. (i) If \( 0 < \inf_{\eta_{mn}} = H_0 \leq \eta_{mn} < 1 \), then \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \subset \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \).

(ii) If \( 1 \leq \eta_{mn} \leq \sup_{\eta_{mn}} = H < \infty \), then \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right] \subset \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \).

(iii) If \( 0 < \eta_{mn} < \mu_{mn} < \infty \) and \( \left\{ \frac{\mu_{mn}}{\eta_{mn}} \right\} \) is double analytic, then \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \subset \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\mu_{mn}} \).

Proof: The proof can be established using standard technique.

The following result is well known.

3.4. Lemma. If a sequence space \( E \) is solid, then it is monotone.

3.5. Theorem. The class of sequence \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \) is not solid and hence not monotone.

Proof: It is routine verification. Therefore we omit the proof.

3.6. Theorem. Let \( f, f_1 \) and \( f_2 \) be modulus functions. Then we have

(i) \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \subset \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \)

(ii) \( \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \cap \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \subset \chi_{\Delta^m} \left[ \left( \| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \).
The ideal convergence of difference strongly of $\chi^2$ ...

$\chi^2_{\Delta f_1+\Delta f_2} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta$

**Proof:** (i) Let $\inf_{m,n} \eta_{mn} = H_0$. For given $\epsilon > 0$, we first choose $\epsilon_0 > 0$ such that $\max \left\{ \epsilon^H_0, \epsilon_0^H \right\} < \epsilon$. Now using the continuity of $f$, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $f(t) < \epsilon_0$. Let $\Delta^m x \in \chi^2_{\Delta f_1} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta$

We observe that

$$A(\delta) = \left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} \geq \delta^H \right\} \in I_2.$$

Thus if $(r, s) \notin A(\delta)$ then

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < \delta^H$$

$$\Rightarrow \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < rs\delta^H,$$

$$\Rightarrow f \left( f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right)^\eta_{mn} < \delta^H,$$

$$\Rightarrow f \left( f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right)^\eta_{mn} < \delta^H,$$

Hence from above inequality and using continuity of $f$, we must have

$$f \left( f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right)^\eta_{mn} < \epsilon,$$

$$\Rightarrow \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < \epsilon.$$

Hence we have

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} \geq \epsilon \right\} \in I_2.$$

(ii) Let $x \in \chi^2_{\Delta f_1} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta \cap \chi^2_{\Delta f_2} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta$. Then the fact that

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ (f_1 + f_2) \left( \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right) \right]^\eta_{mn} \leq$$

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \left( \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right) \right]^\eta_{mn} +$$

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_2 \left( \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right) \right]^\eta_{mn}.$$ This completes the proof.

3.7. **Theorem.** The class of sequence $\Lambda^2_{\Delta f} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta$ is a sequence algebra

**Proof:** Let $(\Delta^m x_{mn}, \Delta^m y_{mn}) \in \Lambda^2_{\Delta f} \left[ \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \right]^\eta$ and $0 < \epsilon < 1$. Then the result follows from the following inclusion relation:

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < \epsilon \right\} \in I_2$$

$$\sup \left\{ \left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m x_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < \epsilon \right\} \in I_2 \right\}$$

$$\cap \left\{ \left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \| (\| \Delta^m y_{mn} \|^{1/m+n}, d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0)) \|_p \right]^\eta_{mn} < \epsilon \right\} \in I_2 \right\}.$$ Similarly we can prove the result for other cases.
References


The ideal convergence of difference strongly of $\chi^2$ ...


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