Spectral Inclusions for $C_0$-Semigroups

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Abstract

In this work we show that the spectral inclusion of semigroups hold for descend, ascent, essential descend, essential ascent, Drazin, Kato, and essential Kato.

Keywords: Banach algebra, Operator, $C_0$-semigroup, descend, ascent, Drazin spectrum, essential Kato spectrum

1 Introduction and preliminaries

Throughout this work, $X$ denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on $X$. Let $T$ a closed operator with domain $D(A)$, we denote by $T^\ast$, $R(T)$, $N(T)$, $R^\infty(T) = \bigcap_{n \geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_k(T)$, $\sigma_{es}(T)$ respectively the adjoint, the range, the kernel, the hyper-range, the resolvent set, the spectrum, the point spectrum, the residual spectrum, the Kato spectrum and the essential kato spectrum of $T$.

The ascent of $T$ is defined by $a(T) = \min\{p : N(T^p) = N(T^{p+1})\}$; if such $p$ does not exist, we let $a(T) = \infty$. Similarly, the descent of $T$ is $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$, if such $q$ does not exist we let $d(T) = \infty$ [6] and [7]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and therefore we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

The descend and ascent spectrum defined by:

$$\sigma_{\text{desc}}(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = \infty\}$$
\[
\sigma_{asc}(T) = \{ \lambda \in \mathbb{C} : a(\lambda - T) = \infty \}
\]
The essential ascent and descend of \(T\) are defined respectively by:

\[
d_e(T) = \min \{ n \in \mathbb{N} : \dim R(T^n)/R(T^{n+1}) < \infty \}
\]
\[
a_e(T) = \min \{ n \in \mathbb{N} : \dim N(T^{n+1})/N(T^n) < \infty \}
\]
The essential ascent and descend spectrum are defined respectively by:

\[
\sigma_{asc}^e(T) = \{ \lambda \in \mathbb{C} : a_e(\lambda - T) = \infty \}
\]
\[
\sigma_{des}^e(T) = \{ \lambda \in \mathbb{C} : d_e(\lambda - T) = \infty \}
\]

Recall that \(T\) is a Drazin invertible if \(d(T) < \infty\) and \(a(T) < \infty\). The Drazin spectrum is \(\sigma_D(T) = \{ \lambda \in \mathbb{C} : d(\lambda - T) = \infty\ \text{and} \ a(\lambda - T) = \infty \}\).

Recall that \(T\) is said to be Kato operator or semi-regular if \(R(T)\) is closed and \(N(T) \subseteq R^\infty(T)\). The set \(\rho_\gamma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is Kato} \}\) denotes the Kato resolvent and \(\sigma_\gamma(T) = \mathbb{C} \setminus \rho_\gamma(T)\) the Kato spectrum of \(T\). It is well known that \(\rho_\gamma(T)\) is an open subset of \(\mathbb{C}\).

\(T\) is essential Kato if \(R(T)\) is closed and there exists a subspace \(L\) in \(X\) with \(\dim L < \infty\) such that \(N(T) \subseteq R^\infty(T) + L\). The essential Kato spectrum of \(T\) is defined by \(\sigma_{\text{ess}}^e(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not a essential Kato operator} \}\).

A one-parameter family \(\{T(t)\}_{t \geq 0}\) of operators on \(X\) is called a \(C_0\)-semigroup of operators if:

i) \(T(0) = I\).
ii) \(T(t + s) = T(t)T(s), \forall t, s \geq 0\).
iii) \(\lim_{t \to 0} T(t)x = x, \forall x \in X\).

Also \(\{T(t)\}_{t \geq 0}\) has a unique infinitesimal generator \(A\) defined in domain \(D(A)\) by:

\[
D(A) = \{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \}
\]

\[
Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \forall x \in D(A)
\]

Let \(\{T(t)\}_{t \geq 0}\) a \(C_0\)-semigroup and \(A\) its infinitesimal generator.

1) \(A\) is a closed operator.
2) If \(x \in D(A)\), then \(T(t)x \in D(A)\) and we have:

\[
T(t)Ax = AT(t)x, \forall t \geq 0
\]

3) The application: \(t \in [0, +\infty[ \rightarrow T(t)x \in X\) is differentiable on \([0, +\infty[\) for all \(x \in D(A)\) and we have:
\[
\frac{d}{dt} T(t)x = T(t)Ax = AT(t)x, \forall t \geq 0
\]

4) For all \( t \geq 0 \) we have:

\[
\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} T(s)x ds = T(t)x
\]

5) If \( x \in X \), then \( \int_{0}^{t} T(s)x ds \in D(A) \) and we have :

\[
A \int_{0}^{t} T(s)x ds = T(t)x - x
\]

6) There exist a \( M \geq 1 \) and \( \omega \geq 0 \) such that \( \forall t \geq 0, \|T(t)\| \leq Me^{\omega t} \).

