Fixed Points for Multiplicative Expansive Mappings in Multiplicative Metric Spaces

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Abstract
In this paper, we introduce the notion of multiplicative expansive mappings in multiplicative metric spaces. Next, we prove common fixed point theorems for these mappings. Finally, we also provide some examples in support of our results.

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1 Introduction and preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] introduced the concept of a multiplicative metric space as follows:
Definition 1.1. Let \( X \) be a nonempty set. A multiplicative metric is a mapping \( d : X \times X \to \mathbb{R}_+ \) satisfying the following conditions:

(i) \( d(x, y) \geq 1 \) for all \( x, y \in X \) and \( d(x, y) = 1 \) if and only if \( x = y \);

(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(iii) \( d(x, y) \leq d(x, z) \cdot d(z, y) \) for all \( x, y, z \in X \) (multiplicative triangle inequality).

Example 1.2. ([3]) Let \( \mathbb{R}_n^+ \) be the collection of all \( n \)-tuples of positive real numbers. Let \( d : \mathbb{R}_n^+ \times \mathbb{R}_n^+ \to \mathbb{R} \) be defined by

\[
d(x, y) = \left( \frac{|x_1|}{y_1} \cdot \frac{|x_2|}{y_2} \cdots \frac{|x_n|}{y_n} \right),
\]

where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}_n^+ \) and \( |\cdot| : \mathbb{R}_+ \to \mathbb{R}_+ \) is defined by

\[
|a| = \begin{cases} 
a & \text{if } a \geq 1; \\
\frac{1}{a} & \text{if } a < 1.
\end{cases}
\]

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore \( (\mathbb{R}_n^+, d) \) is a multiplicative metric space.

Example 1.3. ([6]) Let \( d : \mathbb{R} \times \mathbb{R} \to [1, \infty) \) be defined by \( d(x, y) = a^{|x-y|} \), where \( x, y \in \mathbb{R} \) and \( a > 1 \). Then \( d \) is a multiplicative metric.

Remark 1.4. We note that the Example 1.2 is valid for positive real numbers and Example 1.3 is valid for all real numbers.

One can refer to [2] and [3] for detailed the multiplicative metric topology.

Definition 1.5. Let \( (X, d) \) be a multiplicative metric space. Then a sequence \( \{x_n\} \) in \( X \) said to be

(1) a multiplicative convergent to \( x \) if for every multiplicative open ball \( B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\} \), \( \epsilon > 1 \), there exists a natural number \( N \) such that \( n \geq N \), then \( x_n \in B_\epsilon(x) \), that is, \( d(x_n, x) \to 1 \) as \( n \to \infty \).

(2) a multiplicative Cauchy sequence if for all \( \epsilon > 1 \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon \) for all \( m, n > N \), that is, \( d(x_n, x_m) \to 1 \) as \( n, m \to \infty \).

(3) We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is a multiplicative convergent to \( x \in X \).

Remark 1.6. The set of positive real numbers \( \mathbb{R}_+ \) is not complete according to the usual metric. Let \( X = \mathbb{R}_+ \) and the sequence \( \{x_n\} = \left\{ \frac{1}{n} \right\} \). It is obvious \( \{x_n\} \) is a Cauchy sequence in \( X \) with respect to usual metric and \( X \) is not a complete metric space since \( 0 \not\in \mathbb{R}_+ \). In case of a multiplicative metric space,
we take a sequence \( \{x_n\} = \{a^{\frac{k}{n}}\} \), where \( a > 1 \). Then \( \{x_n\} \) is a Cauchy sequence since for \( n \geq m \),

\[
d(x_n, x_m) = \left| \frac{x_n}{x_m} \right| = \left| \frac{a^{\frac{k}{n}}}{a^{\frac{k}{m}}} \right| = \left| a^{\frac{k}{n} - \frac{k}{m}} \right| = a^{\frac{k}{n} - \frac{k}{m}} < a^{\frac{k}{m}} < \epsilon \quad \text{if} \quad m > \frac{\log a}{\log \epsilon},
\]

where \( |a| = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases} \)

Also, \( \{x_n\} \to 1 \) as \( n \to \infty \) and \( 1 \in \mathbb{R}_+ \). Hence \((X, d)\) is a complete multiplicative metric space.

