A Remark on Attractors for the 2D Navier-Stokes Equations with Weak Damping and Distributed Delay

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Abstract

We shall show the long-time behavior such as the existence of attractors for the 2D Navier-Stokes equation with weak damping, translation bounded functions class and distributed delay.

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1 Introduction

In this present paper, the long-time behavior for the 2D Navier-Stokes equations with weak damping and distributed delay that governs the motion of
incompressible fluid was considered:

\[
\begin{align*}
\begin{cases}
  u_t - \nu \Delta u + (u \cdot \nabla)u + \alpha u + \nabla p &= f(x,t) + \int_{-h}^{0} G(s, u(t + s)) ds, \quad (x,t) \in \Omega_{\tau}, \\
  \text{div} u &= 0, \quad (x,t) \in \Omega_{\tau}, \\
  u(t, x) |_{\partial \Omega} &= \varphi, \quad \varphi \cdot n = 0, \quad (x,t) \in \partial \Omega_{\tau}, \\
  u(\tau, x) &= u_0(x), \quad x \in \Omega, \\
  u(t, x) &= \phi(t - \tau, x), \quad (x,t) \in \Omega_{\tau h},
\end{cases}
\end{align*}
\]

(1)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, $\Omega_{\tau} = \Omega \times (\tau, +\infty)$, $\Omega_{\tau h} = \Omega \times (\tau - h, \tau)$, $\tau \in \mathbb{R}$ is the initial time, $\nu$ is the kinematic viscosity of the fluid, $u = u(t, x) = (u_1(t, x), u_2(t, x))$ is the velocity vector field which is unknown, $p$ is the pressure, $\alpha > 0$ is positive constant, $\alpha u$ is the weak damping, $\varphi$ denotes the velocity in boundary and satisfies $\varphi \in L^\infty(\partial \Omega)$, $f(x, t)$ is the non-autonomous external force, $\int_{-h}^{0} G(s, u(t + s)) ds$ is the distributed delay external force. $\phi$ is the initial state of delay in $(-h, 0)$, $h > 0$ is a constant.

This paper will be organized as follows: in section 2, we shall give some preliminaries; in section 3, the existence and uniqueness of global weak solutions will derived; we shall prove the existence of absorbing ball in section 4, the global attractor will be concluded in last section.

## 2 Preliminaries

Throughout this paper, $C$ will stand for a generic positive constant, depending on $\Omega$ and some constants, but independent of the choice of the initial time $\tau$ and $t$.

We set $E := \{ u | u \in (C_0^\infty(\Omega))^2, \text{div} u = 0 \}$, $H$ is the closure of the set $E$ in $(L^2(\Omega))^2$ topology, $W$ is the closure of the set $E$ in $(H^2(\Omega))^2$ topology, i.e.,

\[
W = \{ u \in W | \| u \|_W = \| u \|_{H^2, u|_{\partial \Omega} = 0} \}.
\]

(2)

For each $t \in (\tau, T)$ when $T > \tau$, we define $u : (\tau - h, T) \to (L^2(\Omega))^2$, here $u_t$ is a function in $(-h, 0)$ satisfying $u_t = u(t + s), s \in (-h, 0)$. Next, we denote $C_H = C^0([-h, 0]; H)$ and $C_V = C^0([-h, 0]; V)$ as two Banach’s space corresponding to the norm

\[
\| u \|_{C_H} = \sup_{\theta \in [-h, 0]} | u(t + \theta) |
\]

and

\[
\| u \|_{C_V} = \sup_{\theta \in [-h, 0]} \| u(t + \theta) \|
\]

respectively, $L^2_H = L^2(-h, 0; H)$, $L^2_V = L^2(-h, 0; V)$.
A remark on attractors for the 2D Navier-Stokes equations

First we construct background function \( \psi \) which satisfying

\[
\text{div}\psi = 0, \quad x \in \Omega, \\
\psi = \varphi, \quad x \in \partial\Omega, \\
\|\psi\|_{L^\infty(\partial\Omega)} \leq c\|\varphi\|_{L^\infty(\partial\Omega)}, \\
|\psi| \leq c\|\varphi\|_{L^\infty(\partial\Omega)}.
\]

Using the background function and \( v = u - \psi \), (1) is equivalent to

\[
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + \alpha u + \nabla p = \bar{f} + g_\psi(v_t),
\]

\[
\text{div}v = 0,
\]

\[
v = 0,
\]

\[
v(\tau, x) = v_0(x),
\]

\[
v(t, x) = \phi(t - \tau, x) - \psi(x) = \eta(t - \tau, x),
\]

here \( \bar{f} = f - \alpha \psi + \nu \Delta \psi - (\psi \cdot \nabla)\psi \), \( g_\psi(v_t) = \int_{-h}^{0} G(s, v(t+s) + \psi)ds \).

