A-Differential of Graphs

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Abstract

Let $G = (V(G), E(G))$ be an arbitrary graph and let $X \subseteq V(G)$. The set $A(X) = X \cap N(X)$ consists of the non-isolates in $X$, where $N(X) = \{y \in V(G) : xy \in E(G) \text{ for some } x \in X\}$. Let $B(X)$ denote the set of vertices in $V(G) \setminus X$ that has a neighbor in $X$. The A-differential of $X$ is given by $\partial_A(X) = |B(X)| - |A(X)|$. The A-differential of $G$, denoted by $\partial_A(G)$, is equal to $\max\{\partial_A(X) : X \text{ is a subset of } V(G)\}$.

This paper gives the A-differential of graphs resulting from some binary operations such as join and composition.

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1 Introduction

Let $G = (V(G), E(G))$ be an arbitrary graph of order $n$. The neighborhood of a vertex $v$ of $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For a set $X \subseteq V(G)$, the neighborhood of $X$ is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$.

The closed neighborhood of $X$ is the set $N_G[X] = N_G(X) \cup X$. The set $A(X) = A_G(X) = X \cap N_G(X)$ consists of the non-isolates in $X$ and the boundary of $X$, denoted by $B(X) = B_G(X)$, is the set $(V(G) \setminus X) \cap N_G(X)$.

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The A-differential of X is given by \( \partial_A(X) = |B_G(X)| - |A_G(X)| \) and the A-differential of a graph G is given by \( \partial_A(G) = \max \{ \partial_A(X) : X \subseteq V(G) \} \). Any subset X of \( V(G) \) with \( \partial_A(X) = \partial_A(G) \) is called the \( \partial_A \)-set of G. The sets \( B(X) \) are considered by Slater in [4]. The parameter \( \partial_A(G) \) is considered by Haynes et.al. in [2] and by Pushpam and Yukesh in [3].

A set \( S \subseteq V(G) \) is a dominating set of G if \( N[S] = V(G) \). The domination number \( \gamma(G) \) of G is the minimum cardinality of a dominating set. If \( S \) is a dominating set with \( |S| = \gamma(G) \), then we call \( S \) a minimum dominating set of G or a \( \gamma \)-set in G. If \( N(S) = V(G) \), then we say that \( S \) is a total dominating set of G. The total domination number \( \gamma_t(G) \) of G is the minimum cardinality of a total dominating set. If \( S \) is a total dominating set with \( |S| = \gamma_t(G) \), then we call \( S \) a minimum total dominating set of G or a \( \gamma_t \)-set in G. Obviously, every total dominating set is a dominating set; hence, \( \gamma(G) \leq \gamma_t(G) \).

A subset \( S \) of \( V(G) \) is an independent set if every pair of distinct elements of \( S \) are non-adjacent. \( S \) is an independent dominating set of G if \( S \) is both independent and dominating set. The independent domination number \( \gamma_i(G) \) of G is the minimum cardinality of an independent dominating set. If \( S \) is an independent dominating set with \( |S| = \gamma_i(G) \), then we call \( S \) a minimum independent dominating set of G or a \( \gamma_i \)-set in G.

Domination and other variants of domination can be found in [1].

2 Results

Lemma 2.1 Let G be any graph of order \( n \geq 2 \). Then \( 0 \leq \partial_A(G) \leq n - 1 \).

Proof: Let \( X = \emptyset \) be subset of \( V(G) \). Then \( B(X) = \emptyset \) and \( A(X) = \emptyset \). Hence, \( \partial_A(X) = 0 \leq \partial_A(G) \).

Next, let \( Y \) be a subset of \( V(G) \) with \( \partial_A(G) = \partial_A(Y) \). If \( Y = \emptyset \), then \( \partial_A(Y) = 0 \leq n - 1 \). If \( Y \neq \emptyset \), then \( |B(Y)| \leq |V(G)\{Y\}| \leq n - 1 \) and \( |A(Y)| \geq 0 \). Therefore, \( \partial_A(G) = \partial_A(Y) = |B(Y)| - |Y| \leq (n - 1) - 0 = n - 1 \). Accordingly, \( 0 \leq \partial_A(G) \leq n - 1 \). \( \square \)

The degree of vertex \( v \) of a graph G is given by \( \text{deg}_G(v) = |N(v)| \). The maximum degree of G, denoted by \( \Delta(G) \), is \( \max \{ \text{deg}_G(v) : v \in V(G) \} \). If \( v \) is a vertex of G with \( \text{deg}_G(v) = \Delta(G) \), then we call \( v \) a vertex of maximum degree in G.

