Almost Everywhere Convex Functions in Generalized Sobolev Spaces

Matloob Anwar and Hina Urooj

School of Natural Sciences
National University of Sciences and Technology
Islamabad, Pakistan

Abstract

In this paper we characterize almost every where convex functions using generalized Sobolev space $W^{1,p(x)}_{loc}(\Omega)$ under some condition imposed on the generalized weak gradient. We assume in this paper that the variable exponent $p(x)$ is bounded on $\Omega$, an open convex subset of $\mathbb{R}^n$.

Keywords: Sobolev Spaces, Generalized Sobolev Spaces, mollification

1. Introduction

Lebesgue and Sobolev spaces are most important spaces in mathematical analysis and these spaces have several interesting properties. We refer [2] and [4] for the basic theory of the variable exponent Lebesgue and Sobolev spaces, but give the main definitions for the reader's convenience.

Let $p(x)$ be a measurable function on $\Omega \subseteq \mathbb{R}^n$ with values in $[1, \infty)$. Denote $p^+ = ess \sup_{x \in \Omega} p(x)$ and $p^- = ess \inf_{x \in \Omega} p(x)$.

The generalized Lebesgue space $L^{p(x)}(\Omega)$ is the space of all measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that

$$I_p(f) := \int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$ 

With respect to the norm given below this is a Banach space:
\[ \|f\|_{L^p(x)} = \inf \{ \lambda > 0 : I^p(f/\lambda) \leq 1 \}. \]

We say that \( \Omega' \) is compactly contained in \( \Omega \) denoted as \( \Omega' \subset \subset \Omega \) if there exists a compact set \( K \) such that \( \Omega' \subset K \subset \Omega \). \( L^p_{loc}(\Omega) \) is the space of functions \( f \) such that \( f \in L^p(\Omega') \) for all \( \Omega' \subset \subset \Omega \). The generalized Sobolev space \( W^{1,p}(x)_{loc}(\Omega) \) is the space of functions \( f \) such that \( f \in L^p(\Omega')_{loc}(\Omega') \) for all \( \Omega' \subset \subset \Omega \). The generalized Sobolev space \( W^{1,p}(x)_{loc}(\Omega) \) is the space of functions in \( L^p(\Omega) \) whose generalized weak gradient \( \nabla f \) exists and lies in \( L^p(\Omega) \), with the norm

\[ \|f\|_{W^{1,p}(x)_{loc}} = \|f\|_{L^p(x)} + \|\nabla f\|_{L^p(x)}. \]

The space \( W^{1,p}(x)_{loc}(\Omega) \) is defined in the same manner as \( L^p_{loc}(\Omega) \).

La Torre [5] has proved the inclusion of convex functions in the usual Sobolev space \( W^{1,p}(x)_{loc}(\Omega) \) for all \( p \geq 1 \), he also proved the converse for a.e. convex functions; see ref. [1] for the basic theory in the usual Sobolev space. This paper investigates the results in [5] for generalized Sobolev space \( W^{1,p}(x)_{loc}(\Omega) \).

## 2. Main Results

We assume, throughout the paper, that the variable exponent \( p(x) \) is bounded that is \( p^+ < \infty \). \( \Omega \) is an open convex subset in \( \mathbb{R}^n \).

### Definition 2.1.

A function defined on \( \Omega \) \( f : \Omega \to \mathbb{R} \) is said to be locally Lipschitz if for each \( x_0 \in \Omega \) there exists a neibourhood \( B(x_0, r) \) of \( x_0 \) and a positive constant \( C \) such that

\[ |f(y) - f(x)| \leq C\|y - x\| \text{ for all } y, x \in B(x_0, r). \]

\( C \) is called a Lipschitz constant, which will vary with \( x_0 \).

### Definition 2.2.

A function \( f \) on \( \Omega \) is said to be locally bounded if for each \( x_0 \in \Omega \), there exists a neibourhood \( U \) of \( x_0 \) and a positive constant \( M \) such that \( |f(x)| \leq M \) for all \( x \in U \).

### Lemma 2.3.

Let \( f : \Omega \to \mathbb{R} \) be a locally Lipschitz function then each of its partial derivatives, if exists, is locally bounded.