Define \( Ru = B(u, \psi) + B(\psi, u) \), let \( v_0 \in H, \eta \in L^2_H \), then the problems (1) and (9)–(13) can be written as the generalized abstract form

\[
v_t + \nu Av + \alpha v + B(v) + B(\psi, v) + B(\psi, v) = P\bar{f} + g_\psi(v_t),
\]

\[
v(\tau) = v_0,
\]

\[
v(t) = \eta(t - \tau),
\]

where the pressure \( p \) has disappeared by virtue of the application of the Leray-Helmholtz projection \( P \).

Suppose the external force \( G \) in (1) satisfies:

(a) \( G \): \([-h, 0] \times \mathbb{R}^2 \to \mathbb{R}^2 \), and measurable;

(b) \( G(s, 0) = 0, s \in [-h, 0] \);

(c) \( \exists r \in L^2(-h, 0) \), such that \( |G(s, u) - G(s, v)|_{\mathbb{R}^2} \leq r(s)|u - v|_{\mathbb{R}^2} \).

From (a) and (c), if \( \xi, \eta \in C_H \) we have:

\[
|g_\psi(\xi) - g_\psi(\eta)|^2 \\
\leq \int_\Omega \left( \int_{-h}^{0} |G(s, \xi(s))(x) - G(s, \eta(s))(x)|_{\mathbb{R}^2} ds \right)^2 dx \\
\leq \int_\Omega \left( \int_{-h}^{0} r(s)|\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2} ds \right)^2 dx \\
\leq \int_\Omega \|r\|_{L^2(-h, 0)}^2 \left( \int_{-h}^{0} |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2} ds \right)^2 dx \\
\leq L_g \|\xi - \eta\|_{C_H}^2.
\]
where $L_g = h\|r\|_{L^2(-h,0)}$, and then $u,v \in C^0([-h,T];H)$, for each $t > 0, m_0 \geq 0 \ (m \in [0, m_0])$ we have

\[
\int_0^t e^{m\tau} |g_\psi(u_\tau) - g_\psi(v_\tau)|^2 d\tau \\
\leq \|r\|_{L^2(-h,0)}^2 \int_0^t e^{m\tau} \left( \int_{-h}^0 |u(s + \tau) - v(s + \tau)|^2 ds \right) d\tau \\
\leq \|r\|_{L^2(-h,0)}^2 \int_{-h}^0 \left( \int_0^t e^{m\tau} |u(s + \tau) - v(s + \tau)|^2 d\tau \right) ds \\
\leq \|r\|_{L^2(-h,0)}^2 \int_{-h}^0 \left( \int_{-h}^0 e^{m(r-s)} |u(r) - v(r)|^2 dr \right) ds \\
\leq C_g^2 \int_{-h}^0 e^{m\tau} |u(r) - v(r)|^2 dr,
\]

where $C_g^2 = \|r\|_{L^2(-h,0)}^2 hel^{m_0 h}$. 

(d) $\nu \lambda_1 > 2c_1 \lambda_1^{1/2} \|\psi\| + 4C_g^2/\alpha$, where $\lambda_1$ is the first eigenvalue of $A$ under the homogeneous Dirichlet boundary condition.

### 3 Existence of Weak Global Solutions

The existence of weak global solutions for (1) can be derived by similar methods as in [3]:

**Theorem 3.1** Let $f \in (L^2(\Omega))^2$ (or translation bounded functions class) and assume (a) $\sim$ (d) hold, when $v_0 \in H$ there exists a unique weak solution $v(t)$ of (9)–(13) satisfies

\[
v(t) \in L^\infty(0,T; H) \cap L^2(0,T; V)
\]

and $\frac{dv}{dt}$ is uniformly bounded in $L^2(0,T; V')$.

