Extreme Values Theory for Dynamical Systems:

Analysis of Stability Behavior in Delayed Kaldor-Kalecki Model

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Abstract

In this paper, we develop a new method for studying the stability of dynamical systems. Our approach based on Extreme Values Theory (EVT), analyses the evolution of return period and uses the Generalized Pareto Distribution (GPD) parameters to determine the fractal dimension of the attractor. After obtaining the solution of the considered system, we applied a $g$-transformation in order to ensure that points are considered extreme, if they are very close to the equilibrium of the dynamical system. In the second step, we adjust a GPD to set of extremes to calculate its fractal dimension and the return
period relative to the equilibrium. In this paper, this process is implemented in order to characterize the stability of the delayed Kaldor-Kalecki model.

Keywords: Extreme Values Theory, Delayed Kaldor-Kalecki model, Dynamical System, Fractal dimension, GPD distribution, Return period

1. Introduction

A dynamical system can be defined as a set of possible states, where the current states of the system are determined uniquely on the basis of his past statements. We can distinguish two types of dynamical systems. The first one is applied to discrete time, named discrete-time dynamical system and it is formulated by:

\[
\begin{aligned}
(x(n+1) &= f(x(n)) \\
x(0) &= x_0 \\
n &\geq 0
\end{aligned}
\]

Where \( f: \Omega \rightarrow \mathbb{R}^n, x \rightarrow f(x) \), is a mapping of class \( C^k, k \geq 1 \), of an open \( \Omega \subseteq \mathbb{R}^n \) in \( \mathbb{R}^n \).

The second type is called continuous-time dynamical system and it can be obtained as limit of discrete systems. The following equations describe the evolution of this system:

\[
\begin{aligned}
\dot{x}(t) &= f(x(t)) \\
x(0) &= x_0
\end{aligned}
\]

\( f: \Omega \rightarrow \mathbb{R}^n, x \rightarrow f(x) \) is a mapping of class \( C^k, k \geq 1 \), of an open \( \Omega \subseteq \mathbb{R}^n \) in \( \mathbb{R}^n \).

In this paper, we propose to study the stability of dynamical system solution, which the existence and local uniqueness are insured by Cauchy-Lipschitz theorem [3]. In fact, stability was probably the first question in classical dynamical systems which is treated by many mathematicians and particularly in control engineering. Thus, there is no single concept of stability, and many different definitions are possible [28]. The question we ask in stability theory is, do small changes in the system conditions (initial conditions, boundary conditions, parameter values etc.) lead to large changes in the solution? Generally speaking, if small changes in the system conditions lead only to small changes in the solution, we say that the solution is stable. More formally, an equilibrium state \( x = 0 \) is said to be:

- **Stable** if for any positive scalar \( \epsilon \) there exists a positive scalar \( \delta \) such that \( \|x(t_0)\| < \delta \) implies \( \|x(t)\| < \epsilon \) for all \( t \geq t_0 \).
- **Asymptotically stable** if it is stable and if in addition \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \).
- **Unstable**, if there exists an \( \epsilon > 0 \) such that for every \( \delta > 0 \) there exists an \( x(t_0) \) with \( \|x(t_0)\| < \delta \), \( \|x(t_1)\| \geq \epsilon \) for some \( t_1 > t_0 \).
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- **Completely unstable** if there exists an \( \varepsilon > 0 \) such that for every \( \delta > 0 \) and for every \( x(t_0) \) with \( \|x(t_0)\| < \delta, \|x(t_1)\| \geq \varepsilon \) for some \( t_1 > t_0 \).

In this paper, we intend to study the stability of dynamical systems using a new method based on extreme values theory; which concerns a modeling of maxima and minima of a random variable. To achieve this, the present paper is organized as follow: in the first section, we recall main concepts of extreme values theory, principally peaks over threshold approach. In the second section, we explicit our methodology, which is suitable to the application of EVT in dynamical systems, and then we provide some details about one case study concerning the Kaldor-Kalecki model. Finally, we conclude some theoretical findings which explain the numerical results.