**Proof.** For \( x_0 \in \Omega \), let \( B(x_0, r) \) be a neibourhood of \( x_0 \). For \( x \in B(x_0, r) \), choose \( h > 0 \) and the unit vectors \( e_i = (0, 0, ..., 1, ..., 0, 0) \) (1 at the \( i^{th} \) position, \( i = 1, \ldots, n \) ) in the direction of each coordinate axes, such that \( x + he_i \in B(x_0, r) \). We can choose such \( h \) for each \( x \); for instance, \( h < \text{dist}(x, \partial B(x_0, r)) \) will work. Given \( f \) is locally Lipschitz with Lipschitz constant say \( C \), we have

\[ |f(x + he_i) - f(x)| \leq C|h| \text{ for all } x \in B(x_0, r). \]

Dividing by \( h \) and applying limit \( h \to 0 \),

\[ \lim_{h \to 0} |f(x + he_i) - f(x)/h| \leq C \text{ for all } x \in B(x_0, r). \]
Using continuity of modulus function and definition of partial derivative, we finally have

\[ \left| \frac{\partial f(x)}{\partial x_i} \right| \leq C \text{ for all } x \in B(x_0, r). \]

\[ \square \]

**Theorem 2.4.** Let the function \( f : \Omega \to \mathbb{R} \) be a locally Lipschitz. Then \( f \in W^{1,p(x)}_{\text{loc}}(\Omega) \).

**Proof.** Since the given function \( f \) is a locally Lipschitz function, by Rademacher’s theorem \( f \) is almost everywhere differentiable and hence the gradient \( \nabla f \) exists a.e in \( \Omega \). The Lipschitz continuity of \( f \) also assures the local boundedness of \( f \) and \( \frac{\partial f(x)}{\partial x_i} \). Considering local boundedness of \( f \), we have for every open set \( U \subset \subset \Omega \) there exists a constant \( m_U \) such that \( |f(x)| \leq m_U \) for all \( x \in U \), and hence we have

\[ I_p(f) = \int_U |f(t)|^{p(t)} dt \leq \int_U (m_U)^{p(x)} dx \leq \max\{m_U^{p^-}, m_U^{p^+}\} \mu(U) < \infty. \]

Thus \( f \in L^{p(x)}_{\text{loc}}(\Omega) \) and on applying the same argument as above on \( \frac{\partial f(x)}{\partial x_i} \), we conclude that \( \frac{\partial f(x)}{\partial x_i} \in L^{p(x)}_{\text{loc}}(\Omega) \). Next we shall show that the classical \( i \)th partial derivative of \( f \) is indeed the weak \( i \)th partial derivative of \( f \). For an arbitrary \( \varphi \in C^\infty_0(\Omega) \) and \( h \) sufficiently small we have

\[ \int_\Omega \frac{f(x + he_i) - f(x)}{h} \varphi(x) dx = -\int_\Omega \frac{\varphi(x) - \varphi(x - he_i)}{h} f(x) dx. \]

Above integrals are defined since the product of a function in \( L^{p(x)}_{\text{loc}}(\Omega) \) with a function in \( C^\infty_0(\Omega) \) is an integrable function. Applying limit \( h \to 0 \) and using the Lebesgue convergence theorem for integrable functions, we have

\[ \int_\Omega \frac{\partial f}{\partial x_i} \phi(x) dx = -\int_\Omega \frac{\partial \phi}{\partial x_i} f(x) dx. \]

Since \( \phi \) was arbitrary we have our desired result. \( \square \)

**Remark 2.5.** Since convex functions on an open set are locally Lipschitz, above theorem also holds for convex functions.

**Theorem 2.6.** Let \( f : \Omega \to \mathbb{R} \) be a function in \( W^{1,p(x)}_{\text{loc}}(\Omega) \) satisfies

\[ f(y) - f(x) \geq < \nabla f(x), y - x > \text{ almost everywhere } y, x \in \Omega \text{ then } < y - x, \nabla f(y) - \nabla f(x) > \geq 0 \text{ almost everywhere } y, x \in \Omega. \nabla f \text{ denotes the weak gradient of } f. \]

**Proof.** The proof follows by adding the given inequality with the inequality obtained from it by interchanging \( x \) and \( y \). \( \square \)
Theorem 2.7. [6] Let the function \( f : \Omega \to \mathbb{R} \) be a Gâteaux differentiable. Then \( f \) is convex iff \( \nabla f(x) - \nabla f(y), x - y \geq 0 \) for all \( x, y \in \Omega \).

Theorem 2.8. [6] Suppose \( f \) is Gâteaux differentiable function on \( \Omega \) then \( f \) is convex if and only if \( f(x) - f(y) \geq \nabla f(y), x - y \) holds for all \( x, y \in \Omega \).

