Certain Generalized Integral Formulas
Involving Chebyshev Hermite Polynomials, Generalized $M$-Series and Aleph-Function, and Their Application in Heat Conduction

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Abstract
We present two generalized integral formulas whose integrands are $M$-series, Aleph-function and Chebyshev Hermite polynomials. We also apply those integrals to give a generalized solution of a partial differential equation arising from heat conduction, whose solution is seen to be specialized to yield some known solutions.
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1 Introduction and Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}_0^+$ and $\mathbb{N}$ be sets of complex numbers, real numbers, nonpositive and positive integers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Aleph-function is defined by means of Mellin-Barnes type integral in the following manner (see, e.g., [6]):

\[
\mathbb{N}[z] = \mathbb{N}_{p_k,q_k,\tau_k;r}^{m,n} \left[ z^{(a_j,A_j)_{1,n}, \left[ \tau_j (a_{jk},A_{jk}) \right]_{n+1,p_k;r}} \right] = \frac{1}{2\pi i} \int_{L}^{m,n} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s) z^{-s} ds,
\]

where $z \in \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$, and

\[
\Omega_{p_k,q_k,\tau_k;r}^{m,n} (s) = \prod_{j=1}^{m} \frac{\Gamma (b_j + B_j s) \cdot \prod_{j=1}^{n} \Gamma (1 - a_j - A_j s)}{\sum_{k=1}^{r} \tau_k \prod_{j=m+1}^{p_k} \Gamma (1 - b_{jk} - B_{jk} s) \cdot \prod_{j=n+1}^{q_k} \Gamma (a_{jk} + A_{jk} s)}.
\]

Here $\Gamma$ denotes the familiar Gamma function (see, e.g., [9, Section 1.1]); The integration path $L = L+i\gamma_\infty$ ($\gamma \in \mathbb{R}$) extends from $\gamma - i\infty$ to $\gamma + i\infty$; The poles of the Gamma functions $\Gamma (1 - a_j - A_j s)$ ($n \in \mathbb{N}$; $1 \leq j \leq n$) do not coincide with those of $\Gamma (b_j + B_j s)$ ($n \in \mathbb{N}$; $1 \leq j \leq m$); The parameters $p_k, q_k \in \mathbb{N}_0$ satisfy the conditions $0 \leq n \leq p_k$, $1 \leq m \leq q_k$, $\tau_k > 0$ ($1 \leq k \leq r$); The parameters $A_j, B_j, A_{jk}, B_{jk} > 0$ and $a_j, b_j, a_{jk}, b_{jk} \in \mathbb{C}$; The empty product in (2) is (as usual) understood to be unity.

For the details of the Aleph-function in (1) such as its existence conditions, the interested reader may be referred (for example) to the earlier work [5].

Remark 1.1. The special case of (1) when $\tau_k = 1$ ($k \in \overline{1,r} := \{1, 2, \ldots, r\}$) is seen to yield the $I$-function, due to Saxena [7], defined by the following manner:

\[
J_{p_k,q_k,\tau_k;r}^{m,n} [z] = \mathbb{N}_{p_k,q_k,\tau_k;r}^{m,n} \left[ z^{(a_j,A_j)_{1,n}, \left[ 1 (a_j,A_j) \right]_{n+1,p_k}} \right] = \frac{1}{2\pi i} \int_{L}^{m,n} \Omega_{p_k,q_k,\tau_k;r}^{m,n} (s) z^{-s} ds,
\]

where the kernel $\Omega_{p_k,q_k,\tau_k;r}^{m,n} (\xi)$ is from that in (2). The existence conditions for the integral in (3) are easily modified from those given in [5] with $\tau_k = 1$ ($k \in \overline{1,r}$).
Remark 1.2. The $I$-function in (3) when $r = 1$ is seen to be further specialized to become the familiar $H$-function (see [4]):

$$H^{m,n}_{p,q} \equiv \Omega^{m,n}_{pk,\omega,1;1}(z) = \mathcal{N}^{m,n}_{pk,\omega,1;1}(z) := \frac{1}{2\pi i} \int_{\mathcal{C}} \Omega^{m,n}_{pk,\omega,1;1}(\xi) z^{-\xi} d\xi,$$

where the kernel $\Omega^{m,n}_{pk,\omega,1;1}(\xi)$ can be obtained from (2).