2. **Extreme Values Theory (EVT)**

When modeling the maxima (minima) of a random variable, extreme values theory plays the same fundamental role as the Central Limit theorem plays when modeling the sum of random variables. In both cases, the theory tells us what the limiting distributions are.

Extreme Values Theory was originally introduced by Fisher and Tippett [6] and formalised by Gnedenko, who showed that under some suitable conditions, the asymptotic distribution of extremes identified on a sample of independent identically distributed (i.i.d) stochastic variables, \( X_1, X_2, \ldots, X_n \) belongs to the type of one of the following three cdfs:

- **Gumbel**: \( G_0(x) = \exp(-e^{-x}), x \in \mathbb{R} \)
- **Fréchet**: \( G_1, \alpha(x) = \exp(-x^{-\alpha}), x \geq 0, \alpha > 0 \)
- **Reversed Weibull**: \( G_2, \alpha(x) = \exp(-(x)^{-\alpha}), x \leq 0, \alpha < 0 \)

The three types of cdfs, given above, can be generalized as members of single family of cdfs. For that, let us introduce the new parameter \( \xi = \frac{1}{\alpha} \), therefore the proposed cdf is done by:

\[
G_\xi(x) = \exp\left(-\left(1 + \xi x\right)^{-1/\gamma}\right), 1 + \xi x > 0. \tag{3}
\]

The cdf \( G_\xi(x) \) is known as the generalized extreme values or as the extreme values cdf in the von Mises form. The parameter \( \xi \) is called the extreme values index.

The limiting case \( \xi \to 0 \) corresponds to the Gumbel distribution, \( \xi > 0 \) corresponds to the Fréchet and if \( \xi < 0 \) the distribution is weibull. Figure 1 below illustrates the three forms of extreme values distributions.
The classical extreme values approach, called “Block Component-Wise”, consists to segment the statistical series on the same size blocks (month, year, quarter...) and to identify the maxima of each block. The selected extremes will be adjusted by Generalized Extreme Values (GEV) distribution. This approach is strongly criticized because the estimation of the distribution based on extracted blocks maxima, involves a loss of information [32]. An alternative to the Block Component-Wise method is the “Peaks-Over-Threshold (POT)” approach. In such method, instead of modeling the maxima, the stochastic structure of the random exceedances over a high threshold value is considered. The POT, essentially related to the results of Pickands [27], Balkema and de Haan [1], is a widely used method [5] [7] [10] [29]. Balkema-de Haan-Pickands theorem states that under some regulatory conditions, the exceedances limiting distribution is a Generalized Pareto Distribution (GPD) [4] [31].

So, Given a suitably large threshold $T \geq 0$ and let $\{t_1, ..., t_n\} \subseteq \{t'_1, ..., t'_n\}$ to denote those time points (in increasing order) for which $X_{t'_1}, ..., X_{t'_n}$ exceed $T$, that is, let $X_{t_1}, ..., X_{t_n}$ denote the exceedances over $T$ with corresponding excesses $Z_{t_i} = X_{t_i} - T$, $i \in \{1, ..., n\}$. It follows from Embrechts et al. (1997, pp. 166) [8] that:

1) The number of exceedances $N_t$ follows approximately a Poisson process with intensity $\lambda$, that is, $N_t \sim \text{Poi}(\Lambda(t))$ with integrated rate function $\Lambda(t) = \lambda t$;

2) The excesses $X_{t_1}, ..., X_{t_n}$ over $T$ approximately follow (independently of $N_t$) a Generalized Pareto Distribution (GPD), denoted by $\text{GPD}(\xi, \sigma)$ for $\xi \in \mathbb{R}$ called the shape parameter and $\sigma > 0$, called the scale parameter with distribution function:

$$F_{\xi, \sigma}(x) = \begin{cases} 
1 - \left(1 + \frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}} & \text{for } \xi \neq 0, \\
1 - \exp\left(-\frac{x}{\sigma}\right) & \text{for } \xi = 0
\end{cases}$$ (4)
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Where $x \geq 0$, if $\xi \geq 0$, and $x \in [0, -\frac{\beta}{\xi}]$, if $\xi < 0$.

The main steps of POT implementation are:

1) Test the “Independent and Identically Distributed (iid)” hypothesis: Data should be a sequence of iid random variables.

