On a Motion of a Line Along Two
Closed Ruled Surfaces

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Abstract

In this study, a closed motion given in $\mathbb{R}^3$ are considered in the dual space $\mathbb{D}^3$. Thus this motion is generalized to the $\mathbb{D}^3$. During this motion, some relations among the dual integral invariants of the closed ruled surfaces are given. In addition, separating real and dual parts of these dual integral invariants, some results and some theorems related to Holditch Theorem are given in the line space $\mathbb{R}^3$ by Study's mapping.

Mathematics Subject Classification: 53A17, 53B30

Keywords: Integral invariants, ruled surface, dual spherical motion

1 INTRODUCTION

Ruled surfaces generally are the trajectories of the oriented lines embedding in a moving rigid body in a spatial motion. So, it is important in spatial geometry mechanism in $\mathbb{R}^3$ and in the study of rational design problem.

A $(x)$- closed trajectory ruled surface generated by an $\overrightarrow{x}$- oriented line fixed in the moving system is characterized by the pitch $l_x$ and the angle of pitch $\lambda_x$, which are real integral invariants. Recently the differential geometry of

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closed ruled surfaces according to invariants has been studied in many papers [1,2,3,4,5,7,8,10,11].

In this paper, a dual closed curve $c(t)$ which is a spherical indicatrix of a closed ruled surface $(\vec{V}_1^*(t))$ with a real parameter $t$, two frames $D^*(\vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*)$ related to the closed ruled surface $(\vec{V}_1^*(t))$ and $D(\vec{V}_1, \vec{V}_2, \vec{V}_3)$ which moves respect to $D^*$ and related to a closed ruled surface drawn by a vector $\vec{V}_1(t)$, and a line $\vec{V}$ which is fixed in the frame $D$ are considered; and dual integral invariants of the closed ruled surfaces which correspond to the dual closed curves drawn by the vectors $\vec{V}, \vec{V}_1$ and $\vec{V}_1^*$ are studied; and it is found some relations among the dual integral invariants of the closed ruled surfaces which correspond to the dual closed curves drawn by the vectors $\vec{V}, \vec{V}_1$ and $\vec{V}_1^*$. In addition, these results are carried to the line space $\mathbb{R}^3$ and are given some theorems by means of Study’s mapping.

Let the dual orthonormal system $\{0, \vec{V}_1, \vec{V}_2, \vec{V}_3\}$ be generating system of a moving unit dual sphere $K$ and $\{0, \vec{V}_1^*, \vec{V}_2^*, \vec{V}_3^*\}$ be generating dual orthonormal system of fixed unit dual sphere.

Including $K$ is a moving sphere and $K_1$ is a fixed sphere, the motion of $K$ in according to $K_1$ is denoted by $K/K_1$.[2]

Let $A = (A_{ij})$ be a dual orthogonal matrices, where $A^T A = I$ ($\det A = \pm 1$). Euclidean motions in $\mathbb{R}^3$ is given by such a dual orthogonal matrices in $\mathbb{D}^3$. According to this, the dual matrix $A(t)$ of the motion $K/K_1$ represents a spatial motion in $\mathbb{R}^3$. If the unit dual spheres $K$ and $K_1$ respectively correspond to the line spaces $H$ and $H_1$, then $K/K_1$ corresponds to the spatial motion which will be denoted by $H/H_1$. Then $H$ is the moving space with respect to the fixed space $H_1$.

Then, the derivative equations of the dual spherical closed motion of the $K$-sphere with respect to the $K_1$-sphere are

$$d\vec{v}_i = \sum_{j=1}^{3} \Omega^j_i \vec{v}_j, \quad \Omega^j_i (t) = w^j_i (t) + \varepsilon w^*_{ij} (t), \quad \Omega^j_i = -\Omega^j_i, \quad t \in I, \quad 1 \leq i \leq 3$$

(1)

The definitions of dual number, dual vector and the dual space $\mathbb{D}^3$ are given in [9]. And the theorem of E.Study is given in [2].

**Definition 1.1** A dual vector

$$\vec{\psi} = w_3 \vec{V}_1 + w_1 \vec{V}_2 + w_1 \vec{V}_3$$

(2)

is called the instantaneous Pfaffian vector of the dual closed spherical $K/K_1$, where $w_{ij} = -w_{ji}$, $1 \leq i,j \leq 3$ are the dual 1-forms of the $K/K_1$. 