For later use, we introduce the following operator acting on \( X \) and depending on the parameters \( \lambda \in \mathbb{C} \) and \( t \geq 0 \):

\[
B_{\lambda}(t)x = \int_{0}^{t} e^{\lambda(t-s)}T(s)x ds, x \in X
\]

It is well known (see [4] and [8]) that \( B_{\lambda}(t) \) is a bounded linear operator on \( X \).

Let \( \{T(t)\}_{t \geq 0} \) be a \( C_{0} \)-semigroup on \( X \) with infinitesimal generator \( A \):

In [1], R. Derndinger and R. Nagel showed that \( e^{t\sigma(A)} \subseteq \sigma(T(t)) \), \( e^{t\sigma_{p}(A)} \subseteq \sigma_{p}(T(t)) \) and \( e^{t\sigma_{r}(A)} \subseteq \sigma_{r}(T(t)) \). Likewise A. El Koutri and A. Taoudi in [2] showed that \( e^{t\sigma_{r}(A)} \subseteq \sigma_{r}(T(t)) \). Later on, a similar result was obtained for the essential Kato spectrum by A.El Koutri and A.Taoudi [3].

These works push to ask the following question: Does this spectral inclusion hold for the other parts of spectrum?

In this work, we show that this spectral inclusion of \( C_{0} \)-semigroups hold for ascent, essential ascent, descend, essential descend and Drazin spectrum, and we will give another proof of this inclusion for Kato and essential Kato spectrum.

2 Main results

Lemma 2.1. [3] Let \( A \) be the generator of a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \). Then, for all \( \lambda \in \mathbb{C}, t \geq 0 \), and \( n \in \mathbb{N} \),

1. \[
(e^{\lambda t} - T(t))^{n}x = (\lambda - A)^{n}B_{\lambda}(t)x, \forall x \in X
\]
2. \[
R^{\infty}(e^{\lambda t} - T(t))^{n}x \subseteq R^{\infty}(\lambda - A)^{n}x, \forall x \in D(A^{n})
\]
3. $N((\lambda - A)^n) \subseteq N(e^{\lambda t} - T(t))^n$.

**Lemma 2.2.** Let $\{T(t)\}_{t \geq 0}$ a $C_0$-semigroup on $X$ with infinitesimal generator $A$. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)xdx$, then:

1. There exist a $M \geq 1$ and $\omega > Re(\lambda)$ such that $F_\lambda(t) \leq \frac{M}{(\omega - Re(\lambda))^2}e^{(\omega - Re(\lambda))t}$.

2. $\forall x \in X$, $F_\lambda(t)x \in D(A)$ and $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$ with $G_\lambda(t) = e^{-\lambda t}I$.

3. The operators $F_\lambda(t)$, $G_\lambda(t)$ and $B_\lambda(t)$ are pairwise commute and for all $x \in D(A)$:

   \[
   (\lambda - A)F_\lambda(t)x = F_\lambda(t)(\lambda - A)x \\
   (\lambda - A)G_\lambda(t)x = G_\lambda(t)(\lambda - A)x \\
   (\lambda - A)B_\lambda(t)x = B_\lambda(t)(\lambda - A)x
   \]

**Proof.**

1. There exist a $M \geq 1$ and $\omega > Re(\lambda)$ such that $\forall t \geq 0$, $\|T(t)\| \leq Me^{\omega t}$. For $x \in X$ we have:

   \[
   \|B_\lambda(t)x\| \leq \int_0^t e^{(t-s)Re(\lambda)}\|T(s)\||x||ds \\
   \leq \frac{M}{\omega - Re(\lambda)}e^{\omega t}\|x\|
   \]

   \[
   \|F_\lambda(t)x\| \leq \int_0^t e^{-Re(\lambda)s}\|B_\lambda(s)\||x||ds \\
   \leq \frac{M}{\omega - Re(\lambda)}\int_0^t e^{(\omega - Re(\lambda))s}ds\|x\| \\
   \leq \frac{M}{(\omega - Re(\lambda))^2}e^{(\omega - Re(\lambda))t}\|x\|
   \]