Definition 2.9. The function

\[
\phi(x) = \begin{cases} 
Ce^\left(\|x\|^2 - 1\right) & \text{if } \|x\| < 1, \\
0 & \text{if } \|x\| \geq 1
\end{cases}
\]

belongs to \( C_0^\infty(\mathbb{R}^n) \) for any constant \( C \), which can be chosen such that \( \int \phi(x)dx = 1 \).

For any \( \varepsilon > 0 \), the function \( \phi(x) \) generates a sequence of smooth functions \( \phi_\varepsilon \) on \( \mathbb{R}^n \) called the standard sequence of mollifiers, defined as \( \phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi \left( \frac{x}{\varepsilon} \right) \), satisfying \( \int \phi_\varepsilon(x)dx = 1 \) and having compact support \( B(0, \varepsilon) \).

Definition 2.10. Let \( f \in L^1_{loc}(\Omega) \), its mollification \( f_\varepsilon : \Omega_\varepsilon \to \mathbb{R} \) is defined using the following convolution as

\[
f_\varepsilon(x) = (\phi_\varepsilon * f)(x) = \int_\Omega \phi(x - t)f(t)dt,
\]

where \( \Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon\} \), that is for \( x \in \Omega_\varepsilon \), \( B(x, \varepsilon) \subset \Omega \) and for each \( x, y \) in \( B(x, \varepsilon) \), the vector \( x - y \) lies in \( B(0, \varepsilon) \).

Theorem 2.11. Let the function \( f : \Omega \to \mathbb{R} \) belongs to \( W^{1,p}(\Omega) \) and satisfies \( \langle y - x, \nabla f(y) - \nabla f(x) \rangle \geq 0 \) for a.e. \( y, x \in \Omega \). Then for any compact set \( K \) in \( \Omega \) there exists \( \varepsilon_0 > 0 \) such that the mollifications \( f_\varepsilon \) are convex for all \( \varepsilon \) such that \( 0 < \varepsilon < \varepsilon_0 \).

Proof. Let \( K \) be a compact set in \( \Omega \) then we can find some \( \varepsilon_0 > 0 \) such that \( K \subset \Omega_{\varepsilon_0} \subset \Omega \). This can be done by choosing \( \varepsilon_0 \) such that \( \varepsilon_0 < \min_{x \in K}\{\text{dist}(x, \partial \Omega)\} \).

Also it follows from the definition of \( \Omega_\varepsilon \) that \( \Omega_{\varepsilon_0} \subset \Omega_\varepsilon \) for \( 0 < \varepsilon < \varepsilon_0 \) and hence for such \( \varepsilon, K \subset \Omega_\varepsilon \). We have for all \( x, y \in K \) and \( \varepsilon \in (0, \varepsilon_0) \):

\[
\langle x - y, \nabla f_\varepsilon(x) - \nabla f_\varepsilon(y) \rangle = \sum_{i=1}^{n} (x_i - y_i) \left( \frac{\partial f_\varepsilon}{\partial x_i}(x) - \frac{\partial f_\varepsilon}{\partial x_i}(y) \right)
\]

\[
= \sum_{i=1}^{n} (x_i - y_i) \left\{ \int_\Omega (\phi_\varepsilon(x - z) - \phi_\varepsilon(y - z)) \frac{\partial f}{\partial x_i}(z)dz \right\}
\]

\[
= \sum_{i=1}^{n} (x_i - y_i) \frac{1}{\varepsilon^n} \left( \int_\Omega \phi \left( \frac{x - z}{\varepsilon} \right) f(z)dz - \phi \left( \frac{y - z}{\varepsilon} \right) f(z)dz \right)
\]
\[
= - \sum_{i=1}^{n} (x_i - y_i) \left( \int_{B(0,1)} \phi(s) \left( \frac{\partial f}{\partial x_i} (x - \varepsilon s) - \frac{\partial f}{\partial x_i} (y - \varepsilon s) \right) ds \right)
= \int_{B(0,1)} \phi(s) < x - y, \nabla f(x) - \nabla f(y) > ds \geq 0.
\]

Hence \(f_\varepsilon\) are convex on \(K\), follows from Theorem 2.7.

\textbf{Corollary 2.12.} If \(f\) satisfies \(f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle\) almost everywhere \(y, x \in \Omega\) and is continuous then \(f\) is convex.