Definition 1.2 The dual vector [7]
\[ \vec{D} = \vec{d} + \varepsilon \vec{d}^* = \oint \vec{\psi} = \vec{V}_1 \oint w^3 + \vec{V}_2 \oint w^1 + \vec{V}_3 \oint w^2 \] (3)

will be called the dual Stenier vector of a dual orbit \((X)\).

Since the orbits of the points of \(K\) and \(K_1\), as the fixed and moving polodes (centrodes), are closed dual curves, then the corresponding ruled surfaces are closed ruled surfaces during the closed dual motion \(K/K_1\) [2].

A dual spherical curve \(c : x(t) , t \in \mathbb{R}\) is called dual spherical indicatrix of a closed ruled surface in the line space \(H_1\).

The definitions of the pitch and the angle of pitch are given in [5]. These are integral invariants of a closed ruled surface [2,5] and denoted by \(\lambda\) and \(l\), respectively.

Definition 1.3 Let \(\vec{X}(t) = \vec{x}(t) + \varepsilon \vec{x}^*(t)\), \(\|\vec{X}\| = 1\) represents an orientable closed ruled surface. The dual angle of pitch of the closed ruled surface \((X)\) is given as following
\[ \wedge_x = -\left< \vec{D}, \vec{X} \right> = -\left< \vec{d} + \varepsilon \vec{d}^*, \vec{x} + \varepsilon \vec{x}^* \right> \\
\wedge_x = -\left< \vec{d}, \vec{x} \right> = -\varepsilon \left( \left< \vec{d}^*, \vec{x} \right> + \left< \vec{d}, \vec{x}^* \right> \right) \] (4)

[1].

2 Dual Angles of Pitch of Closed Ruled Surfaces Drawn on the Fixed Dual Sphere \(K_1\)

Let us choose a fixed dual point \(\vec{V}_1^*\) which is on the unit dual moving sphere \(K\). During the one-parameter dual spherical closed motion \(K/K_1\), for every \(t\), there is a constant dual angle \(\phi\) between the vector \(\vec{V}_1^*\) and unit dual vector \(\vec{V}_1\) of the moving sphere \(K\), i.e., let us choose
\[ \left< \vec{V}_1(t), \vec{V}_1^*(t) \right> = \cos \phi \quad (first \ \phi \neq k\pi , \ k \in \mathbb{Z}) \] (5)

During the dual closed spherical motion \(K/K_1\), \(\vec{V}_1(t)\) and \(\vec{V}_1^*(t)\) draw two closed spherical curves on the unit dual fixed sphere \(K_1\). These curves correspond to two closed ruled surfaces in the fixed line space \(H_1\) by Study’s mapping. Let us denote the closed ruled surfaces drawn by the vectors \(\vec{V}_1\) and \(\vec{V}_1^*\) by \((V)\) and \((V^*)\) , respectively.
Consider \( \{ \vec{V}_1(t), \vec{V}_2(t), \vec{V}_3(t) \} \) and \( \{ \vec{V}_1^*(t), \vec{V}_2^*(t), \vec{V}_3^*(t) \} \) two sets of orthonormal unit dual vectors that are rigidly linked to the spheres \( K \) and \( K_1 \), and denoted by

\[
D \{ \vec{V}_1(t), \vec{V}_2(t), \vec{V}_3(t) \}, \quad D^* \{ \vec{V}_1^*(t), \vec{V}_2^*(t), \vec{V}_3^*(t) \}
\]

respectively, where frames \( D \) and \( D^* \) are taken form by the vectors \( \vec{V}_1 \) and \( \vec{V}_1^* \), respectively.