2. Let $F(t)\lambda x = \int_0^t e^{-\lambda s}B_\lambda(s)xdx$. For $x \in X$, $F(t)x \in D(A)$.

   Indeed for all $h \in [0, 1]$:

   \[
   \frac{T(h) - I}{h}F(t)\lambda x = \frac{T(h) - I}{h} \int_0^t e^{-\lambda s}B_\lambda(s)xdx \\
   = \frac{1}{h} \int_0^t \int_0^s e^{-\lambda u}T(u + h)xdu - \frac{1}{h} \int_0^t \int_0^s e^{-\lambda u}T(u)xdu \\
   = \frac{1}{h} \int_0^t (\int_0^s e^{-\lambda u}T(u + h)xdu - \int_0^s e^{-\lambda u}T(u)xdu)ds
   \]
Let:
\[
\varphi(h, s)x = \frac{1}{h} \left( \int_0^h e^{-\lambda u}T(u + h)xdu - \int_0^h e^{-\lambda u}T(u)xdu \right) = \frac{e^{\lambda h}}{h} \int_0^{h+s} e^{-\lambda u}T(u)xdu - \frac{1}{h} \int_0^s e^{-\lambda u}T(u)xdu
\]
\[
= \frac{e^{\lambda h} - 1}{h} \int_0^s e^{-\lambda u}T(u)xdu + \frac{e^{\lambda h}}{h} \int_s^{h+s} e^{-\lambda u}T(u)xdu - \frac{1}{h} \int_0^h e^{-\lambda u}T(u)xdu
\]

Therefore, \( \lim_{h \to 0} \varphi(h, s)x = \lambda e^{-\lambda s}B_\lambda(s)x + e^{-\lambda s}T(s)x - x. \)

Moreover, the function \( h \mapsto \varphi(h, s)x, \) is bounded on \([0, 1]\). Thus:
\[
\lim_{h \to 0} \frac{T(h) - I}{h}F_\lambda(t)x = \lambda \int_0^t e^{-\lambda s}B_\lambda(s)xds + e^{-\lambda t} \int_0^t e^{-\lambda s}T(s)xds - tx
\]

Then for all \( x \in X, F_\lambda(t)x \in D(A) \) and we have
\[
AF_\lambda(t)x = \lambda F_\lambda(t)x + e^{-\lambda t}B_\lambda(t)x - tx
\]

Then \( (\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI \) with \( G_\lambda(t) = e^{-\lambda t}I. \)

3. For all \( t \geq 0, F_\lambda(t) \) and \( B_\lambda(t) \) commuting.

   Indeed, for \( t, s \geq 0 \) we have:
\[
B_\lambda(t)B_\lambda(s)x = \int_0^t e^{\lambda(t-u)}T(u)B_\lambda(s)xdu
= \int_0^t e^{\lambda(t-u)}T(u) \int_0^s e^{\lambda(s-v)}T(v)xvdvdu
= \int_0^t \int_0^s e^{\lambda(t-u)}e^{\lambda(s-v)}T(u)T(v)xvdvdu
= \int_0^s e^{\lambda(s-v)}T(v) \int_0^t e^{\lambda(t-u)}T(u)xvdvdu
= B_\lambda(s)B_\lambda(t)x
\]

Therefore:
\[
F_\lambda(t)B_\lambda(t)x = \int_0^t e^{-\lambda u}B_\lambda(u)B_\lambda(t)xdu
= \int_0^t e^{-\lambda u}B_\lambda(t)B_\lambda(u)xdu
= B_\lambda(t) \int_0^t e^{-\lambda u}B_\lambda(u)xdu
= B_\lambda(t)F_\lambda(t)x
\]
For all \( x \in D(A) \) we have:

\[
F_\lambda(t)(\lambda - A)x = \int_0^t e^{-\lambda s}B_\lambda(s)(\lambda - A)x ds \\
= \int_0^t e^{-\lambda s}(e^{\lambda s} - T(s))x ds \\
= tx - \int_0^t e^{-\lambda s}T(s)x ds \\
= tx - G_\lambda(t)B_\lambda(t)x \\
= (\lambda - A)F_\lambda(t)x
\]

For all \( x \in D(A) \) \((\lambda - A)G_\lambda(t)x = G_\lambda(t)(\lambda - A)x\) trivial
For all \( x \in D(A) \) \((\lambda - A)B_\lambda(t)x = B_\lambda(t)(\lambda - A)x\) see lemma 2.1