\textbf{Corollary 2.13.} Let the function \(f: \Omega \to \mathbb{R}\) belongs to \(W^{1,p(x)}_{\text{loc}}(\Omega)\) and satisfies \(< x - y, \nabla f(x) - \nabla f(y) \rangle \geq 0\) almost everywhere in \(\Omega\). Then \(f\) satisfies \(f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle\) almost everywhere in \(\Omega\).

\textbf{Proof.} The given function \(f\) belongs to \(W^{1,p(x)}_{\text{loc}}(\Omega)\), hence being locally integrable function, \(f\) and its weak gradient \(\nabla f\) can be approximated by the mollified functions \(f_\varepsilon\) and \(\nabla f_\varepsilon\) almost everywhere in \(\Omega\), that is \(f_\varepsilon \to f\) and \(\nabla f_\varepsilon \to \nabla f\) everywhere in \(\Omega\) except a set \(A\) of measure zero. Also we have given \(< x - y, \nabla f(x) - \nabla f(y) \rangle \geq 0\) for all \(x, y \in \Omega\setminus A\). Let \(x, y \in \Omega\setminus A\). Let \(K\) be the compact set containing \(x\) and \(y\). For this compact set \(K\), by using Theorem 2.11, we can find a number \(\varepsilon_0 > 0\) such that the \(f_\varepsilon\) are convex for all \(\varepsilon \in (0, \varepsilon_0)\), and differentiable will satisfy \(f_\varepsilon(x) - f_\varepsilon(y) \geq \langle \nabla f_\varepsilon(y), x - y \rangle\). On applying limit \(\varepsilon \to 0\) to this inequality we get the desired result.

\textbf{Theorem 2.14.} Let \(f: \Omega \to \mathbb{R}\) be an almost everywhere convex function then \(f\) belongs to \(W^{1,p(x)}_{\text{loc}}(\Omega)\) and satisfies \(f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle\) almost everywhere in \(\Omega\).

\textbf{Proof.} The given function \(f\) is almost everywhere convex, hence can be written as a sum of two functions \(g\) and \(h\), that is, \(f = g + h\), where \(g\) is a convex function while \(h\) is a zero function almost everywhere in \(\Omega\). The function \(g\) being a convex function is locally Lipschitz and hence almost everywhere differentiable on \(\Omega\). Since \(g\) is differentiable on \(\Omega\setminus A\), where \(\mu(A)\) is zero, using Theorem 2.7 we have

\[g(x) - g(y) \geq \langle \nabla g(y), x - y \rangle \quad \text{for all } x, y \in \Omega \setminus A,
\]

where \(\nabla g(y)\) is the weak gradient of \(g\). Since \(f\) and \(g\) are equivalent the weak gradient of \(f\) is the same as for \(g\), hence \(f \in W^{1,p(x)}_{\text{loc}}(\Omega)\) and above equation can be written as

\[f(x) - f(y) \geq \langle \nabla g(y), x - y \rangle \quad \text{for all } x, y \in \Omega \setminus A.
\]
Definition 2.15. A function $f$ defined on an open set $\Omega$ of $\mathbb{R}^n$ is said to be differentiable at $c$ if there exists a linear functional $df(c; v)$, defined for all $v$ in $\mathbb{R}^n$ such that for all $\delta > 0$ satisfies the following:

$$|f(x) - f(c) - df(c; x - c)| \leq \delta \|x - c\| \text{ for } x \in \Omega.$$ 

Theorem 2.16. Suppose that $f : \Omega \rightarrow \mathbb{R}$ be a function in $W^{1,p(x)}(\Omega)$ which satisfies $f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle$, $f_\epsilon(x) \rightarrow f(x)$ and $\nabla f_\epsilon(x) \rightarrow \nabla f(x)$ for all $x$ in $\Omega \setminus A$, with $\mu(A) = 0$. Then $f$ is almost everywhere convex on $\Omega$.