**Proof.**

Step 1. Set the orthogonal base in H of A is $w_j$, and $Aw_j = \lambda_j w_j, \forall j$, using Faedo-Galerkin to find the approximate solution $v_n(t) = \sum_{j=0}^n a_{nj}(t)w_j$ of (9)–(13) and $v_n(t)$ satisfies:

\[
\frac{dv_n}{dt} + \nu Av_n + \alpha v_n + B(v_n) + R(v_n) = P_n \bar{f} + g_\psi(v_{nt}), \\
v_n(\tau) = v_{n0}, \\
v_n(t) = \eta(t - \tau), t \in (\tau - h, \tau)
\]
where \( R(v) = B(v, \psi) + B(\psi, v) \), \( a_n(t) \) is to be determined.

Step 2. By the generalized Lions-Aubin Lemma, the sequence \( \{v_n(t)\} \) is uniformly bounded in \( L^\infty(0,T; H) \cap L^2(0,T; V) \).

Step 3. We shall prove the uniqueness and continuous dependence of global solution (see [3]).

4 Existence of Absorbing set

In this section, we shall prove the existence of absorbing set for the 2D Navier-Stokes equation with weak damping and distributed delay. From Theorem 3.1, we obtained the solution semigroup

\[
S(t)(v_0, \eta) = v_t(\cdot; (v_0, \eta)),
\]

where \((v_0, \eta) \in H \times L^2_H\), the corresponding norm can be described as

\[
\|(v_0, \eta)\|^2_{H \times L^2_H} = |v_0|^2 + \int_{-h}^0 |\eta(s)|^2 ds, (v_0, \eta).
\]

**Theorem 4.1** Assume \( \bar{f} \in (L^2(\Omega))^2 \) (or translation bounded functions class) and \((v_0, \eta) \in H \times L^2_H\), the conditions \((a) \sim (d)\) hold, then the semigroup \( S(t) \) possesses an absorbing set in \( C_H \) for the problem (9)-(13).

**Proof.** see, e.g. [3].

**Theorem 4.2** Assume \( \bar{f} \in (L^2(\Omega))^2 \) (or translation bounded functions class), and \((v_0, \eta) \in H \times L^2_H\), the conditions \((a) \sim (d)\) hold, then the semigroup \( S(t) \) possesses an absorbing set in \( C_V \) for the problems (9)-(13).

**Proof.** see, e.g. [3].

5 Existence of Global Attractors

The main results in our paper can be stated as

**Theorem 5.1** Assume \( \bar{f} \in (L^2(\Omega))^2 \), and \((v_0, \eta) \in H \times L^2_H\), the conditions \((a) \sim (d)\) hold, then the system (9)-(13) which is equivalent to (1) possesses a global attractor.

Moreover, if \( f(x) = f(x,t) \) which satisfies \((a) f(x,t) \in L^2_{trc}(R, H), (b) f(x,t) \in L^2_n(R, H), (c) f(x,t) \in L^2_n(R, H)\), we can derive the existence of uniform attractors for the 2D Navier-Stokes equations with delay also.
Proof. Step 1: The theorem 3.1 guarantee that the solution semigroup generated by the system (9)–(13) is continuous. Theorem 4.1 and Theorem 4.2 prove that the semigroup $S(t)$ possesses two bounded balls $B(0, \rho_H)$ and $B_V(0, \rho_V)$ in $C_H$ and $C_V$ respectively. If we can prove $B(0, \rho_H)$ is compact in $C_H$, then $S(t)$ possesses a global attractors in $C_H$, this is equivalent to prove the next two properties by the generalized Arzelà-Ascoli theorem:

1. $\bigcup_{\phi \in B_V} S(t)(\phi)(\theta)$ is relative compact in $H$ for all $\theta \in [-h, 0]$. Since $V \subset C_H$ is compact which implies this conclusion.
2. $S(t)V_0(0, \rho_V)$ is equicontinuous.

From the fundamental theory of existence of global attractor (See [1]), $\{S(t)\}$ generated by the system (9)–(13) which is equivalent to (1) possesses a global attractors in $C_H$.

Step 2: Similar to [3], we can derive the existence of uniqueness weak solution which can generate a continuous process $U(t, \tau)$, then choosing appropriate symbol space, using the property of translation bounded class functions class or the normal functions class, we can prove the existence of uniform attractors, more details, please refer to [2].

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References


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