1-Movable Total Dominating, Connected Dominating, and Double Dominating Sets in the Composition of Graphs

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Abstract

This study aimed to characterize 1-movable total dominating sets and 1-movable connected dominating sets in the composition $G[H]$ of arbitrary connected nontrivial graphs $G$ and $H$ and determine the corresponding value of the parameters. This also aimed to determine the bounds of the 1-movable double domination number in the composition $G[H]$ of arbitrary connected nontrivial graphs $G$ and $H$.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph with $n = |V(G)|$ and $m = |E(G)|$. For any vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighborhood of $S$ is the set $N_G[S] = N[S] = S \cup N(S)$.

A set $S \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. It is a connected dominating set of $G$ if it is a dominating set and the subgraph $\langle S \rangle$ induced by $S$ is connected. It is a total dominating set of $G$ if $N_G(S) = V(G)$. That is, every vertex in $V(G)$ is adjacent to some vertex in $S$. It is a double dominating set of $G$ if for each $x \in V(G)$, $|N_G[x] \cap S| \geq 2$. A total dominating set $X$ in $G$ is a 1-movable total dominating set in $G$ if for every $v \in X$, either $X \setminus \{v\}$ is a total dominating set, or there exists a vertex $u \in (V(G) \setminus X) \cap N(v)$ such that $(X \setminus \{v\}) \cup \{u\}$ is a total dominating set. A connected dominating set $C$ in $G$ is a 1-movable connected dominating set of $G$ if for every $v \in C$, either $C \setminus \{v\}$ is a connected dominating set, or there exists a vertex $u \in (V(G) \setminus C) \cap N(v)$ such that $(C \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G$. A double dominating set $D$ in a connected nontrivial graph $G$ is a 1-movable double dominating set of $G$ if for every $v \in D$, either $D \setminus \{v\}$ is a double dominating set, or there exists a vertex $u \in (V(G) \setminus D) \cap N(v)$ such that $(D \setminus \{v\}) \cup \{u\}$ is a double dominating set of $G$. Furthermore, the domination number $\gamma(G)$ (resp. connected domination number $\gamma_c(G)$, total domination number $\gamma_t(G)$, double domination number $\gamma_{\times 2}(G)$, 1-movable total domination number $\gamma_{\times 2}^1(G)$, 1-movable connected domination number $\gamma_{\times 2}^1(G)$, 1-movable double domination number $\gamma_{\times 2}^1(G)$) of $G$ equals the minimum cardinality of a dominating (resp. connected dominating, total dominating, double dominating, 1-movable total dominating, 1-movable connected dominating, and 1-movable double dominating) set of $G$. Moreover, domination in the composition of graphs was studied in [3] and 1-movable domination, 1-movable independent domination, 1-movable total domination, and 1-movable connected domination in graphs are introduced and investigated in [4], [5], and [6], respectively.

2 1-movable Total Domination in the Composition of Graphs

The composition $G[H]$ of two graphs $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$. Observe that any nonempty subset $C$ of $V(G[H]) = \ldots$
V(G) × V(H) can be expressed as \( C = \bigcup_{x \in S} (\{x\} \times T_x) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \). And so we shall use this form for a subset \( C \) of \( V(G[H]) = V(G) \times V(H) \).

To illustrate the above definition, consider \( G = P_3 \) and \( H = P_4 \). The graph \( G[H] = P_3[P_4] \) is the graph shown below.

![Graph](image)

**Figure 1:** The composition \( P_3[P_4] \)

**Theorem 2.1** [3] Let \( G \) and \( H \) be connected graphs. Then \( C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H]) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a total dominating set in \( G[H] \) if and only if

(i) \( S \) is a total dominating set in \( G \) or

(ii) \( S \) is a dominating set in \( G \) and \( T_x \) is a total dominating set in \( H \) for every \( x \in S \setminus N_G(S) \)

**Remark 2.2** [3] Let \( G \) and \( H \) be nontrivial connected graphs. Then \( \gamma_t(G[H]) = \gamma_t(G) \).

**Theorem 2.3** Let \( G \) and \( H \) be connected nontrivial graphs. A subset \( C = \bigcup_{x \in S} (\{x\} \times T_x) \) of \( V(G[H]) \) is a 1-movable total dominating set of \( G[H] \) if and only if it is a total dominating set of \( G[H] \).