The following result is due to Haynes, et.al. in [2].

Theorem 2.2. For any graph G, \( \Delta(G) \leq \partial_A(G) \).

Theorem 2.3. Let G be a graph of order \( n \). Then \( \partial_A(G) = 0 \) if and only if every component of G is trivial.
Proof: Suppose $\partial_A(G) = 0$. Then $\Delta(G) = 0$ by Theorem 2.2. This implies that every component of $G$ is isomorphic to $K_1$.

The converse is clear. \qed

Theorem 2.4 Let $G$ be any graph of order $n \geq 2$. Then $\partial_A(G) = n - 1$ if and only if $\Delta(G) = n - 1$.

Proof: Suppose that $\partial_A(G) = n - 1$ and let $X \subseteq V(G)$ such that $\partial_A(G) = \partial_A(X)$. If $|X| \geq 2$, then $|V(G)\setminus X| \leq n - 2$. Hence, $\partial_A(G) = \partial_A(X) = |B(X)| - |A(X)| \leq |B(X)| \leq n - 2$, contrary to the assumption that $\partial_A(G) = n - 1$. This implies that $|X| = 1$, say $X = \{v\}$; hence $B(X) = V(G)\setminus \{v\}$. Therefore, $v \in N(x)$ for all $x \in V(G)\setminus \{v\}$. Accordingly, $\Delta(G) = \deg_G(v) = n - 1$.

For the converse, assume that there exists a vertex $v$ of $G$ such that $v \in N(x)$ for all $x \in V(G)\setminus \{v\}$. Set $X = \{v\}$. Then $B(X) = V(G)\setminus \{v\}$ and $A(X) = \emptyset$. Thus, $\partial_A(G) \geq n - 1$. By Lemma 2.1, $\partial_A(G) = n - 1$. \qed

The following result is a direct consequence of Theorem 2.4.

Corollary 2.5 $\partial_A(K_n) = n - 1$ for any positive integer $n$.

The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}.$$ 

We now give the $A$-differential of the join of two graphs.

Theorem 2.6 Let $G$ and $H$ be graphs of orders $n$ and $m$, respectively.

(i) If either $G$ or $H$ is complete, then $\partial_A(G + H) = n + m - 1$.

(ii) If $G$ and $H$ are non-complete, then

$$\partial_A(G + H) = \max\{n + m - 4, \partial_A(G) + m, \partial_A(H) + n\}.$$ 

Proof: (i) Assume that $H = K_m$ and pick $a \in V(H)$. By the adjacency of $G + H$, it follows that $v \in N(a)$ for all $v \in V(G)\setminus \{a\}$. Hence, by Theorem 2.4, $\partial_A(G + H) = n + m - 1$.

(ii) Let $k = \max\{n + m - 4, \partial_A(G) + m, \partial_A(H) + n\}$. Consider the following cases:

Case 1. Assume that $k = n + m - 4$. Pick $a \in V(G)$ and $b \in V(H)$. Set $X = \{a, b\}$. Then $B_{G+H}(X) = V(G + H)\setminus X$ and $A_{G+H}(X) = \{a, b\}$. This implies that $\partial_A(X) = n + m - 4 \leq \partial_A(G + H)$. 

Next, let $Y \subseteq V(G + H)$ be such that $\partial_A(G + H) = \partial_A(Y)$. If $Y \cap V(G) \neq \emptyset$ and $Y \cap V(H) \neq \emptyset$, then $B_{G+H}(Y) = V(G + H) \setminus Y$ and $A_{G+H}(Y) = Y$. Hence, $\partial_A(G+H) = \partial_A(Y) = n + m - 2|Y| \leq n + m - 4$. Suppose now that either $Y \subseteq V(G)$ or $Y \subseteq V(H)$, say $Y \subseteq V(G)$. Then $B_{G+H}(Y) = V(H) \cup (N_G(Y) \setminus Y)$ and $A_{G+H}(Y) = A_G(Y)$. Therefore, $\partial_A(G + H) = \partial_A(Y) = m + |N_G(Y) \setminus Y| - |A_G(Y)| \leq m + \partial_A(G) \leq n + m - 4$.