These two frames are rigidly linked to \( D \) and \( D^* \) by

\[
\begin{align*}
\vec{V}_3(t) &= \frac{\vec{V}_1(t) \wedge \vec{V}_1^*(t)}{\sin \phi}, \\
\vec{V}_2(t) &= \vec{V}_3(t) \wedge \vec{V}_1(t), \\
\vec{V}_3^*(t) &= -\vec{V}_3^*(t), \\
\vec{V}_2^*(t) &= \vec{V}_3^*(t) \wedge \vec{V}_1^*(t).
\end{align*}
\]

Thus, for the vectors \( \vec{V}_i \) and \( \vec{V}_i^* \) we have

\[
\begin{align*}
\vec{V}_1^*(t) &= \cos \phi \vec{V}_1(t) + \sin \phi \vec{V}_2(t), \\
\vec{V}_2^*(t) &= \sin \phi \vec{V}_1(t) - \cos \phi \vec{V}_2(t), \\
\vec{V}_3^*(t) &= -\vec{V}_3(t)
\end{align*}
\]

or

\[
\begin{align*}
\vec{V}_1(t) &= \cos \phi \vec{V}_1^*(t) + \sin \phi \vec{V}_2^*(t), \\
\vec{V}_2(t) &= \sin \phi \vec{V}_1^*(t) - \cos \phi \vec{V}_2^*(t), \\
\vec{V}_3(t) &= -\vec{V}_3^*(t)
\end{align*}
\]

\((\phi = \varphi + \varepsilon \varphi^*)\). Let an unit dual vector \( \vec{V}(t) \) be fixed in the orthonormal frame. For \( \vec{V}(t) \),

\[
\vec{V}(t) = \cos \Delta \vec{V}_1(t) + \sin \Delta \cos \Theta \vec{V}_2(t) + \sin \Delta \sin \Theta \vec{V}_3(t)
\]

\((\Delta = \alpha + \varepsilon \alpha^*, \Phi = \theta + \varepsilon \theta^* = \text{constant}, \text{see Figure 1})\).

In addition, unit dual vector \( \vec{V}(t) \) with respect to dual orthonormal frame \( D^* \) by equations (9) and (10) also is written as follows:

\[
\vec{V}(t) = (\cos \Delta \cos \phi + \sin \Delta \sin \phi \cos \Theta) \vec{V}_1^*(t) + (\cos \Delta \sin \phi - \sin \Delta \cos \phi \cos \Theta) \vec{V}_2^*(t) - \sin \Delta \sin \Theta \vec{V}_3^*(t)
\]

By means of equation (10)

\[
\langle \vec{V}_1(t), \vec{V}(t) \rangle = \cos \Delta.
\]

\(\langle \cdot, \cdot \rangle\) denotes the scalar product of two vectors.
In the same way

\[ \langle \vec{V}_1^* (t), \vec{V} (t) \rangle = \cos W \]  

(see Figure 1). Equation (13) by means of equation (11) is written as follows:

\[ \cos W = \cos \Delta \cos \phi + \sin \Delta \sin \phi \cos \Theta. \]

For \( \Theta = 0 \), from equation (13)

\[ \cos W = \cos \Delta \cos \phi + \sin \Delta \sin \phi = \cos (\phi - \Delta), \]

or \( W = \phi - \Delta \) is obtained (See Figure 1).

In addition, constant dual angles \( \Delta_i \) and \( W_i \) which are among the unit dual vectors \( (\vec{V}_i (t), \vec{V} (t)) \) and \( (\vec{V}_i^* (t), \vec{V} (t)) \) are given as follows:

\[ \cos \Delta_i = \langle \vec{V}_i (t), \vec{V} (t) \rangle, \]

and

\[ \cos W_i = \langle \vec{V}_i^* (t), \vec{V} (t) \rangle, \quad (i = 1, 2, 3) \]

respectively.

From equations (10) and (11), the relations among constant angles \( \Delta_i \) and \( W_i \) as depend on the angles \( \Delta, \phi \) and \( \Theta \) are obtained.

Thus, during the motion \( K/K_1 \) we may consider the closed ruled surface drawn by the vector \( \vec{V} (t) \) which is fixed in the orthonormal frames \( D \) and \( D^* \). This closed ruled surface is expressed as \( (V(t)) \).