The generalized $M$-series (see [8]) is defined as follows:

$$pM^{\alpha,\beta}_{q}(z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p+1} \Gamma(a_j)} \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n z^n}{(b_1)_n \cdots (b_q)_n} \Gamma(\alpha n + \beta),$$

where $z, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $(a_k)_n$ $(k \in \mathbb{N}, p)$ and $(b_j)_n$ $(j \in \mathbb{N}, q)$ are the familiar Pochhammer symbols (see, e.g., [9, pp. 4-6]). The series (5) can be defined whenever the denominator parameters $b_j \in \mathbb{C} \setminus \mathbb{Z}$ $(j \in \mathbb{N}, q)$. If any numerator parameter $a_j$ is a negative integer or zero, then the series (5) terminates to a polynomial in $z$. The series (5) is convergent for all $z$ if $p \leq q$; If $p = q + 1$, it is convergent for $|z| < \delta = \alpha^\alpha$; If $p > q + 1$, it is divergent. When $p = q + 1$ and $|z| = \delta$, the series (5) can converge on some additional conditions depending on the parameters. Further detailed account of the $M$-Series can be found in [8].

The generalized $M$-series can be represented as special cases of the generalized Wright hypergeometric function $p \psi_q(z)$ and the Fox $H$-function (see [4]), respectively, as follows:

$$pM^{\alpha,\beta}_{q}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p+1} \Gamma(a_j)} \left[ \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n z^n}{(b_1)_n \cdots (b_q)_n} \Gamma(\alpha n + \beta) \right],$$

and

$$pM^{\alpha,\beta}_{q}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p+1} \Gamma(a_j)} H_{p+1,q+1}^{1,1} \left[ \sum_{n=0}^{\infty} \frac{(1 - \alpha_j)_n \cdots (1 - \beta_j)_n z^n}{(0, 1, \ldots, 0, 1)_{p,q}} \Gamma(\alpha n + \beta) \right].$$

Furthermore, setting $p = q = 1$, $b = 1$ and $a = \gamma \in \mathbb{C}$ in (5) gives the generalized Mittag-Leffler function:

$$E_{\alpha,\beta}^{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(1)_n} \frac{z^n}{\Gamma(\alpha n + \beta)} = _1M_{1}^{1,\beta}(\gamma; 1; z).$$
We also recall the following known integrals:
\[ \int_{-\infty}^{\infty} x^{2\mu} e^{-\frac{x^2}{2}} H_{2n}(x) \, dx = \frac{2^{n+\frac{3}{2}} \Gamma \left( \mu + \frac{1}{2} \right) \Gamma (\mu + 1)}{\Gamma (\mu - n + 1)} \]  
and
\[ \int_{-\infty}^{\infty} x^{2\mu+1} e^{-\frac{x^2}{2}} H_{2n+1}(x) \, dx = \frac{2^{n+\frac{3}{2}} \Gamma \left( \mu + \frac{3}{2} \right) \Gamma (\mu + 1)}{\Gamma (\mu - n + 1)}, \]
where \( \mu \leq n \) (\( \mu, n \in \mathbb{N}_0 \)) and \( H_m(x) := 2^{-m/2} H_m(x/\sqrt{2}) \), \( H_m(x) \) are the Hermite polynomials and \( H_n \) Chebyshev Hermite polynomial.

Here, we aim at presenting two integral formulas whose integrands are the generalized \( M \)-series and the Aleph-function in addition to those in (9) and (10). Bajpai [1] gave a solution of the partial differential equation (19), very recently, whose generalized solution was provided by Srivastava and Mishra [10]. Motivated essentially by and modifying these two works [1] and [10], we also apply those integrals in Theorem 2.1 to solve the partial differential equation (19), whose solution is seen to be specialized to give the earlier results.

2 Two generalized integrals

By adding to insert the generalized \( M \)-series and the Aleph-function to the integrands of the integral formulas (9) and (10), we can present two generalized integral formulas (11) and (12) asserted by Theorem 2.1.