Similarly, if $Y \subseteq V(H)$, then $\partial_A(G + H) = \partial_A(Y) \leq n + \partial_A(H) \leq n + m - 4$. Therefore, $\partial_A(G + H) = n + m - 4$.

Case 2. Assume that $k = \partial_A(G) + m$. Let $X \subseteq V(G)$ such that $\partial_A(G) = \partial_A(X)$. Then $B_{G+H}(X) = V(H) \cup (N_G(X) \setminus X) = V(H) \cup B_G(X)$ and $A_{G+H}(X) = A_G(X)$. Hence, $\partial_A(G + H) \geq \partial_A(X) = m + \partial_A(X) = m + \partial_A(G)$.

Next, let $Y \subseteq V(G + H)$ be such that $\partial_A(G + H) = \partial_A(Y)$. If $Y \cap V(G) \neq \emptyset$ and $Y \cap V(H) \neq \emptyset$, then $B_{G+H}(Y) = V(G + H) \setminus Y$ and $A_{G+H}(Y) = Y$. It follows that $\partial_A(G + H) = \partial_A(Y) = n + m - 2|Y| \leq n + m - 4 \leq \partial_A(G) + m$.

Suppose now that either $Y \subseteq V(G)$ or $Y \subseteq V(H)$, say $Y \subseteq V(G)$, then $B_{G+H}(Y) = V(H) \cup (N_G(Y) \setminus Y)$ and $A_{G+H}(Y) = A_G(Y)$. Thus, $\partial_A(G + H) = \partial_A(Y) = m + \partial_A(G) \leq m + \partial_A(G)$. Therefore, $\partial_A(G + H) = \partial_A(G) + m$.

Case 3. Assume that $k = \partial_A(H) + n$. Then by following the arguments of the proof of the preceding case, we have $\partial_A(G + H) = \partial_A(H) + n$. □

The following result, which is a direct consequence of Theorem 2.6(i), gives the $A$-differentials of the wheel, fan and star.

**Corollary 2.7** Let $n$ be a positive integer. Then

(i.) $\partial_A(W_n) = n, \forall n \geq 3$;

(ii.) $\partial_A(F_n) = n, \forall n \geq 2$;

(iii.) $\partial_A(S_n) = n, \forall n \geq 1$;

where $W_n$, $F_n$ and $S_n$ are the wheel, fan and star of order $n + 1$.

The next result, which is a direct consequence of Theorem 2.6(ii), gives the $A$-differential of the complete bipartite.

**Corollary 2.8** Let $n \geq 2$ and $m \geq 2$ be integers. Then

$$\partial_A(K_{m,n}) = \max \{n + m - 4, m, n\}.$$
Let $S \subseteq V(G[H])$. The $G$-projection $S_G$ of $S$ and the $H$-projection $S_H$ of $S$ are defined as follows:

$$S_G = \{ u : (u, v) \in S \text{ for some } v \in V(H) \},$$

$$S_H = \{ v : (u, v) \in S \text{ for some } u \in V(G) \}.$$ 

**Theorem 2.9** Let $G$ be a connected graph of order $n$ and $m \geq 2$. Then $\partial_A(G[K_m]) = nm - k$, where

$$k = \min \{|T| + |A_G(T)| : T \text{ is a dominating set in } G\}.$$ 

**Proof:** Let $S$ be a subset of $V(G[K_m])$ with $\partial_A(G[K_m]) = \partial_A(S)$. Suppose $S_G$ is not a dominating set of $G$. Then there exists a vertex $x \in V(G) \setminus N[S_G]$. This implies that $x \notin S_G$ and $x \notin N(v)$ for all $v \in S_G$. Pick $a \in V(K_m)$ and consider the set $S^* = S \cup \{ (x, a) \}$. Let $K = \{ (x, u) : u \in V(K_m) \setminus \{ a \} \}$. Then $|K| \geq 1$ and $K \cap B_{G[K_m]}(S) = \emptyset$. Also, $B_{G[K_m]}(S^*) = B_{G[K_m]}(S) \cup K$ and $A_{G[K_m]}(S^*) = A_{G[K_m]}(S)$. Thus

$$\partial_A(S^*) = |B_{G[K_m]}(S)| + |K| - |A_{G[K_m]}(S)| = \partial_A(S) + |K| > \partial_A(G[K_m]).$$