2) Select an appropriate threshold level.

3) Estimate the parameters using the most appropriate method for the considered excesses dataset.

Note that the step 2 is the most critical one because the theory states that the threshold T should be high in order to satisfy Pickands-Balkema-de Haan theorem, but the higher the threshold is, the less observations are left for the estimation. Thus, the choice of threshold should be presents a tradeoff between bias and variance. In this context, many threshold selection methods are developed, such as Mean Residual Life Plot (MRL plot), Square Error Method (SEM) [2], Automated threshold selection method (ATSM) [30] and Multiple threshold method (MTM). The last one was shown substantial performances among all others methods. It was developed by Deidda [6] to infer the parameters of the GPD, underlying the exceedances of daily rainfall records over a wide range of thresholds. The motivation for this method resides in the needs of an appropriate technique, to overcome the difficulties arising from irregularly discretized rainfall records, or the site-to-site variability of the exceedances distribution parameters. It is shown that the MTM, based on the concept of parameters threshold-invariance, is particularly suitable for regional analysis where “optimum” thresholds may depend on the data collection site. As we expected, we found it also appropriate in our case study where the data are subject to different sources of perturbation.

3. Methodology and application

3.1. Methodology

In this paper, we consider a dynamical system $(\Omega, \mathcal{B}, \nu, f)$ where $\Omega$ is the invariant set in some manifold (usually considered $\mathbb{R}^d$), $\mathcal{B}$ is the Borel $\sigma$-algebra, $f: \Omega \to \Omega$ is a measurable map and $\nu$ an $f$-invariant Borel measure. We consider the stationary stochastic process $X_0, X_1, ...$ given by [12][24][25]:

$$X_m(x) = g(\text{dist}(f^m(x), \xi)) \quad \forall m \in \mathbb{N}$$

$$M_m = \max\{X_0, X_1, ..., X_{m-1}\}$$

Where ‘dist’ is a distance in the space $\Omega$, $\xi$ is a given point, $g$ is an observable function and $M_m$ is the partial maximum in the Gnedenko approach.

This formulation is taken in order to adapt the extreme values theory to dynamical systems, where the observable is expressed as a distance of the orbit $x$ from a point in the attractor $\xi$, usually taken as the initial condition [12][24][25].

Defining $r = \text{dist}(x, \xi)$, we consider the observable $g(r) = C - r^\beta$, where $C$ is a constant and $\beta > 0 \in \mathbb{R}$. 


At first, if the a dynamical system has a regular (periodic or quasi-periodic) behavior, we do not expect, to find convergence to GEV distributions for the extremes [11] [26]. Considering dynamical systems obeying suitable mixing conditions, which guarantee the independence of subsequent maxima, we can prove that the distribution of the block maxima of the observables converges to a member of the GEV family. The resulting parameters of the GEV distributions can be expressed as simple functions of the local dimension of the attractor [12] [24] [25].

Recently, it has been shown how to obtain results which are independent on whether the underlying dynamics of the system is mixing or, instead, regular. The key ingredient relies on using the Pareto rather than the Gnedenko approach. V. Lucarini and all [24] show that, in order to obtain a universal behavior of the extremes, the requirement of a mixing dynamics can be relaxed if the Pareto approach is used, based upon considering the exceedances over a given threshold.

As defined above (extreme values paragraph), the exceedance $z = X - T$ measures by how much $X$ exceeds the threshold $T$ and under some conditions the exceedances $z$ are asymptotically distributed according to the Generalised Pareto Distribution.