**Proof.** Suppose that \( C \) is a 1-movable total dominating set of \( G[H] \). Then \( C \) is a total dominating set of \( G[H] \).

Conversely, suppose that \( C \) is a total dominating set of \( G[H] \). Then (i) or (ii) of Theorem 2.1 holds. Suppose first that (i) holds. Let \((x, a) \in C\). If \(|T_x| \geq 2\), then \( C \setminus \{(x, a)\} \) is a total dominating set of \( G[H] \) by Theorem 2.1 (i). Suppose that \(|T_x| = 1\), say \( T_x = \{a\} \) for some \( a \in V(H) \). Since \( H \) is a connected nontrivial graph, there exists \( b \in V(H) \setminus T_x \) such that \( ab \in E(H) \). Thus, \((x, b) \notin C\) and \( C \setminus \{(x, a)\} \cup \{(x, b)\} \) is a total dominating set in \( G[H] \) by Theorem 2.1 (i). This shows that \( C \) is a 1-movable total dominating set in \( G[H] \). Suppose now that (ii) holds. Then by Theorem 2.1, \( C \) is a total dominating set of \( G[H] \). Again, let \((x, a) \in C\). Consider the following cases:
Case 1. $x \in N(S)$

If $|T_x| \geq 2$, then $C \setminus \{(x, a)\}$ is a total dominating set of $G[H]$ by Theorem 2.1 (ii). If $T_x = \{a\}$ for some $a \in V(H)$, then there exists $(x, b) \in (V(G[H]) \setminus C) \cap N_{G[H]}((x, a))$ and $[C \setminus \{(x, a)\}] \cup \{(x, b)\}$ is a total dominating set of $G[H]$ by Theorem 2.1 (ii).

Case 2. $x \notin N(S)$

Then $T_x$ is a total dominating set of $H$ (hence, $|T_x| \geq 2$). Since $G$ is a nontrivial connected graph, there exists $y \in V(G) \cap N_G(x)$. By assumption, $y \in V(G) \setminus S$. Hence, $(y, a) \notin C$ and $[C \setminus \{(x, a)\}] \cup \{(y, a)\}$ is a total dominating set of $G[H]$ by Theorem 2.1 (ii). Therefore, $C$ is a 1-movable total dominating set of $G[H]$. □

The next result follows from Theorem 2.1, Remark 2.2, and Theorem 2.3.

**Corollary 2.4** Let $G$ and $H$ be connected nontrivial graphs. Then $\gamma_{mt}^1(G[H]) = \gamma_t(G)$.

## 3 1-movable Connected Domination in the Composition of Graphs

**Theorem 3.1** [3] Let $G$ and $H$ be connected graphs. Then $C = \bigcup_{x \in S_s} (\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a connected dominating set in $G[H]$ if and only if $S$ is a connected dominating set in $G$, where $T_x$ is a connected dominating set in $H$ whenever $|S| = 1$, that is, $S = \{x\}$.

**Theorem 3.2** Let $G$ and $H$ be connected nontrivial graphs. A subset $C = \bigcup_{x \in S_s} (\{x\} \times T_x)$ of $V(G[H])$ is a 1-movable connected dominating set of $G[H]$ if and only if it satisfies the following properties:

(i) $S$ is a connected dominating set of $G$;

(ii) If $S = \{x\}$, then $T_x$ is a connected dominating set of $H$; and

(iii) If $S = \{x\}$ and $|T_x| = 1$, then either $S$ is a 1-movable connected dominating set of $G$ or $T_x$ is a 1-movable connected dominating set of $H$.

**Proof.** Suppose that $C$ is a 1-movable connected dominating set of $G[H]$. Then properties (i) and (ii) hold by Theorem 3.1. Suppose that $S = \{x\}$ and $T_x = \{a\}$ for $x \in V(G)$ and $a \in V(H)$. Then $C = \{(x, a)\}$. Since $C$ is a 1-movable connected dominating set, there exists $(y, b) \in (V(G[H]) \setminus C) \cap N_{G[H]}((x, a))$ such that $(C \setminus \{(x, a)\}) \cup \{(y, b)\} = \{(y, b)\}$ is a connected dominating set in $G[H]$. If $y = x$, then $b \in (V(H) \setminus T_x) \cap N_H(a)$. Hence, $(T_x \setminus \{a\}) \cup \{b\} = \{b\}$ is
a (connected) dominating set in $H$ by Theorem 3.1. This implies that $T_x$ is a 1-movable (connected) dominating set of $H$. If $y \neq x$, then $y \in (V(G) \setminus S) \cap N_G(x)$ and $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a (connected) dominating set of $G$. This implies that $S$ is a 1-movable (connected) dominating set of $G$. Thus, (iii) holds.