This contradicts the property of $S$. Therefore, $S_G$ is a dominating set of $G$. Consequently, $B_{G[K_m]}(S) = V(G[K_m]) \setminus S$ and

$$\partial_A(G[K_m]) = \partial_A(S) = |B_{G[K_m]}(S)| - |A_{G[K_m]}(S)| = nm - |S| - |A_{G[K_m]}(S)|.$$ 

Next, let $y \in V(K_m)$ and consider $S_1 = S_G \times \{ y \}$. Then $B_{G[K_m]}(S_1) = V(G[K_m]) \setminus S_1$ and

$$\partial_A(S_1) = nm - |S_1| - |A_{G[K_m]}(S_1)|.$$ 

Now $(x, y) \in A_{G[K_m]}(S_1)$ if and only if there exist $(x', y) \in S_1$ such that $(x, y)(x', y) \in E(G[K_m])$. That is, there exists $x' \in S_G$ such that $xx' \in E(G)$. Thus, $(x, y) \in A_{G[K_m]}(S_1)$ if and only if $x \in A_G(S_G)$. This implies that $|A_{G[K_m]}(S_1)| = |A_G(S_G)|$. Since $|S_1| = |S_G|$, it follows that

$$\partial_A(S_1) = nm - |S_G| - |A_G(S_G)|.$$ 

Let $x \in A_G(S_G)$. Then $x \in S_G$ and there exists $x' \in S_G$ such that $xx' \in E(G)$. Let $a, b \in V(K_m)$ such that $(x, a), (x', b) \in S$. Then $(x, a), (x', b) \in E_G(G[K_m])$. This implies that $(x, a) \in A_{G[K_m]}(S)$, i.e., $x \in (A_{G[K_m]}(S))_G$ (the $G$-projection of $A_{G[K_m]}(S)$). Therefore, $|A_G(S_G)| \leq \sum_{x \in (A_{G[K_m]}(S))_G} |A_{G[K_m]}(S)| \leq |A_{G[K_m]}(S)|$. Hence,

$$\partial_A(S_1) = nm - |S_G| - |A_G(S_G)| \geq nm - |S| - |A_{G[K_m]}(S)| = \partial_A(G[K_m]).$$
This forces \(|S| + |A_G[K_m]| \leq |S| + |A_G(S_G)|\).

Finally, let \(k = \min \{|T| + |A_G(T)| : T \text{ is a dominating set in } G\}\). If \(T^*\) is a dominating set in \(G\) such that \(k = |T^*| + |A_G(T^*)|\), then \(k \leq |S| + |A_G(S_G)|\).

Consider \(C = T^* \times \{z\}\) for some \(z \in V(K_m)\). Then \(B_{G[K_m]}(C) = V(G[K_m])\) and

\[
\partial_A(C) = nm - |T^*| - |A_G(T^*)| \geq nm - |S| - |A_G(S_G)| = \partial_A(G[K_m]).
\]

Therefore, \(\partial_A(G[K_m]) = nm - k\). \(\square\)

**Corollary 2.10** Let \(G\) be a connected graph of order \(n\) and \(m \geq 2\). If \(G\) has a minimum independent dominating set, then \(\partial_A(G[K_m]) = nm - \gamma(G)\).

**Proof:** Suppose \(T^*\) is a minimum independent dominating set in \(G\). Then \(A_G(T^*) = \emptyset\). Also, \(|T^*| \leq |T|\) for any dominating set \(T\) in \(G\); hence, \(|T^*| \leq |T| + |A_G(T)|\) for any dominating set \(T\) in \(G\). It follows that \(\min \{|T| + |A_G(T)| : T \text{ is a dominating set in } G\} = |T^*| + |A_G(T^*)| = |T^*| = \gamma(G)\). By Theorem 2.9, \(\partial_A(G[K_m]) = nm - \gamma(G)\). \(\square\)