In this study, we search the solutions which are close to $\zeta$, this means that the $\text{dist}(x, \zeta) = r$ should be as small as possible. Since $g(r) = C - r^\beta$ is monotonically decreasing, this means that $g(r)$ becomes higher for smaller values of $r$. According to POT approach, we must seek observations which exceed a threshold $T$ defined as $T = g(r^*)$. So, in this case, we obtain an exceedance every time the distance between the orbits of the dynamical system and $\zeta$ are smaller than $r^*$. Therefore, we define the exceedances $z = g(r) - T$. By the Bayes’ theorem [24], we have that:

$$P\left(r < g^{-1}(z + T) \mid r < g^{-1}(T)\right) = \frac{p(r < g^{-1}(z + T))}{p(r < g^{-1}(T))}.$$

Consequently, the probability $H_{g,T}(z)$ of observing an exceedance of at least $z$ given that an exceedance occurs is given by:

$$H_{g,T}(z) = \frac{v(B_{g^{-1}(z+T)}(\zeta))}{v(B_{g^{-1}(T)}(\zeta))}. \quad (5)$$

And the corresponding cdf is given by $F_{g,T}(z) = 1 - H_{g,T}(z)$.

In order to calculate the fractal dimension of the attractor, we assume the existence of the following limit $\lim_{r \to 0} \frac{\log v(B_r(\zeta))}{\log(r)} = D(\zeta)$, for a chosen $\zeta$.

Where $D(\zeta)$ is the local dimension of the attractor [9]. Therefore, we can obtain:

$$H_{g,T}(z) \sim \left(\frac{g^{-1}(z+T)}{g^{-1}(T)}\right)^{D(\zeta)}.$$

By substituting $g$ by its form; $g(r) = C - r^\beta$ where $r = \text{dist}(x, \zeta)$ we derive as extreme values distribution law one member of the Generalised Pareto Distribution family. Results are detailed below:
\[ \sigma = \frac{(C-T)\beta}{D}, \quad \xi = -\frac{\beta}{D}. \]

Our proposed methodology consists to resolve the final system of Kaldor-Kalecki (see paragraph below) using Matlab, by considering a set of initial condition and delay values around \( E^* \) and \( \tau_0 \) to generate a different scenarios. It should be noted that resulting solutions are continuous functions of \( Y \) and \( K \), so to obtain time series, we implement a subroutine for a random discretization.

In order, to use extreme values theory, which represents our analytic study of system stability, we calculate \( g(r) = C - r^\beta \) where \( r = \text{dist}(x, E^*) \), \( x \) is done by a pair \( (Y, K) \) obtained by a discretization of system solution, as mentioned above, \( r \) is a distance between \( x \) and \( E^* \), \( C \) and \( \beta \) are constants. This \( g \)-transformation ensures that points are considered extremes if they are very close to the equilibrium points \( E^* \) of the dynamical system.

After that, we adjust GPD to each obtained time series. For this reason, we select a suitable threshold basing on MTM method [10] [23]. So, the optimal GPD is the distribution which maximizes the p-value of adequation test (Anderson Darling test).

To conclude results of stability analysis, we calculate the return period and the fractal dimension of each attractor using the following formulas:

\[ t = \frac{1}{1 - F}, \quad D = \frac{(C-T)\beta}{\sigma} = -\frac{\beta}{\xi}. \]

Where \( t \) is the return period, \( F \) is the distribution function of extremes, characterized by the scale \( \sigma \) and the shape \( \xi \) parameters, \( D \) is the fractal dimension and \( T \) is a given threshold.

### 3.2. Application: Delayed Kaldor-Kalecki model

To test our proposed methodology, we propose in the present paper the analysis of an interesting economic model, namely called delayed Kaldor-Kalecki model. The use of this model is justified by two principal reasons. The first one corresponds to its usefulness in many recent economic applications [14] [16] [18] [19] [21]. The second one is comparing results and testing our methodology on the same model analyzed by Kaddar and al. [15], who consider the existence of local Hopf bifurcation and establish an explicit algorithm, for determining the direction the stability or instability of the considered delayed Kaldor Kalecki model [16] [17]. As presented by Matsumoto and Szidarovszky (2014), Kalecki (1935) [16] assumes a lag between “investment order” and “investment installation” and call it a gestation lag of investment. Thus, he constructs a macro dynamic model as a delay differential equation of retarded type. Kaldor (1940) [17], on the other hand, studies the evolution of production and capital formation and believes that nonlinearities of behavioral equations could be a clincher in such cyclic oscillations.
The main innovation in the Kaldor-Kalecki approach is to model national income and capital with nonlinear investment and saving functions. In this context, nonlinearity and delay are treated as two principal ingredients for endogenous cycles.