**Proof.**

1. If \( d(e^\lambda t - T(t)) < \infty \), then there exists \( n \in \mathbb{N} \) such that:

\[
R(e^\lambda t - T(t))^n = R(e^\lambda t - T(t))^{n+1}
\]

There exist two operators \( H_n(t) \) and \( L_n(t) \) such that:

\[
(\lambda - A)^n H_n(t) + L_n(t)B_\lambda^n(t) = I
\] (1)

\( H_n(t), L_n(t) \) and \( B_\lambda(t) \) commuting and for all \( x \in D(A) \) we have: \((\lambda - A)H_n(t)x = H_n(t)(\lambda - A)x \) and \((\lambda - A)L_n(t)x = L_n(t)(\lambda - A)x\).

Indeed, according to lemma 2.2 there exists tow bounded operators \( H_1(t) \) 

**Theorem 1.** Let \( \{T(t)\}_{t \geq 0} \) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). Then for all \( t > 0 \) we have:

1. \( d(e^\lambda t - T(t)) < \infty \Rightarrow d(\lambda - A) < \infty \).
2. \( a(e^\lambda t - T(t)) < \infty \Rightarrow a(\lambda - A) < \infty \).
3. \( d_e(e^\lambda t - T(t)) < \infty \Rightarrow d_e(\lambda - A) < \infty \).
4. \( a_e(e^\lambda t - T(t)) < \infty \Rightarrow a_e(\lambda - A) < \infty \).
5. \( e^\lambda t - T(t) \) is a Drazin invertible \( \Rightarrow \lambda - A \) is a Drazin invertible.
6. \( e^\lambda t - T(t) \) is a Kato operator \( \Rightarrow \lambda - A \) is a Kato operator.
7. \( e^\lambda t - T(t) \) is a essential Kato operator \( \Rightarrow \lambda - A \) is a essential Kato operator.

**Proof.**

1. If \( d(e^\lambda t - T(t)) < \infty \), then there exists \( n \in \mathbb{N} \) such that:

\[
R(e^\lambda t - T(t))^n = R(e^\lambda t - T(t))^{n+1}
\]

There exist two operators \( H_n(t) \) and \( L_n(t) \) such that:

\[
(\lambda - A)^n H_n(t) + L_n(t)B_\lambda^n(t) = I
\] (1)

\( H_n(t), L_n(t) \) and \( B_\lambda(t) \) commuting and for all \( x \in D(A) \) we have: \((\lambda - A)H_n(t)x = H_n(t)(\lambda - A)x \) and \((\lambda - A)L_n(t)x = L_n(t)(\lambda - A)x\).

Indeed, according to lemma 2.2 there exists tow bounded operators \( H_1(t) \)
and \( L_1(t) \) such that \((\lambda - A)H_1(t) + L_1(t)B_\lambda(t) = I \).

Let \( i \in \{1, \ldots, n-1\} \) and \( x \in X \) we have:

\[
(\lambda - A)^i H_1^n(t)x = (\lambda - A)H_1(t)(\lambda - A)^{-i}H_1^n(t)x = H_1(t)(\lambda - A)(\lambda - A)^{-i}H_1^n(t)x \in D(A).
\]

hence \( \forall n \in \mathbb{N}^*, H_1^n(t)x \in D(A^n) \).

Moreover

\[
[(\lambda - A)H_1(t)]^n = [I - L_1(t)B_\lambda(t)]^n \\
(\lambda - A)^n H_1^n(t) = I - L_{1,n}(t)B_\lambda(t)
\]

with \( L_{1,n}(t) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} L_1^k(t)B_{\lambda}^{k-1}(t) \).

hence \( (\lambda - A)^n H_1^n(t) + L_{1,n}(t)B_\lambda(t) = I \)

Similarly

\[
L_{1,n}(t)B_\lambda^n(t) = [I - (\lambda - A)^n H_1^n(t)]^n = I - (\lambda - A)^n \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (\lambda - A)^{n(k-1)}H_1^{nk}(t)
\]

Let \( H_n(t) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (\lambda - A)^{n(k-1)}H_1^{nk}(t) \) and \( L_n(t) = L_{1,n}(t) \),

then \((\lambda - A)^n H_n(t) + L_n(t)B_\lambda^n(t) = I\), moreover \( H_n(t), L_n(t) \) and \( B_\lambda(t) \)

commuting and for all \( x \in D(A) \) we have:

\((\lambda - A)H_n(t)x = H_n(t)(\lambda - A)x \) and \((\lambda - A)L_n(t)x = L_n(t)(\lambda - A)x \).