**Theorem 2.11** Let \(G\) and \(H\) be non-trivial connected graphs. If \(C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V([G])\), where \(S \subseteq V(G)\) and \(T_x \subseteq V(H)\) for each \(x \in S\), then

\[
(i) \quad B_{G[H]}(C) = [B_G(S) \times V(H)] \cup \bigcup_{x \in S \setminus N(S)} \{x\} \times B_H(T_x) \cup \bigcup_{x \in S \cap N(S)} \{x\} \times (V(H) \setminus T_x);
\]

\(\quad (ii) \quad A_{G[H]}(C) = (\bigcup_{x \in S \setminus N(S)} \{x\} \times T_x) \cup (\bigcup_{y \in S \setminus N(S)} \{y\} \times A_H(T_y))\); and

\(\quad (iii) \quad \partial_A(C) = |B_G(S)| |V(H)| + \sum_{x \in S \setminus N(S)} \partial_A(T_x) + \sum_{x \in S \setminus N(S)} (|V(H)| - 2|T_x|)\).

**Proof:** (i) Let \((x, p) \in B_{G[H]}(C)\). Then \((x, p) \in N_{G[H]}(C) \setminus C\).

Consider the following cases:

**Case 1.** \(x \notin S\)

Since \((x, p) \in N_{G[H]}(C)\), there exists \((y, q) \in C \cap N_{G[H]}((x, p))\). This implies that \(y \in S \cap N_G(x)\). Thus, \(x \in N_G(S) \setminus S = B_G(S)\). Therefore, \((x, p) \in B_G(S) \times V(H)\).

**Case 2.** \(x \in S\)

Since \((x, p) \notin C\), it follows that \(p \notin T_x\). Hence, \((x, p) \in \{x\} \times (V(H) \setminus T_x)\). If \(x \in S \setminus N(S)\), then there exists \((x, a) \in C \cap N_{G[H]}((x, p))\). It follows that \(a \in T_x \cap N_H(p)\). Hence, \(p \in N_H(T_x) \setminus T_x = B_H(T_x)\). Therefore, \((x, p) \in \{x\} \times B_H(T_x)\). Thus, \(B_{G[H]}(C)\) is contained in the union of the
given sets.

Next, let \((y,t) \in B_G(S) \times V(H)\). Then \(y \in B_G(S) = N_G(S) \setminus S\). Hence, \(y \notin S\) and there exists \(z \in N_G(y) \cap S\). Pick any \(b \in T_x\). Then \((z,b) \in N_{G[H]}((y,t)) \cap C\). This implies that \((y,t) \in N_{G[H]}(C) \setminus C = B_{G[H]}(C)\).

Thus, \(B_G(S) \times V(H) \subseteq B_{G[H]}(C)\). Now, let \(x \in S \setminus N(S)\) and let \((x,q) \in \{x\} \times B_H(T_x)\). Then \(q \in N(T_x) \setminus T_x\), that is, \(q \notin T_x\) and there exists \(r \in N_H(q) \cap T_x\). Hence, \((x,r) \in C \cap N_{G[H]}((x,q))\). Thus, \((x,q) \in N_{G[H]}(C) \setminus C = B_{G[H]}(C)\). Therefore, \(\{x\} \times B_H(T_x) \subseteq B_{G[H]}(C)\) for every \(x \in S \setminus N(S)\).

Finally, let \(y \in S \cap N(S)\) and let \((y,d) \in \{y\} \times (V(H) \setminus T_y)\). Then, \((y,d) \notin C\) and there exists \(w \in S \cap N_G(y)\). Choose any \(c \in T_w\). Then \((w,c) \in C \cap N_{G[H]}((y,d))\). Thus, \((y,d) \in N_{G[H]}(C) \setminus C = B_{G[H]}(C)\). Therefore, \(\{y\} \times (V(H) \setminus T_y) \subseteq B_{G[H]}(C)\) for each \(y \in S \cap N(S)\). This establishes the desired equality.

\((ii)\) Let \((x,t) \in A_{G[H]}(C)\). Then \((x,t) \in C \cap N(C)\). This implies that \(x \in S\) and \(t \in T_x\). Thus, \((x,t) \in \{x\} \times T_x\). Suppose \(x \in S \setminus N(S)\). Since \((x,t) \in N(C)\) and \(x \notin N(S)\), there exists \((x,j) \in C \cap N_{G[H]}((x,t))\). This implies that \(j \in T_x \cap N_H(t)\). Therefore, \(t \in T_x \cap N_H(T_x) = A_H(T_x)\), that is, \((x,t) \in \{x\} \times A_H(T_x)\). This shows that \(A_{G[H]}\) is contained in the union of the given sets.