More formally, Kaldor-Kalecki model is given by the following two equations:

\[
\begin{align*}
\dot{Y}(t) &= \alpha \left[ \Phi(Y(t), K(t)) - S(Y(t)) \right] \\
\dot{K}(t) &= \Phi(Y(t), K(t)) - \delta K(t)
\end{align*}
\]

Where \(Y(t)\) is national income at time \(t\), \(K(t)\) denotes capital, \(\Phi(Y(t), K(t))\) is an investment function. \(S(Y(t))\) is a savings function and the parameters \(\alpha\) and \(\delta\) denotes the adjustment coefficient and the rate of depreciation.

Kaleckian investment function with delay can be presented by:

\[
I(t) = \Phi(Y(t - \tau), K(t))
\]

Where \(\tau\) denotes the gestation delay. It should be noted that there are several extension of the delay investment function. Kaddar and Talibi (2009) [15] introduce time delay also in capital accumulation equation, i.e.,

\[
I(t) = \Phi(Y(t - \tau), K(t - \tau))
\]

So, the complete delayed Kaldor-Kalecki model formulated in [15] is presented by:

\[
\begin{align*}
\dot{Y}(t) &= \alpha \left[ I(Y(t), K(t)) - S(Y(t), K(t)) \right] \\
\dot{K}(t) &= I(Y(t - \tau), K(t - \tau)) - \delta K(t)
\end{align*}
\]

As in Krawiec and Szydlowski (2000) [20] and Kaddar (2008) [13], we consider some assumptions on the investment and saving functions:

\[
I(Y, K) = I(Y) - \beta K,
\]

and

\[
S(Y, K) = \gamma Y,
\]

Where \(\beta > 0\) and \(\gamma \in (0,1)\). Then the final model becomes:

\[
\begin{align*}
\dot{Y}(t) &= \alpha \left[ I(Y(t) - \beta K(t) - \gamma Y(t)) \right] \\
\dot{K}(t) &= I(Y(t - \tau) - \beta K(t - \tau) - \gamma Y(t)) - \delta K(t)
\end{align*}
\]

### 3.3. Numerical analysis

For further numerical investigations, we consider the following Kaldor-type investment function [15]:

\[
I(Y) = \frac{\exp(Y)}{1 + \exp(Y)}
\]
and we assume that : \( \alpha = 3, \beta = 0.2, \delta = 0.1 \) and \( \gamma = 0.2 \) to have a positive equilibrium of the system (see Kaddar (2009)) [15]:

\[
E^* = (1.31346, 2.62699) \quad \text{With } \tau_0 = 2.9929
\]

To investigate the properties of the constructed dynamical system above, we represent in the figures below, a phase diagrams based on some scenarios, obtained from different delay and initial condition values. These diagrams help to convey the dynamic properties of differential equations. By their nature, phase diagrams are a feature of dynamical systems. We can remark that for delay values less than 2.9929, the dynamical system has a stable equilibrium point \( E^* \).

**Figure 2: Illustration of a stable equilibrium point E.**

The system becomes unstable when the delay is more than 2.9929.

**Figure 3: Unstable dynamical system**

In these cases ( \( \tau > 2.9929 \) and \( \tau < 2.9929 \) ), the obtained results are independent of initial conditions values.
In other case, for the delay equals to 2.9929, we obtain a stable periodic solution, but the form of the phase diagram change within the initial conditions.

![Figure 4: Illustration of a stable periodic solution](image)

For our elaborated approach, we propose to use extreme values theory which represents our analytic study of system stability.

To implement our methodology, we calculate \( g(r) = C - r^\beta \) where \( r = \text{dist}(x, E^*) \), \( C = 1 \) and \( \beta = 2 \).

To adjust GPD to each obtained time series, we select a suitable threshold basing on MTM method and we calculate the fractal dimension of each attractor using the following formula:

\[
D = \frac{(C - T)\beta}{\sigma} = -\frac{\beta}{\xi}
\]

The table below summarizes the obtained results for which we can easily deduce that fractal dimension value depends on the delay and initial conditions. For some scenario, the adjusted GPD is not obtained (NAN), which means that none of the tested distributions gives an acceptable p-value. We recall that \( n \) is the total number of observations and \( l \) represents the number of those which exceed the threshold (T).