For the converse, suppose that $C$ satisfies properties (i), (ii), and (iii). Then by Theorem 3.1, $C = \bigcup \{\{x\} \times T_x\}$ is a connected dominating set of $G[H]$. Let $(x, a) \in C$. Suppose that $|S| \geq 2$. If $|T_x| \geq 2$, then $C \setminus \{(x, a)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Suppose that $|T_x| = 1$, that is, $T_x = \{a\}$ for some $a \in V(H)$. Since $H$ is a connected nontrivial graph, there exists $b \in V(H) \setminus T_x$ such that $ab \in E(H)$. Hence, $(x, b) \notin C$ and $[C \setminus \{(x, a)\}] \cup \{(x, b)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Hence, in this case, $C$ is a 1-movable connected dominating set of $G[H]$. Suppose that $|S| = 1$, say $S = \{x\}$. Suppose further that $|T_x| \geq 2$. Since $G$ is a connected nontrivial graph, there exists $y \in V(G) \setminus S$ such that $xy \in E(G)$. Hence, $(y, a) \notin C$ and $C \setminus \{(x, a)\} \cup \{(y, a)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Suppose that $|T_x| = 1$, say $T_x = \{a\}$. Then $C = \{(x, a)\}$. Suppose first that $S$ is a 1-movable connected dominating set of $G$. Then there exists $y \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a connected dominating set of $G$. Hence, $(y, a) \notin C$ and $C \setminus \{(x, a)\} \cup \{(y, a)\} = \{(y, a)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Suppose that $S$ is not a 1-movable connected dominating set of $G$. Then, by assumption, $T_x$ is a 1-movable connected dominating set of $H$. Hence, there exists $b \in (V(H) \setminus T_x) \cap N_H(a)$ such that $(T_x \setminus \{a\}) \cup \{b\} = \{b\}$ is a connected dominating set of $H$. Thus, $(x, b) \notin C$ and $C \setminus \{(x, a)\} \cup \{(x, b)\} = \{(x, b)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Consequently, $C$ is a 1-movable connected dominating set of $G[H]$. □

**Corollary 3.3** Let $G$ and $H$ be connected nontrivial graphs. Then

$$
\gamma_{mc}^1(G[H]) = \begin{cases} 
1, & \text{if } \gamma(G) = 1 = \gamma_{mc}^1(H) \text{ or } \gamma_{mc}^1(G) = 1 = \gamma(H) \\
2, & \text{if } \gamma(G) = 1 \\
\gamma_c(G), & \text{if } \gamma(G) \neq 1.
\end{cases}
$$

**Proof.** Consider the following cases:

Case 1. $\gamma(G) = 1$.

Suppose that $\gamma(H) = 1$. Let $S = \{x\}$ and $D = \{a\}$ be subsets of $V(G)$ and $V(H)$, respectively. If $S$ is a $\gamma_{mc}^1$-set of $G$ and $D$ is a $\gamma$-set of $H$, then $C = \{(x, a)\}$ is a 1-movable connected dominating set of $G[H]$ by Theorem 3.2. Hence, $\gamma_{mc}^1(G[H]) = |C| = 1$. If $D$ is a 1-movable connected dominating set of $H$ and $S$ is a $\gamma$-set of $G$, then $C = \{(x, a)\}$ is a 1-movable connected dominating set of $G[H]$ by Theorem 3.2. Hence, $\gamma_{mc}^1(G[H]) = |C| = 1$. 


Suppose that \( \gamma_{mc}^1(G) \neq 1 \) and \( \gamma_{mc}^1(H) \neq 1 \). Let \( S_1 = \{x\} \) be a \( \gamma \)-set of\( G \) and \( D_1 = \{a\} \) be a \( \gamma \)-set of \( H \) for some \( x \in V(G) \) and \( a \in V(H) \). Since \( G \) is a connected nontrivial graph, there exists \( y \in (V(G) \setminus S_1) \cap N_G(x) \).