Let \( y \in R(\lambda - A)^n \) and \( x \in D(A^n) \) such that \( y = (\lambda - A)^nx \). According to (1) we have:

\[
(\lambda - A)^nx = (\lambda - A)^nH_n(t)(\lambda - A)^nx + L_n(t)B_\lambda^n(t)(\lambda - A)^nx = (\lambda - A)^{n+1}H_n(t)(\lambda - A)^{n-1}x + L_n(t)(e^M - T(t))^{n+1}x
\]

Let \( z \in X \) such that \((e^M - T(t))^{n+1}x = (e^M - T(t))^{n+1}z \), then:

\[
(\lambda - A)^nx = (\lambda - A)^{n+1}[(\lambda - A)^{n-1}H_n(t)x + L_n(t)B_\lambda^{n+1}(t)z]
\]

Therefore \( R(\lambda - A)^n = R(\lambda - A)^{n+1} \), hence \( d(\lambda - A) < \infty \).

2. If \( a(e^M - T(t)) < \infty \), there exist \( n \in \mathbb{N} \) such that:

\[
N(e^M - T(t))^n = N(e^M - T(t))^{n+1}
\]
Let $x \in N(\lambda - A)^{n+1}$ then $(\lambda - A)^{n+1}x = 0$

$$(\lambda - A)^{n+1}x = 0 \implies (e^{\lambda t} - T(t))^{n+1}x = 0 \implies (e^{\lambda t} - T(t))^{n}x = 0$$

$$(\lambda - A)^{n}x = (\lambda - A)^{n}H_{n}(t)(\lambda - A)^{n}x + L_{n}(t)(e^{\lambda t} - T(t))^{n}x = 0$$

Therefore $N(\lambda - A)^{n} = N(\lambda - A)^{n+1}$, hence $a(\lambda - A) < \infty$.

3. If $d_{e}(e^{\lambda t} - T(t)) < \infty$, there exists $n \in \mathbb{N}$ such that:

$$\dim R(e^{\lambda t} - T(t))^{n}/R(e^{\lambda t} - T(t))^{n+1} < \infty$$

$$\dim R(\lambda - A)^{n}/R(\lambda - A)^{n+1} < \infty.$$ 

Indeed consider the application:

$$\psi : R(\lambda - A)^{n} \to R(e^{\lambda t} - T(t))^{n}/R(e^{\lambda t} - T(t))^{n+1}$$

$$(\lambda - A)^{n}x \mapsto (e^{\lambda t} - T(t))^{n}x + R(e^{\lambda t} - T(t))^{n}$$

$\psi$ is well defined, linear, surjective. According to (1) we have:

$$N(\psi) \subseteq R(\lambda - A)^{n+1} \subseteq R(\lambda - A)^{n}$$

According to the theorem of isomorphism $R(\lambda - A)^{n}/N(\psi)$ and $R(e^{\lambda t} - T(t))^{n}/R(e^{\lambda t} - T(t))^{n+1}$ are isomorphic, then $\dim R(\lambda - A)^{n}/N(\psi) < \infty$, $R(\lambda - A)^{n+1}/N(\psi) \subseteq R(\lambda - A)^{n}/N(\psi)$, hence $\dim R(\lambda - A)^{n+1}/N(\psi) < \infty$ and $R(\lambda - A)^{n}/R(\lambda - A)^{n+1}$ and $(R(\lambda - A)^{n}/N(\psi))/(R(\lambda - A)^{n+1}/N(\psi))$ are isomorphic, therefore $\dim R(\lambda - A)^{n}/R(\lambda - A)^{n+1} < \infty$ hence $d_{e}(\lambda - A) < \infty$.

4. If $a_{e}(e^{\lambda t} - T(t)) < \infty$, there exist $n \in \mathbb{N}$ such that:

$$\dim N(e^{\lambda t} - T(t))^{n+1}/N(e^{\lambda t} - T(t))^{n} < \infty$$

We have $\dim N(\lambda - A)^{n+1}/N(\lambda - A)^{n} < \infty$.