Now, suppose that \(x \in S \cap N(S)\) and \(a \in T_x\). Then \((x,a) \in C \cap N_{G[H]}((x,a))\). Thus, \((x,a) \in C \cap N_{G[H]}(C) = A_{G[H]}(C)\). Hence, \(\{x\} \times T_x \subseteq A_{G[H]}(C)\) for each \(x \in S \cap N(S)\). Next, let \(z \in S \setminus N(S)\) and let \(p \in A_H(T_x)\). Then \(p \in T_z\) and there exists \(q \in T_z \cap N_H(p)\). Then \((z,q) \in C \cap N_{G[H]}((z,p))\). It follows that \((z,p) \in C \cap N_{G[H]}(C) = A_{G[H]}(C)\). This establishes the desired equality.

\((iii)\) By definition, \((i)\) and \((ii)\),

\[
\partial_A(C) = |B_{G[H]}(C)| - |A_{G[H]}(C)| = |B_G(S)||V(H)| + \sum_{x \in S \setminus N(S)} |B_H(T_x)| + \sum_{x \in S \cap N(S)} |V(H) \setminus T_x| - \sum_{x \in S \cap N(S)} |T_x| - \sum_{x \in S \cap N(S)} |A_H(T_x)| = |B_G(S)||V(H)| + \sum_{x \in S \setminus N(S)} \partial_A(T_x) + \sum_{x \in S \cap N(S)} (|V(H)| - 2|T_x|)
\]

This completes the proof of the theorem. \(\square\)

**Theorem 2.12** Let \(G\) and \(H\) be connected non-trivial graphs. If \(C = \bigcup_{x \in S} \{x\} \times T_x\) is a \(\partial_A\)-set of \(G[H]\), then \(S\) is a dominating set of \(G\), \(T_x\) is a \(\partial_A\)-set of \(H\) for each \(x \in S \setminus N(S)\) and \(|T_x| = 1\) for each \(x \in S \cap N(S)\).
Proof: Suppose $S$ is not a dominating set of $G$. Then there exists $x \in V(G) \setminus N_G[S]$. Thus, $x \notin S$ and $x \notin N_G(v)$ for all $v \in S$. Choose any $a \in V(H)$ and let $C^* = C \cup \{(x, a)\}$. Let $K = \{(x, p) : p \in V(H) \setminus \{a\} \cap N_H(a)\}$. Since $H$ is a non-trivial connected graph, $|K| \geq 1$ and $K \cap B_{G[H]}(C) = \emptyset$. Moreover, $B_{G[H]}(C^*) = B_{G[H]}(C) \cup K$ and $A_{G[H]}(C^*) = A_G[H](C)$. Thus,

$$\partial_A(C^*) = |B_{G[H]}(C)| + |K| - |A_{G[H]}(C)| = |K| + \partial_A(C),$$

contrary to our assumption that $C$ is a $\partial_A$-set of $G[H]$. Therefore, $S$ is a dominating set of $G$.

Next, let $x \in S \setminus N(S)$. Suppose that $T_x$ is not a $\partial_A$-set of $H$. Then $\partial_A(T_x) < \partial_A(H)$. Let $D \subseteq V(H)$ be such that $\partial_A(H) = \partial_A(D)$. Let $C_1 = \bigcup_{y \in S} \{y\} \times T_y^*$, where $T_y^* = T_y$ if $y \neq x$ and $T_x^* = D$.