Dual angle of pitch of the closed ruled surface \( (V(t)) \) in according to the definition angle of pitch and by means of equation (4) is written as follows:

\[ \wedge_V = \lambda_V - \varepsilon l_V = - \oint (w_2^3) V. \]
In addition, equation (16) is written as

\[ \wedge V = - \langle \vec{D}, \vec{V} \rangle , \]  

(17)

where

\[ \vec{D} = \vec{V}_1 \oint w^3 + \vec{V}_2 \oint w^1 + \vec{V}_3 \oint w^2 \]  

(18)

is the dual Steiner vector of the motion \( K/K_1 \). Thus, according to the equations (10) and (18), the dual angle of pitch of the closed ruled surface may be expressed as follows:

\[ \wedge V = \cos \triangle \oint (-w^3) + \sin \triangle \cos \Theta \oint (-w^1) + \sin \triangle \sin \Theta \oint (-w^2) . \]  

(19)

The integral \( \oint (-w^3) \) is the dual angle of pitch of the closed ruled surface \( \vec{V}_1 \). In this case

\[ \wedge_{\vec{V}_1} = \oint (-w^3) = - \langle \vec{D}, \vec{V}_1 \rangle \]  

(20)

may be written.

In addition, because of this is also stated the unit dual vector \( \vec{V} \) according to the frame \( D^* \), in the same way

\[ \wedge V = - \langle \vec{D}^*, \vec{V}^* \rangle \]  

(21)

may be written, where \( \vec{D}^* \) is given as follows:

\[ \vec{D}^* = \vec{V}_1^* \oint (w^3)^* + \vec{V}_2^* \oint (w^1)^* + \vec{V}_3^* \oint (w^2)^* \]  

(22)

The integral \( \oint (-w^3) \) may be consider as a function of the dual angles of pitch \( \wedge_{\vec{V}_1} \) and \( \wedge_{\vec{V}_1^*} \) of the closed ruled surfaces \( \vec{V}_1(t) \) and \( \vec{V}_1^*(t) \), respectively. For this case, we consider the dual angle of pitch of the closed ruled surface \( (V_1^*) \). Thus, from the equations (8) and (17), \( \wedge_{V_1^*} \) is written as follows

\[ \wedge_{V_1^*} = \cos \phi \oint (-w_2^3) + \sin \phi \oint (-w_3^1) . \]  

(23)

Thus, equation (23) is written as follows

\[ \wedge_{V_1^*} = \wedge_{V_1} \cos \phi + \sin \phi \oint (-w_3^1) . \]  

(24)
From equation (24), for the integral \( \oint (-w_3^1) \) we have
\[
\oint (-w_3^1) = \frac{1}{\sin \phi} (\wedge V_1 - \wedge V_1 \cos \phi) .
\] (25)

For meaning of the integral \( \oint (-w_1^2) \) we use equalities in the following:
\[
[d, w^i] = \sum_{j=1}^{3} [w^j, w^i] ,
\]
\[
[d, w_i^k] = \sum_{j=1}^{3} [w_i^j, w^k] ,
\]
[6]. Thus, for the \( w_1^2 \), \( [d, w_1^2] = [w_1^3, w_3^2] \) can be used. According to Stokes theorem
\[
\oint (-w_1^2) = - \oint_{\partial G_3} [d, w_1^2] = \oint_{\partial G_3} [w_3^1, w_1^2] = a_{V_3}
\] (26)
is obtained, where \( \partial G_3 \) is the dual spherical indicatrix of the vector \( \vec{V}_3 \), and \( a_{V_3} \) is the dual spherical area bounded on the fixed sphere by the dual spherical indicatrix of the vector \( \vec{V}_3 \). Thus, from equations (19), (25) and (26)
\[
\wedge V = \wedge V_1 \cos \Delta + \frac{\sin \Delta \cos \Theta}{\sin \phi} (\wedge V_1 - \wedge V_1 \cos \phi) + \sin \Delta \sin \Theta \cos \phi a_{V_3} ,
\] (27)
or
\[
\wedge V = \frac{1}{\sin \phi} (\cos \Delta \sin \phi - \sin \Delta \cos \Theta \cos \phi) \wedge V_1 + \frac{1}{\sin \phi} \sin \Delta \cos \Theta \wedge V_1^* + \sin \Delta \sin \Theta a_{V_3}
\] (28)
is obtained.

Equation (28) may be also written by the angles \( \Delta_i \) and \( W_i \) as follows
\[
\wedge V = \wedge V_1 \frac{\cos W_2}{\sin \phi} + \wedge V_1 \frac{\cos \Delta_2}{\sin \phi} + \cos \Delta_3 a_{V_3} .
\] (29)

If dual angle of pitch \( \wedge V \) is expressed according to the frame \( D^* \) from equations (11) and (21) the same results is obtained.