**Theorem 2.1.** For \( \nu, \rho > 0 \), \( \mu \geq n \) (\( \mu, n \in \mathbb{N}_0 \)), \( \gamma, \eta \in \mathbb{C} \) and \( \Re (\gamma) > 0 \), the following integral formulas hold true:
\[ \int_{-\infty}^{\infty} x^{2\mu} e^{-\frac{x^2}{2}} H_{2n}(x) \, dx \, M^{\gamma,\eta}_q \left[ g_p; h_q; \alpha x^{2p} \right] _{p_i + 2q_i + 1, r_i; r} \left[ z x^{2\nu} \right] \, dx = 2^{n+\frac{1}{2}} \]
\[ \times \sum_{k=0}^{\infty} \mathcal{M}(k) \mathcal{K}_{p_i + 2q_i + 1, r_i; r}^{m,n+2} \left[ \frac{1}{2} - \mu - \rho k, \nu, \alpha_1, \nu_1, \gamma, \eta \right] \]
\[ \times \sum_{k=0}^{\infty} \mathcal{M}(k) \mathcal{K}_{p_i + 2q_i + 1, r_i; r}^{m,n+2} \left[ 2^{2\nu} \left[ \frac{1}{2} - \mu - \rho k, \nu, \alpha_1, \nu_1, \gamma, \eta \right] \right], \]
and
\[ \int_{-\infty}^{\infty} x^{2\mu+1} e^{-\frac{x^2}{2}} H_{2n+1}(x) \, dx \, M^{\gamma,\eta}_q \left[ g_p; h_q; \alpha x^{2p} \right] _{p_i + 2q_i + 1, r_i; r} \left[ z x^{2\nu} \right] \, dx = 2^{n+\frac{3}{2}} \]
\[ \times \sum_{k=0}^{\infty} \mathcal{M}(k) \mathcal{K}_{p_i + 2q_i + 1, r_i; r}^{m,n+2} \left[ 2^{2\nu} \left[ \frac{1}{2} - \mu - \rho k, \nu, \alpha_1, \nu_1, \gamma, \eta \right] \right], \]
where
\[ \mathcal{M}(k) := \prod_{k=1}^{p} \frac{(g_j)_k}{(h_j)_k} \frac{2^{pk} \mathcal{C}^k}{\Gamma (\gamma k + \eta)}. \]
and whose existence conditions are given as follows:

\[ \varphi_l > 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_l \quad (l \in \mathbb{I}, r); \quad (14) \]

\[ \varphi_l \geq 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_l \quad \text{and} \quad \Re\{\zeta_l\} + 1 < 0, \quad (15) \]

where

\[ \varphi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left( \sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \quad (16) \]

\[ \zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left( \sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2} (p_l - q_l), \quad (l \in \mathbb{I}, r). \quad (17) \]

**Proof.** We prove only (11). Let \( L \) be the left-hand side of (11). Then, using the series defining the generalized \( M \)-series and the Mellin-Barnes type contour integral in the Aleph-function, and interchanging the order of integration and summation, which is guaranteed by uniform convergence of the involved series, we arrive at

\[ L = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Omega(s) \left\{ \sum_{k=0}^{\infty} \mathcal{M}(k) \left\{ \int_{-\infty}^{\infty} x^{2\mu+2\rho k+2\mu s} e^{-x^2/2} He_{2n}(x) \, dx \right\} \right\} z^s \, ds. \quad (18) \]

Evaluating the integral in (18) with the help of the integral (9), in view of (1), we obtain the desired integral formula (11). A similar argument will establish the formula (12), whose detailed account of the proof is omitted. \( \square \)