Proof. To reach the desired conclusion, we will first prove that $f$ is an element of $W^{1,\infty}_loc(\Omega)$, that is, $f$ and its weak gradient $\nabla f$ are essentially locally bounded on $\Omega$. We prove that $f$ is essentially locally bounded on $\Omega$ by proving the local boundedness of $f_\epsilon$ on some compact set in $\Omega$. Choose a cube $Q$ in $\Omega \setminus A$ centered at $x_0$ with vertices $v_1, v_2, \ldots, v_{2^n}$. Since $Q$ is compact there is a positive number $\epsilon_0$ such that $f_\epsilon$ are convex on $Q$ for all $\epsilon$ in $(0, \epsilon_0)$. Also, it is given that $f_\epsilon$ converges to $f$ a.e in $\Omega \setminus A$, there exists a constant $C_1$ such that $f_\epsilon(x_i) \leq C_1$ for all $\epsilon$ in $(0, \epsilon_0)$ $i = 1, 2, \ldots, 2^n$. Using convexity of $f_\epsilon$ on $Q$ and the fact that every $x$ in $Q$ is a convex combination of its vertices, we have $f_\epsilon(x) \leq C_1$ for all $x$ in $Q$ and for all $\epsilon$ in $(0, \epsilon_0)$. Hence $f_\epsilon$ is bounded above on $Q$. Now by symmetry of $Q$, for every $x \in Q$ there exists $y \in Q$ such that $x_0 = (x + y)/2$. Hence $f_\epsilon(x_0) \leq (f_\epsilon(x) + f_\epsilon(y))/2$ which implies $f_\epsilon(x) \geq 2f_\epsilon(x_0) - f_\epsilon(y) \geq 2f_\epsilon(x_0) - C_1 = C_2$. Hence $f_\epsilon$ is bounded on $Q$ by the constant $C = \max(C_1, C_2)$, and there exists an open ball $B(x_0, \delta)$ such that $f$ is bounded a.e. on $B(x_0, \delta)$, that is $\|f\|_{L^\infty(B(x_0, \delta))} < C$.

To prove the essential local boundedness of $\nabla f$ we will show that for sufficiently small $\epsilon > 0$, $f_\epsilon$ are Lipschitzian on $B(x_0, \delta)$. Since for each sufficiently small $\epsilon$, the functions $f_\epsilon$ are defined, also bounded on $B(x_0, \delta)$, we can apply the definition of differentiability at $x_0$ to each $f_\epsilon$ as follows: For some $\eta > 0$ and $x \in B(x_0, \delta)$, we have

$$|f_\epsilon(x) - f_\epsilon(x_0) - df_\epsilon(x_0, x - x_0)| \leq \eta \|x - x_0\| \text{ for all } x \in B(x_0, \delta),$$

which implies

$$|f_\epsilon(x) - f_\epsilon(x_0)| < |df_\epsilon(x_0, x - x_0)| + \eta \|x - x_0\|. \quad (1)$$

Since $df_\epsilon(x_0; x - x_0) = \langle \nabla f_\epsilon(x_0), x - x_0 \rangle$, the Cauchy Schwartz inequality implies that $|df_\epsilon(x_0; x - x_0)| \leq \|\nabla f_\epsilon(x_0)\| \|x - x_0\|$, hence inequality (2.1) becomes

$$|f_\epsilon(x) - f_\epsilon(x_0)| \leq \|\nabla f_\epsilon(x_0)\| \|x - x_0\| + \eta \|x - x_0\| = C' \|x - x_0\|,$$

for all $x \in B(x_0, \delta)$, where $C' = \eta + \|\nabla f(x_0)\|$. Hence for $\epsilon$ small enough, $f_\epsilon$ are Lipschitz on $B(x_0, \delta)$ and therefore have gradient $\nabla f_\epsilon$ which are bounded by $C'$. This implies that the weak gradient $\nabla f$ is essentially locally bounded on $B(x_0, \delta)$. Hence it is proved that $f$ is an element of $W^{1,\infty}_loc(\Omega)$. Since functions
Almost everywhere convex functions in $W^{1,\infty}_{loc}(\Omega)$ can be identified as locally Lipschitz functions (see [3]), we can find a continuous function $F$ on $\Omega$ such that $F = f$ almost everywhere on $\Omega$. Also, the function $F$ satisfies the inequality given in the hypothesis almost everywhere $x$ in $\Omega$, hence convex by Corollary 2.12, which in turn implies that $f$ is almost every where convex.

Conclusion. In this paper we have discussed some interesting properties of almost every where convex functions in Generalized Sobolev spaces. These results are useful to study some further applications of convex functions in Generalized Sobolev Spaces. Like in Sobolev Spaces convex functions are used to solve differential equations.

Acknowledgements. This research work is funded by School of Natural Sciences, National University of Sciences and Technology, Islamabad Pakistan.

REFERENCES


Received: June 23, 2015; Published: August 12, 2015