Thus, from equation (29) we get the following theorem.

**Theorem 2.1** There is the relation (29) in the space \( \mathbb{D}^3 \), during the one-parameter dual closed motion \( K/K_1 \), among the dual angle of pitch of the closed ruled surface drawn by the vector \( \vec{V} \) which moves along the closed ruled surfaces \( (V_1) \) and \( (V_1^*) \), the dual angles of pitch of the closed ruled surfaces \( (V_1) \), \( (V_1^*) \) and closed spherical area bounded on the fixed sphere by the indicatrix of the vector \( \vec{V}_3 \).
Now, let us consider some special cases.

If the vectors $\mathbf{V}_1$ and $\mathbf{V}^*_1$ draw the same ruled surface, then $\wedge \mathbf{V}_i = \wedge \mathbf{V}^*_i$. In this case, we have only related to the moving along the closed ruled surface $(V_1)$. Therefore from equations (28) and (29) it is obtained in the following results:

$$\wedge \mathbf{V} = \frac{1}{sin\phi} (cos \triangle sin\phi - sin \triangle cos\Theta cos\phi + sin \triangle cos\Theta) \wedge \mathbf{V}_1 + sin \triangle sin\Theta a \mathbf{V}_3$$

(30)

or

$$\wedge \mathbf{V} = \frac{cos W_2 + cos \triangle_2}{sin\phi} \wedge \mathbf{V}_1 + cos \triangle_3 a \mathbf{V}_3.$$ (31)

Thus, we have the following theorem.

**Theorem 2.2** There is the relation (30) or (31) in the $\mathbb{D}^3$ during the motion $K/K_1$, among the dual angle of pitch of the closed ruled surface drawn by the vector $\mathbf{V}$ which is fixed in the frame $D$, the angle of pitch of the closed ruled surface drawn by the vector $\mathbf{V}_1$ and closed dual spherical area bounded on the fixed dual sphere $K_1$ by the indicatrix of the vector $\mathbf{V}_3$.

Considering all the case, we choose fixed the vector $\mathbf{V}$ according to frames $D$ and $D^*$. Now, let us take the vectors $\mathbf{V}_1$, $\mathbf{V}^*_1$ and $\mathbf{V}$ on a great circle of the moving unit dual sphere $K$. In this case,

$$\Theta = 0.$$ (32)

Thus,

$$\left\langle \mathbf{V}, \mathbf{V}_3(t) \right\rangle = 0$$ (33)

by equation (7). From equation (14)

$$cosW = cos \triangle cos\phi + sin \triangle sin\phi = cos(\phi - \triangle)$$ (34)

or

$$\phi = W + \triangle$$ (35)

is obtained. (See figure 1)

In this case, during the one-parameter dual closed motion $K/K_1$ we again study the motion of the vector $\mathbf{V}(t)$ which moves along the closed ruled surfaces $(V_1)$ and $(V^*_1)$. When $\Theta = 0$, from equations (10) and (11) for the vector $\mathbf{V}(t)$, we have

$$\mathbf{V}(t) = cos \triangle \mathbf{V}_1(t) + sin \triangle \mathbf{V}_2(t),$$ (36)

or

$$\mathbf{V}(t) = cos(\phi - \triangle) \mathbf{V}^*_1(t) + sin(\phi - \triangle) \mathbf{V}^*_2(t).$$ (37)
Thus, from equations (28), (32) and (35) the following result is obtained.

\[ \wedge_V = \frac{\sin (\phi - \Delta)}{\sin \phi} \wedge_{V_1} + \frac{\sin \Delta}{\sin \phi} \wedge_{V_1^*}. \]  

(38)

Hence the following theorem is given.

**Theorem 2.3** There is the relation (38) in the \( \mathbb{D}^3 \) during the motion \( K/K_1 \) related to the closed dual spherical indicatrix of the closed ruled surface \( (V_1^*) \), among the dual angle of pitch of the closed ruled surface drawn by the vector \( \vec{V}(t) \) which is fixed in the frame \( D \), the angles of pitch of the closed ruled surfaces drawn by the vectors \( V_1 \) and \( V_1^* \), respectively.