In 2012, Özavşar and Çevikel [3] gave the concept of multiplicative contraction mappings and proved the Banach contraction principle in a multiplicative metric space.

**Definition 1.7.** Let \( f \) be a mapping of a multiplicative metric space \((X, d)\) into itself. Then \( f \) is called a multiplicative contraction if there exists a constant \( \lambda \in [0, 1) \) such that \( d(fx, fy) \leq \lambda d(x, y) \) for all \( x, y \in X \).

**Theorem 1.8.** Let \( f \) be a multiplicative contraction mapping of a complete multiplicative metric space \((X, d)\) into itself. Then \( f \) has unique fixed point.

Rhoades [4, 5] and Wang et al. [7] proved some fixed point theorems for expansion mappings, which corresponds to some contractive mappings in metric spaces.

Now we define expansive mappings in the setting of multiplicative metric spaces as follows:

**Definition 1.9.** Let \( f \) be a mapping of a multiplicative metric space \((X, d)\) into itself. Then \( T \) is said to be a multiplicative expansive mapping if there exists a constant \( a \) \((a > 1)\) such that \( d(Tx, Ty) \geq \lambda d(x, y) \) for all \( x, y \in X \).

**Example 1.10.** Let \( X = \mathbb{R}_+ \) and \( T : X \to X \) defined by \( Tx = x^3 \). Then

\[
d(Tx, Ty) = d(x^3, y^3) = \left| \frac{x^3}{y^3} \right| \geq \left| \frac{x}{y} \right|^3 = d^3(x, y).
\]

Here \( a = 2 > 1 \). Therefore, \( T \) is a multiplicative expansive mapping.

## 2 Main results

Now we prove some fixed point theorems for multiplicative expansive mappings as follows:

**Theorem 2.1.** Let \( T \) be a surjective multiplicative expansive mapping of a complete multiplicative metric space \((X, d)\) into itself. Then \( T \) has a fixed point.
Proof. Since $T$ is a multiplicative expansive mapping there exists a constant $a > 1$ such that for all $x, y \in X$, we have

$$d(Tx, Ty) \geq d^a(x, y).$$

If $Tx = Ty$, then $1 = d(Tx, Ty) \geq d^a(x, y)$, which implies that $d(x, y) = 1$ and hence $x = y$. This implies $T$ is injective and invertible.

Let $h$ be an inverse mapping of $T$. Then

$$d(x, y) = d(T(hx), T(hy)) \geq d^a(hx, hy).$$

This implies for all $x, y \in X$, we have $d(hx, hy) \leq d^\frac{1}{a}(x, y)$ and hence

$$d(hx, hy) \leq d^p(x, y),$$

where $p = \frac{1}{a}$. So, $h$ is a multiplicative contraction on $X$. By Theorem 1.8, $h$ has a unique fixed point, say $z \in X$. Since $z = T(hz) = Tz$, $z$ is fixed point of $T$.

Finally, suppose that there exists another fixed point $w$ ($\neq z$) such that $Tw = w$. Then we have

$$d(z, w) = d(Tz, Tw) \geq d^a(z, w),$$

which is a contradiction since $a > 1$. Hence $z = w$, that is, $T$ has a unique fixed point. This completes the proof. \qed

Example 2.2. Proceeding Example 1.10, we see that clearly $T$ is a surjective multiplicative expansive mapping. So by Theorem 2.1, $T$ has a unique fixed point 1.

Corollary 2.3. Let $T$ be a surjective mapping of a complete multiplicative metric space $(X, d)$ into itself satisfying for some positive integer $n$,

$$d(T^n x, T^n y) \geq d^a(x, y)$$

for all $x, y \in X$, where a constant $a$ ($a > 1$). Then $T$ has a unique fixed point.