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>( \tau )</th>
<th>( (Y_0; K_0) )</th>
<th>( g(r) = 1 - r^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( n )</td>
</tr>
<tr>
<td>scenario 1</td>
<td>2.9929</td>
<td>(1.5 ; 2.3)</td>
<td>687</td>
</tr>
</tbody>
</table>
## Extreme values theory for dynamical systems

### Table 1: Some results of the proposed methodology

<table>
<thead>
<tr>
<th>Scenario</th>
<th>r</th>
<th>Region</th>
<th>N</th>
<th>Scale</th>
<th>Shape</th>
<th>p-value</th>
<th>Scale</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 2</td>
<td>2</td>
<td>(1.15 ; 2.3)</td>
<td>560</td>
<td>-</td>
<td>-</td>
<td>NAN</td>
<td>NAN</td>
<td>0.9</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>2</td>
<td>(1.15 ; 2.3)</td>
<td>119</td>
<td>-</td>
<td>-</td>
<td>NAN</td>
<td>NAN</td>
<td>1.9</td>
</tr>
<tr>
<td>Scenario 4</td>
<td>2</td>
<td>(1.15 ; 2.9)</td>
<td>438</td>
<td>0.75</td>
<td>51</td>
<td>p-value=0.1</td>
<td>scale=0.3427</td>
<td>shape=-1.3693</td>
</tr>
<tr>
<td>Scenario 5</td>
<td>2.3</td>
<td>(1.5 ; 2.3)</td>
<td>133</td>
<td>0.83</td>
<td>117</td>
<td>p-value=0.22</td>
<td>scale=0.0742</td>
<td>shape=-1.0584</td>
</tr>
<tr>
<td>Scenario 6</td>
<td>4</td>
<td>(1.31 ; 2.62)</td>
<td>1367</td>
<td>-5</td>
<td>322</td>
<td>p-value=0.85</td>
<td>scale=13.5347</td>
<td>shape=-2.2554</td>
</tr>
<tr>
<td>Scenario 7</td>
<td>4</td>
<td>(1.5 ; 2.9)</td>
<td>1731</td>
<td>-10.5</td>
<td>721</td>
<td>p-value=0.35</td>
<td>scale=9.8161</td>
<td>shape=0.8581</td>
</tr>
<tr>
<td>Scenario 8</td>
<td>4</td>
<td>(1.5 ; 2.3)</td>
<td>1773</td>
<td>-11</td>
<td>964</td>
<td>p-value=0.61</td>
<td>scale=7.6565</td>
<td>shape=0.6329</td>
</tr>
<tr>
<td>Scenario 9</td>
<td>4</td>
<td>(1.15 ; 2.3)</td>
<td>1734</td>
<td>-11</td>
<td>963</td>
<td>p-value=0.612</td>
<td>scale=7.9438</td>
<td>shape=0.6544</td>
</tr>
<tr>
<td>Scenario 10</td>
<td>4</td>
<td>(1.15 ; 2.9)</td>
<td>1739</td>
<td>-12.5</td>
<td>983</td>
<td>p-value=0.47</td>
<td>scale=10.9609</td>
<td>shape=0.8113</td>
</tr>
<tr>
<td>Scenario 11</td>
<td>2.9929</td>
<td>(0.6 ; 0.8)</td>
<td>1399</td>
<td>-2.8</td>
<td>543</td>
<td>p-value=0.75</td>
<td>scale=2.393</td>
<td>shape=0.6291</td>
</tr>
<tr>
<td>Scenario 12</td>
<td>2.9929</td>
<td>(1.31 ;0.7)</td>
<td>1398</td>
<td>-3.1</td>
<td>562</td>
<td>p-value=0.42</td>
<td>scale=3.1463</td>
<td>shape=0.7665</td>
</tr>
<tr>
<td>Scenario 13</td>
<td>2.9929</td>
<td>(3 ; 4)</td>
<td>1398</td>
<td>-3.8</td>
<td>633</td>
<td>p-value=0.22</td>
<td>scale=5.2693</td>
<td>shape=1.0974</td>
</tr>
<tr>
<td>Scenario 14</td>
<td>5</td>
<td>(1.31 ; 1.62)</td>
<td>1722</td>
<td>-23</td>
<td>1093</td>
<td>p-value=0.94</td>
<td>scale=15.8775</td>
<td>shape=0.6612</td>
</tr>
</tbody>
</table>