By Theorem 2.11(iii),

$$\partial_A(C_1) = |B_{G[H]}(C_1)| - |A_{G[H]}(C_1)|
= |B_G(S)||V(H)| + \sum_{x \in S \setminus N(S)} \partial_A(T_y^*) + \sum_{x \in S \setminus N(S)} (|V(H)| - 2|T_y^*|)
> |B_G(S)||V(H)| + \sum_{x \in S \setminus N(S)} \partial_A(T_y) + \sum_{x \in S \setminus N(S)} (|V(H)| - 2|T_y|)
= \partial_A(C),$$

This contradicts our assumption that $C$ is a $\partial_A$-set of $G[H]$. Thus, $T_x$ is a $\partial_A$-set of $H$ for each $x \in S \setminus N(S)$. Finally, from Theorem 2.11(iii) and the fact that $C$ is a $\partial_A$-set of $G[H]$, it follows that $|T_x| = 1$ for all $x \in S \cap N(S)$. □

Corollary 2.13 Let $G$ and $H$ be connected non-trivial graphs of orders $m$ and $n$, respectively. Then

$$\partial_A(G[H]) = \max \Omega,$$

where $\Omega = \{n|B_G(S)| + \partial_A(H)|S \setminus N(S)| + (n - 2)|S \cap N(S)| : S$ is a dominating set of $G\}$. In particular,

$$\max \{mn - 2\gamma_t(G), mn + \partial_A(H)\gamma_t(H) - n\gamma_t(H)\} \leq \partial_A(G[H]).$$

Proof: Let $S$ be a dominating set of $G$ and let $D$ be a $\partial_A$-set of $H$. Set $T_x = D$ for each $x \in S \setminus N(S)$ and $T_x = \{a\}$, where $a \in V(H)$, for each $x \in S \cap N(S)$. If $C = \bigcup_{x \in S} \{x\} \times T_x$, then by Theorem 2.11(iii),

$$\partial_A(C) = n|B_G(S)| + \partial_A(H)|S \setminus N(S)| + (n - 2)|S \cap N(S)| \leq \partial_A(G[H]).$$

On the other hand, if $C^* = \bigcup_{y \in S^*} \{y\} \times R_y$ is a $\partial_A$-set of $G[H]$, then $S^*$ is a dominating set of $G$, $R_y$ is a $\partial_A$-set of $H$ for each $y \in S^* \setminus N(S^*)$ and $|R_y| = 1$.
for each $y \in S^* \cap N(S^*)$ by Theorem 2.12. Thus, by Theorem 2.11(iii),

$$\partial_A(G[H]) = \partial_A(C^*)$$
$$= n|B_G(S^*)| + \partial_A(H)|S^* \setminus N(S^*)| + (n - 2)|S^* \cap N(S^*)|$$
$$\leq \max \Omega$$

This proves the assertion.

Next, let $S_1$ and $S_2$ be a $\gamma_t$-set and $\gamma_i$-set of $G$, respectively and let $D$ be a $\partial_A$-set of $H$. Then $N(S_1) \setminus S_1 = V(G) \setminus S_1 = B_G(S_1)$, $S_1 \cap N(S_1) = S_1$, $S_1 \setminus N(S_1) = \emptyset$, $S_2 \cap N(S_2) = \emptyset$, $S_2 \setminus N(S_2) = S_2$, and $N(S_2) \setminus S_2 = V(G) \setminus S_2 = B_G(S_2)$. Let $a \in V(H)$ and $T_x = \{a\}$ for each $x \in S_1$. Also, let $D_x = D$ for each $x \in S_2$. Consider $C_1 = \bigcup_{x \in S_1} \{ \{x\} \times T_x \}$ and $C_2 = \bigcup_{x \in S_2} \{ \{x\} \times D_x \}$. Then by Theorem 2.11(iii),

$$\partial_A(C_1) = n(m - |S_1|) + (n - 2)|S_1|$$
$$= n(m - \gamma_t(G)) + n\gamma_t(G) - 2\gamma_i(G)$$
$$= mn - 2\gamma_t(G)$$

and

$$\partial_A(C_2) = n(m - |S_2|) + |S_2|\partial_A(H)$$
$$= mn - n\gamma_i(H) + \partial_A(H)\gamma_i(H).$$

Since $\max\{\partial_A(C_1), \partial_A(C_2)\} \leq \partial_A(G[H])$, the assertion holds. \(\square\)

**Remark 2.14** The lower bound in Corollary 2.13 is sharp. To see this, consider $G = H = P_4$. It can be verified that $\partial_A(G[H]) = 12 = 4(4) - 2\gamma_t(P_4) = 4(4) + |\partial_A(P_4) - 4\gamma_i(P_4)|$.

**References**


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