Proof. From Theorem 2.1, $T^n$ has unique fixed point say $z \in X$. Since $T^n(Tz) = T(T^n z) = Tz$, $Tz$ is fixed point of $T^n$. By uniqueness of the fixed point, we have $Tz = z$. This implies $z$ is a fixed point of $T$ and also of $T^n$. Hence fixed point of $T$ is unique. \qed

Theorem 2.4. Let $T$ be a mapping of a multiplicative metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \geq (d(x, Ty) \cdot d(y, Tx))^{\frac{1}{2}}$$

for all $x, y \in X$. Then $T$ is an identity mapping on $X$ and hence every point is fixed point of $T$. 

$$d(Tx, Ty) \geq (d(x, Ty) \cdot d(y, Tx))^{\frac{1}{2}}$$ (2.1)
Proof. Let $x$ be an arbitrary point in $X$. Then from (2.1), we have

$$1 = d(Tx, Tx) \geq (d(x, Tx) \cdot d(x, Tx))^\frac{1}{2},$$

which implies that $d(x, Tx) = 1$ and hence $Tx = x$ for all $x \in X$. Hence $T$ is an identity mapping.

Remark 2.5. We note that Theorem 2.4 required that (2.1) held when $x = y$. This essential condition, as we can see by considering the following conditions

$$X = \{1, 2, \ldots, n, \ldots\} \quad \text{and} \quad d(x, y) = \left|\frac{x}{y}\right|.$$

Suppose that $Tx = x + 2$. Then (2.1) holds for $x \neq y$. But $T$ has no fixed point.

Theorem 2.6. Let $T$ be a surjective mapping of a complete multiplicative metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \geq d^a(x, y) \cdot d^b(x, Tx) \cdot d^c(y, Ty)$$

for all $x, y \in X$, where constants $a, b, c$ ($a + b + c > 1$, $a > 1$ and $b < 1$). Then $T$ has a unique fixed point.

Proof. Let $x_0 \in X$. Since $T$ is surjective, there exists $x_1 \in X$ such that $x_1 \in T^{-1}x_0$.

Continuing this we get $x_n \in T^{-1}x_{n-1}$.

If $x_n = x_{n-1}$, then $x_n$ is a fixed point of $T$.

Assume that $x_n \neq x_{n-1}$. Then we have

$$d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1}) \geq d^a(x_n, x_{n+1}) \cdot d^b(x_n, Tx_n) \cdot d^c(x_{n+1}, Tx_{n+1}),$$

which implies that

$$d(x_{n-1}, x_n) \geq d^a(x_n, x_{n+1}) \cdot d^b(x_n, x_{n-1}) \cdot d^c(x_{n+1}, x_n).$$

Thus $d(x_n, x_{n+1}) \leq \frac{1-b}{a+c} (x_{n+1}, x_n)$ and hence

$$d(x_n, x_{n+1}) \leq d^k(x_{n+1}, x_n),$$

where $k = \frac{1-b}{a+c}$. Repeating this, we have

$$d(x_n, x_{n+1}) \leq d^{kn}(x_0, x_1).$$
Now for \( m, n \in \mathbb{N} \) with \( n < M \), we have
\[
d(x_m, x_n) \geq d(x_m, x_{m-1}) \cdot d(x_{m-1}, x_{m-2}) \cdots d(x_{n+1}, x_n)
\]
\[
\geq d^m(x_1, x_0) \cdot d^{m-1}(x_1, x_0) \cdots d^n(x_1, x_0)
\]
\[
= d^{m+k^{m-1}+\cdots+k^n}(x_1, x_0)
\]
\[
\leq d^{k^n}(x_1, x_0).
\]
Letting \( n \to \infty \), \( \{x_n\} \) is Cauchy sequence. By the completeness of \((X, d)\), there exists \( z \in X \) such that \( \{x_n\} \) converges to \( z \).

Let \( y \in T^{-1}z \). Then we have
\[
d(x_n, z) = d(Tx_{n+1}, Ty)
\]
\[
\geq d^a(x_{n+1}, y) \cdot d^b(x_{n+1}, Tx_{n+1}) \cdot d^c(y, Ty)
\]
\[
= d^a(x_{n+1}, y) \cdot d^b(x_{n+1}, x_n) \cdot d^c(y, z).
\]
Letting \( n \to \infty \), we have \( d(y, z) = 1 \) and hence \( y = z \). Since \( Ty = z \), \( z \) is a fixed point of \( T \).