Set \( S = S_1 \cup \{y\} = \{x, y\} \). Then \( S \) is a connected dominating set of \( G \).

Now, set \( T_x = \{a\} = T_y \). By Theorem 3.2(i), \( C = \{(x, a), (y, a)\} \) is a 1-movable connected dominating set of \( G[H] \). Hence, \( \gamma_{mc}^1(G[H]) \leq |C| = 2 \).

Case 2. Let \( u, p \in (V(G[H]) \setminus C) \cap N_{G[H]}((u, p)) \) such that \( C \setminus \{(w, q)\} \cup \{(u, p)\} = \{(u, p)\} \) is a dominating set of \( G[H] \). If \( u = w \), then \( p \in (V(H) \setminus T_u) \cap N_H(q) \).

Hence, \( T_w \setminus \{q\} \cup \{p\} = \{p\} \) is a dominating set in \( H \) by Theorem 2.1. Hence, \( T_w \) is a 1-movable (connected) dominating set of \( H \). Thus, \( \gamma_{mc}^1(H) = 1 \), contrary to our assumption. If \( u \neq w \), then \( u \in (V(G) \setminus S) \cap N_G(w) \). Thus, \( S \setminus \{w\} \cup \{u\} = \{u\} \) is a dominating set in \( G \) by Theorem 2.1. Thus, \( S \) is a 1-movable (connected) dominating set of \( G \). Hence, \( \gamma_{mc}^1(G[H]) = |S| = 1 \), contrary to our assumption. Hence, \( \gamma_{mc}^1(G[H]) = 1 \). Thus, \( \gamma_{mc}^1(G[H]) = 2 \).

Now, suppose \( \gamma(H) \neq 1 \). Let \( S_1 = \{x\} \) be a \( \gamma \)-set of \( G \) for some \( x \in V(G) \). Since \( G \) is connected nontrivial graph, there exists \( y \in V(G) \) with \( x \neq y \) such that \( xy \in E(G) \). Since \( S_1 \) is a dominating set, it follows that \( S_2 = S_1 \cup \{y\} = \{x, y\} \) is a connected dominating set of \( G \). Set \( T_x = T_y = \{p\} \), where \( p \in V(H) \).

Then by Theorem 3.2, \( C = \{(x, p), (y, p)\} \) is a 1-movable connected dominating set of \( G[H] \). Hence, \( \gamma_{mc}^1(G[H]) \leq |C| = 2 \). Next, let \( C^* = \{(z, b)\} \subseteq V(G[H]) \), where \( z \in V(G) \) and \( b \in V(H) \). Since \( \gamma(H) \neq 1 \), there exists \( c \in V(H) \) such that \( bc \notin E(H) \).

Hence, \( (z, b)(z, c) \notin E(G[H]) \). Thus, \( C^* \) is not a dominating set of \( G[H] \). Since \( (z, b) \) was arbitrarily chosen, it follows that \( \gamma_{mc}^1(G[H]) \neq 1 \). Therefore, \( \gamma_{mc}^1(G[H]) = 2 \).

Case 2. Now, suppose \( \gamma(G) \neq 1 \). Let \( C = \bigcup_{x \in S} \{x\} \times T_x \) be a \( \gamma_{mc}^1 \)-set of \( G[H] \). Then, by Theorem 3.2, \( S \) is a connected dominating set of \( G \). Since \( \gamma(G) \neq 1 \), \( |S| \neq 1 \). Hence, \( \gamma_{mc}^1(G[H]) = |C| \geq |S| \geq \gamma_c(G) \).

Also, let \( S_1 \) be a \( \gamma_c \)-set of \( G \). Since \( \gamma(G) \neq 1 \), \( |S_1| \neq 1 \). Now, for each \( x \in S_1 \), set \( T_x = \{a\} \), where \( a \in V(H) \). Then by Theorem 3.2, \( C^* = \bigcup_{x \in S_1} \{x\} \times T_x \) is a 1-movable connected dominating set of \( G[H] \). Hence, \( \gamma_{mc}^1(G[H]) \leq |C^*| = |S_1| = \gamma_c(G) \).