Indeed consider the application:

$$\varphi : N(\lambda - A)^{n+1} \to N(e^{\lambda t} - T(t))^{n+1}/N(e^{\lambda t} - T(t))^{n}$$

$$x \mapsto x + N(e^{\lambda t} - T(t))^{n}$$

$\varphi$ is well defined, linear, and $N(\varphi) \subseteq N(\lambda - A)^{n} \subseteq N(\lambda - A)^{n+1}$. According to the theorem of isomorphism $N(\lambda - A)^{n+1}/N(\varphi)$ and $Im(\varphi) \subseteq$
N(e^{\lambda t} - T(t))^{n+1}/N(e^{\lambda t} - T(t))^{n+1} are isomorphic, therefore \dim N(\lambda - A)^{n+1}/N(\phi) < \infty, we have N(\lambda - A)^n/N(\phi) \subseteq N(\lambda - A)^{n+1}/N(\psi), then \dim N(\lambda - A)^n/N(\phi) < \infty and N(\lambda - A)^{n+1}/N(\lambda - A)^n and (N(\lambda - A)^{n+1}/N(\phi))/(N(\lambda - A)^n/N(\phi)) are isomorphic, therefore \dim N(\lambda - A)^{n+1}/N(\lambda - A)^n < \infty therefore a_{e}(\lambda - A) < \infty.

5. If e^{\lambda t} - T(t) is invertible Drazin, then d(e^{\lambda t} - T(t)) < \infty and a(e^{\lambda t} - T(t)) < \infty then d(\lambda - A)) < \infty and a(\lambda - A) < \infty, hence \lambda - A is invertible Drazin.

6. Suppose that e^{\lambda t} - T(t) is a Kato operator, then for all n \in \mathbb{N} N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^n and R(e^{\lambda t} - T(t)) is closed. According to lemma 2.1 we have: N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^n \subseteq R(\lambda - A)^n. R(\lambda - A) is closed.

Indeed, let y_n = (\lambda - A)x_n be a convergent sequence with limit y \in X, according to (1) we have: x_n = (\lambda - A)H_1(t)x_n + L_1(t)B_\lambda(t)x_n and y_n = (\lambda - A)H_1(t)y_n + (e^{\lambda t} - T(t))L_1(t)x_n. Then:
\(\text{(e}^{\lambda t} - T(t))L_1(t)x_n = y_n - (\lambda - A)H_1(t)y_n\) tends to \(y = -(\lambda - A)H_1(t)y \in R(e^{\lambda t} - T(t))\) since \(\lambda - A)H_1(t)\) is a linear bounded operator and \(R(e^{\lambda t} - T(t))\) is closed. Then there exists \(z \in X\) such that \(y - (\lambda - A)H_1(t)y = (e^{\lambda t} - T(t))z\) then \(y = (\lambda - A)[H_1(t)y + (B_\lambda(t)z)\), hence \(y \in R(\lambda - A)\).

7. Suppose that e^{\lambda t} - T(t) is a essential Kato operator, then there exists a subspace \(L\) in \(X\) such that \(\dim L < \infty, N(e^{\lambda t} - T(t)) \subseteq R^\infty(e^{\lambda t} - T(t) + L\) and \(R(e^{\lambda t} - T(t))\) is closed. According to lemma 2.1 we have: \(N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R^\infty(e^{\lambda t} - T(t) + L \subseteq R^\infty(\lambda - A) + L\). We have \(R(\lambda - A)\) is closed, hence \(\lambda - A\) is a essential Kato operator.

\[\square\]

**Corollary 1.** Let \(T(t)\) be a \(C_0\)-semigroup on \(X\) with infinitesimal generator \(A\). Then for all \(t > 0:\)
\[
\begin{align*}
e^{t\sigma_{\text{desc}}(A)} & \subseteq \sigma_{\text{desc}}(T(t)), \quad e^{t\sigma_{\text{asc}}(A)} \subseteq \sigma_{\text{asc}}(T(t)), \quad e^{t\sigma_{\text{asc}}^e(A)} \subseteq \sigma_{\text{desc}}^e(T(t)) \\
e^{t\sigma_{\text{asc}}^e(A)} & \subseteq \sigma_{\text{asc}}^e(T(t)), \quad e^{t\sigma_{D}(A)} \subseteq \sigma_D(T(t)), \quad e^{t\sigma_{\gamma}(A)} \subseteq \sigma_{\gamma}(T(t)) \\
e^{t\sigma_{\gamma}^e(A)} & \subseteq \sigma_{\gamma}^e(T(t))
\end{align*}
\]

**Proof.** Immediately comes from Theorem 1 \[\square\]

**References**


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