Uniqueness follows easily. This completes the proof.

**Remark 2.7.** In Theorem 2.6, if \( a < 1 \), then the fixed point is not unique.

**Corollary 2.8.** Let \( T \) be a surjective mapping of a complete multiplicative metric space \((X, d)\) into itself satisfying
\[
d(Tx, Ty) \geq d^a(x, y) \cdot d^b(x, Tx) \cdot d^c(y, Ty)
\]
for all \( x, y \in X \), where constants \( a, b \) (\( a + 2b > 1 \), \( a > 1 \) and \( b < 1 \)). Then \( T \) has a fixed point.

**Theorem 2.9.** Let \( T \) be a surjective continuous mapping of a complete multiplicative metric space \((X, d)\) into itself satisfying
\[
d(Tx, T^2x) \geq d^a(x, Tx)
\]
for all \( x, y \in X \), where a constant \( a \) (\( a > 1 \)). Then \( T \) has a fixed point.

**Proof.** In view of Theorem 2.6, there exists a sequence \( \{x_n\} \) such that \( x_{n-1} \neq x_n \) and \( Tx_n = x_{n-1} \). Then we have
\[
d(x_n, x_{n-1}) = d(Tx_{n+1}, T^2x_{n+1})
\]
\[
\geq d^a(x_{n+1}, Tx_{n+1})
\]
\[
= d^a(x_{n+1}, x_n),
\]
which implies that
\[
d(x_n, x_{n+1}) \leq d^b(x_{n-1}, x_n),
\]
where $p = \frac{1}{a} < 1$. From the proof of Theorem 2.6, $\{x_n\}$ is Cauchy sequence. Since $(X, d)$ is complete so $\{x_n\}$ converges to $z \in X$. Also $T$ is continuous, we have $Tx_n = x_{n-1} \to Tz$. As $n \to \infty$, we get $Tz = z$ and hence $z$ is a fixed point of $T$. 

**Example 2.10.** Let $X = \mathbb{R}_+$ and $T : X \to X$ defined by $Tx = x^2$. Then

$$d(Tx, T^2x) = d(x^2, x^4) = \left|\frac{x^2}{x^2}\right|^2 \geq \left|\frac{x}{x^2}\right|^a = d^a(x, Tx),$$

where $1 < a \leq 2$. Clearly $T$ is surjective and continuous. By Theorem 2.9, $T$ has fixed point 1.

**Theorem 2.11.** Let $T$ be a surjective mapping of a complete multiplicative metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \geq \left[\max\{d(x, Tx), d(y, Ty)\}\right]^r$$

for all $x, y \in X$, where a constant $r$ ($r > 1$). Then $T$ has a fixed point.

**Proof.** In view of Theorem 2.6, there exists a sequence $\{x_n\}$ such that $x_{n-1} \neq x_n$ and $Tx_n = x_{n-1}$. Then we have

$$d(x_n, x_{n-1}) = d(Tx_{n+1}, Tx_n)$$
$$\geq \left[\max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}\right]^r$$
$$= \left[\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\}\right]^r$$
$$= d^r(x_n, x_{n+1}),$$

which implies that

$$d(x_n, x_{n+1}) \leq d^r(x_{n-1}, x_n),$$

where $q = \frac{1}{r} < 1$. From the proof of Theorem 2.6, $\{x_n\}$ is Cauchy sequence. Since $(X, d)$ is complete so $\{x_n\}$ converges to $z \in X$.

Let $y \in T^{-1}z$. Then we have

$$d(x_n, z) = d(Tx_{n+1}, Ty)$$
$$\geq \left[\max\{d(x_{n+1}, Tx_{n+1}), d(y, Ty)\}\right]^r$$
$$= \left[\max\{d(x_{n+1}, x_n), d(y, z)\}\right]^r.$$ 

Letting $n \to \infty$, we have $d(y, z) = 1$ and hence $y = z$. Since $Ty = z$, $z$ is a fixed point of $T$. This completes the proof. "$\blacksquare$
References


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