Therefore, \( \gamma_{mc}^1(G[H]) = \gamma_c(G) \). \( \square \)

4 1-movable Double Domination in the Composition of Graphs

**Theorem 4.1** [2] Let \( G \) and \( H \) be any connected nontrivial graphs. A non-
empty subset $C = \bigcup_{x \in S} \{(x) \times T_x\}$ of $V(G[H])$ is a double dominating set of $G[H]$ if and only if $S$ is a dominating set of $G$ and satisfies each of the following:

(a) For each $x \in V(G) \setminus S$ such that $|N_G(x) \cap S| = 1$, $|T_y| \geq 2$ for $y \in N_G(x) \cap S$.

(b) $T_x$ is a double dominating set of $H$ for each $x \in S \setminus N(S)$.

(c) For each $z \in S$ with $|N_G(z) \cap S| = 1$, either $T_z$ is a dominating set of $H$ or $|T_w| \geq 2$ for $w \in N_G(z) \cap S$.

**Theorem 4.2** Let $G$ and $H$ be nontrivial connected graphs such that $|V(H)| \geq 3$. Then $\gamma_t(G) \leq \gamma_{m \times 2}^1(G[H]) \leq 2\gamma_t(G)$.

**Proof.**

Let $S$ be a $\gamma_t$-set of $G$. Suppose that $H = K_n$, where $n \geq 3$. Pick any $a, b \in V(H)$ and let $D = \{a, b\}$. Then $C = S \times D$ is a double dominating set by Theorem 4.1. Moreover, for any $x \in S$ and $p \in D$, there exists $(x, q) \in V(G[H]) \setminus C$, where $q \in V(H) \setminus D$, such that $(C \setminus \{(x, p)\}) \cup \{(x, q)\}$ is a double dominating set of $G[H]$. Thus, $C$ is a $1$-movable double dominating set and $\gamma_{m \times 2}^1(G[H]) \leq |C| = 2\gamma_t(G)$. If $H$ is noncomplete, then choose $c, d \in V(H)$ such that $d_G(c, d) = 2$. Let $D_1 = \{c, d\}$. Then $C_1 = S \times D_1$ is a $1$-movable double dominating set of $G[H]$. Hence, $\gamma_{m \times 2}^1(G[H]) \leq 2\gamma_t(G)$.

Next, let $C^* = \bigcup_{x \in S^*} \{(x) \times T_x\}$ be a $\gamma_{m \times 2}^1$-set of $G[H]$. Then $S^*$ is a dominating set in $G$ and $T_x$ is a double dominating set in $H$ for each $x \in S^* \setminus N(S^*)$. Thus by Remark 2.2, $\gamma_{m \times 2}^1(G[H]) = |C| \geq |S^* \cap N(S^*)| + 2|S^* \setminus N(S^*)| \geq \gamma_t(G)$. □

The strict inequality in Theorem 4.2 can be attained. However, the given bounds are tight. To see that the bounds are tight, consider the graphs $P_3[P_2]$, $P_2[K_3]$ and $P_4[C_4]$ in Figure 2. Consider first the graph, $P_3[P_2]$. It can be verified that $\gamma_{m \times 2}^1(P_3[P_2]) = 3$, $\gamma_t(P_3) = 2$ and $2\gamma_t(P_3) = 4$. Thus, $\gamma_t(P_3) < \gamma_{m \times 2}^1(P_3[P_2]) < 2\gamma_t(P_3)$. Also, if we consider the graph $P_2[K_3]$, it can be verified that $\gamma_{m \times 2}^1(P_2[K_3]) = 2$, $\gamma_t(P_2) = 2$ and $2\gamma_t(P_2) = 4$. Hence, $\gamma_{m \times 2}^1(P_2[K_3]) = \gamma_t(P_2)$. Finally, if we consider the graph $P_4[C_4]$, it can be shown that $\gamma_{m \times 2}^1(P_4[C_4]) = 4$, $\gamma_t(P_4) = 2$ and $2\gamma_t(P_4) = 4$. Hence, $\gamma_{m \times 2}^1(P_4[C_4]) = 2\gamma_t(P_4)$. 
Figure 2: A graph $G[H]$ with $\gamma_t(G) \leq \gamma_{m1}(G[H]) \leq 2\gamma_t(G)$

References


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