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Table 1: Some results of the proposed methodology.
The table above provides interesting findings. Firstly, we obtain for all scenarios; all delay and initial conditions values, that the best adjusted distribution is the Weibull one. This distribution is characterized by a negative shape and it is generally used to describe various types of observed failures of components and phenomena, in reliability and survival analysis [22].

The other finding corresponds to the relation between the fractal dimension and delay values. We can conclude that, when \( \tau < 2.9929 \), the dynamical system has an equilibrium solution and for this case, the fractal dimension is less than 2 for all initial conditions.

Similarly, for \( \tau > 2.9929 \), the dynamical system is unstable and has a local dimension greater than 2 for all initial conditions.

In the transition regime, which occurs for \( \tau = 2.9929 \), the ensemble spread is much higher, because the scaling properties of the measure is different among the various initial conditions, for this case, characterized by stable periodic solution, we have different scales of fractal dimension (for some case, we have values less than 2 and for others we obtain higher values).

To decide in this case, the return period can be very useful. So, we calculate for each scenario, the return period of a set of values around \( E^* = (1.31346, 2.62699) \). The set of values corresponds to peaks which exceeds the identified threshold of each scenario. The behavior of the convergence related to the set of return period (see graphics below) allows to characterize the stability of the system.

![Figure 5: Evolution of return period for equilibrium systems.](image)

For scenarios which characterize systems with equilibrium point, the graphs of return period shows a rapid decrease toward 0 which corresponds to:

\[
t(g(x) = 1) = t \left(1 - \left(d(x,E^*)\right)^2 = 1\right) = t \left(\left(d(x,E^*)\right)^2 = 0\right) = 0
\]
For stable periodic solutions, we have a slow decrease of return period graph without convergence to 0. This is similar to oscillations of pendulum around equilibrium point. This propriety characterizes an unstable periodic system.

\[ t(g(x) = 1) = t\left(1 - (d(x,E^*))^2 = 1\right) = t\left((d(x,E^*))^2 = 0\right) > 0 \]

The unstable systems are characterized by no convergence to 0, represented by a roughly horizontal curve of return period, with nonnegative values (upper to 1). Thus, the system never reaches the equilibrium point \( E^* \). So the system is unstable.
As conclusion, we highlight that the main contribution of our paper consists to adopt extreme values theory as interesting tools to study stability of equilibrium point in dynamical system. As output of our analysis, we provide decision criteria to distinguish between many possible cases of stability. The criteria are based simultaneously on return period behavior over time and fractal dimension values. Thus, for a stable dynamical system, we obtain a value of fractal dimension of the attractor less than 2, but it is higher than 2 if the dynamical system has an unstable equilibrium point. In the case of unstable period solution, the fractal dimension remains inconclusive, this is meaning that in this special case, we can obtain fractal dimension higher, lower or equal to 2. Thus, a need to have additional criteria to decide. In this paper, we suggest to analyze the return period behavior. So, when the evolution of the returns period is characterized by a rapid decrease toward 0, we can confirm that the dynamical system has certainly a stable equilibrium point. When the returns period decrease slowly, without convergence to 0, we talk, in this case about a stable periodic solutions. The third case corresponds to unstable equilibrium point which is characterized by no convergence of return period to 0, represented by a roughly horizontal curve, with nonnegative values (upper to 1).

It should be noted that this methodology can be applied to any dynamical system, other than Kaldor-Kalecki model studied in this paper.

References


http://dx.doi.org/10.1023/a:1009914915709

http://dx.doi.org/10.1007/978-3-642-33483-2


http://dx.doi.org/10.1016/j.physd.2011.11.005


http://dx.doi.org/10.2307/2225740


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