Separating real and dual parts of equation (38), we find

\[ \lambda_v = \frac{\sin (\varphi - \alpha)}{\sin \varphi} \lambda_{v_1} + \frac{\sin \alpha}{\sin \varphi} \lambda_{v_1^*}; \]  

(39)

and

\[ l_v = l_{v_1} \frac{\sin (\varphi - \alpha)}{\sin \varphi} + l_{v_1^*} \frac{\sin \alpha}{\sin \varphi} + \varphi^* \lambda_v \frac{\cos \varphi}{\sin \varphi} - \alpha^* \lambda_{v_1} \frac{\cos \alpha}{\sin \varphi} \]

\[ - \lambda_{v_1} (\varphi^* - \alpha^*) \frac{\cos (\varphi - \alpha)}{\sin \varphi}. \]  

(40)

Thus, we give the following theorem in the line space \( \mathbb{R}^3 \):

**Theorem 2.4** There are the relations (39) and (40) in the line space during the motion \( H/H_1 \) which corresponds to the motion \( K/K_1 \) related to the closed dual spherical indicatrix of the closed ruled surface \( (V_1^*) \), among the real angles of pitch and pitchs of the closed ruled surfaces drawn in \( H_1 \) by the line \( \vec{V} \) which is fixed in \( H \) and the lines \( \vec{V}_1 \) and \( \vec{V}_1^* \), respectively.

Now, we suppose that the closed ruled surfaces \( (V_1) \) and \( (V_1^*) \) are the same. In this case, from equation (38) we get

\[ \wedge_V = \frac{\sin (\phi - \Delta)}{\sin \phi} \wedge_{V_1}, \]  

(41)

or

\[ \frac{\wedge_V}{\wedge_{V_1}} = \frac{\sin (\phi - \Delta) + \sin \Delta}{\sin \phi}. \]  

(42)

This result is important for us, because it is corresponds to Holditch Theorem for chosen motion. Therefore the following theorem is given in dual space \( \mathbb{D}^3 \).
Theorem 2.5 The ratio of the dual angle of pitch of the closed ruled surface drawn on $K_1$ by the fixed vector $\vec{V}$ in the moving frame $D$ to the dual angle of pitch of the closed ruled surface drawn on $K_1$ by the vector $V_1$ is constant and independent of the motion during the motion $K/K_1$ related to the closed dual spherical indicatrix of the closed ruled surface $(V_1^*)$

If we separate real and dual parts of equation (41). In this case, we get

$$\lambda_v \sin \varphi = \lambda_{v_1} [\sin (\varphi - \alpha) + \sin \alpha] \quad (43)$$

or

$$\frac{\lambda_v}{\lambda_{v_1}} = \frac{\sin (\varphi - \alpha) + \sin \alpha}{\sin \varphi} \quad (44)$$

and

$$l_v = l_{v_1} \left[ \frac{\sin (\varphi - \alpha) + \sin \alpha}{\sin \varphi} \right] - \lambda_{v_1} \left[ \frac{(\varphi^* - \alpha^*) \cos (\varphi - \alpha) + \alpha^* \cos \alpha}{\sin \varphi} \right] \quad (45)$$

Then in the line space $\mathbb{R}^3$, we have the following theorems:

Theorem 2.6 When the closed ruled surfaces $(V_1)$ and $(V_1^*)$ are the same, during the motion $H/H_1$ which corresponds to the motion $K/K_1$, the ratio of the real angle of pitch of the closed ruled surface drawn on $H_1$ by the fixed vector $\vec{V}$ in the moving frame $D$ to the real angle of pitch of the closed ruled surface drawn on $H_1$ by the vector $\vec{V}_1$ is constant and independent of the motion and equal to the

$$\frac{\lambda_v}{\lambda_{v_1}} = \frac{\sin (\varphi - \alpha) + \sin \alpha}{\sin \varphi} \quad .$$

Theorem 2.7 When the closed ruled surfaces $(V_1)$ and $(V_1^*)$ are the same, during the motion $H/H_1$ which corresponds to the motion $K/K_1$, there is the relation (45), among the pitches and the real angles of pitch of the closed ruled surfaces drawn in $H_1$ by the line $\vec{V}$ which is fixed in $H$ and the line $\vec{V}_1$.

References


Received: April 29, 2015; Published: July 14, 2015