### 3 Application to Heat Conduction

We apply those results in Section 2 to give a generalized solution of the partial differential equation, which arises in conduction of heat in solids and have been treated in several earlier works (see, e.g., [1], [2, p. 134, Eq. (4)], [10]), given as follows:

\[ \frac{\partial u}{\partial t} = \kappa \left( \frac{\partial^2 u}{\partial x^2} \right) - Hu \quad (-\infty < x < \infty), \quad (19) \]

where \( H := \frac{1}{2} \kappa (x^2/2 - 1) \), and \( u(x, t) \) tends to zero, if the value of \( t \) is maximum and when \( |x| \to \infty \), and \( u(x, 0) := u(x) \).
Theorem 3.1. The following generalized solution of the partial differential equation (19) holds true:

\[
\begin{align*}
  u_1 (x, t) &= \frac{2^\mu}{\sqrt{\pi}} \sum_{k,n=0}^{\infty} \left( \frac{2^n}{(2n)!} \right) e^{-2knt-x^2/4} M(k) \\
  \times \mathcal{N}^{m,n+2}_{p_i+2,q_i+1,\tau_i} \left[ z^{2^n} \left( \frac{1}{2-k-\mu,\nu},(n-\mu-\rho,\nu),(a_j,A_j)_{1,m},[\tau_j(b_j,\nu)],n+1,p_i,r \right) \right] H_{2n} (x),
\end{align*}
\]

and

\[
\begin{align*}
  u_2 (x, t) &= \frac{2^{\mu+1}}{\sqrt{\pi}} \sum_{k,n=0}^{\infty} \left( \frac{2^n}{(2n+1)!} \right) e^{-k(2n+1)t-x^2/4} M(k) \\
  \times \mathcal{N}^{m,n+2}_{p_i+2,q_i+1,\tau_i} \left[ z^{2^n} \left( \frac{1}{2-k-\mu,\nu},(n-\mu-\rho,\nu),(a_j,A_j)_{1,m},[\tau_j(b_j,\nu)],n+1,p_i,r \right) \right] H_{2n+1} (x),
\end{align*}
\]

which are valid for \( \nu, \rho > 0, \mu \geq n \) (\( \mu, n \in \mathbb{N}_0 \)), \( \gamma, \eta \in \mathbb{C}, \Re(\gamma) > 0 \) and the given conditions in (14)-(17).

Proof. We assume that a general solution of the partial differential equation (19) may be given as follows (see, e.g., [1, p. 32, Eq. (5.1) and p. 33, Eq. (5.3)]):

\[
\begin{align*}
  u_1 (x, t) &= \sum_{n=0}^{\infty} \left. C_{2n} e^{-2knt-x^2/4} H_{2n} (x) \right|, \quad (22)
\end{align*}
\]

where

\[
\begin{align*}
  C_{2n} &= \frac{1}{(2n)! \sqrt{2\pi}} \int_{-\infty}^{\infty} u_1 (x) e^{-x^2/4} H_{2n} (x) \, dx, \quad (23)
\end{align*}
\]

and

\[
\begin{align*}
  u_2 (x, t) &= \sum_{n=0}^{\infty} \left. C_{2n+1} e^{-k(2n+1)t-x^2/4} H_{2n+1} (x) \right|, \quad (24)
\end{align*}
\]

where

\[
\begin{align*}
  C_{2n+1} &= \frac{1}{(2n+1)! \sqrt{2\pi}} \int_{-\infty}^{\infty} u_2 (x) e^{-x^2/4} H_{2n+1} (x) \, dx, \quad (25)
\end{align*}
\]

and \( H_{2n} \) and \( H_{2n+1} \) are Chebyshev Hermite polynomials. Now we determine \( u_1 (x, t) \) and \( u_2 (x, t) \), where

\[
\begin{align*}
  u_1 (x, 0) := u_1 (x) &= x^{2\mu} e^{-x^2/4} p M_q^{\gamma,\eta} \left[ g_p; h_q; c x^{2p} \right] \mathcal{N}^{m,n}_{p_i,q_i,\tau_i} \left[ z^{2^n} \right], \quad (26)
\end{align*}
\]
Similarly, combining (25) and (27) with the integral (12) yields

\[ u_2(x, 0) := u_2(x) = x^{2\mu+1}e^{-x^2/4} m_q^{\gamma,\eta} \left[ g_p; h_q; cx^{2\rho} \right] \mathcal{N}_p^{m,n,r} \left[ z x^{2\nu} \right] . \] (27)

Combining (23) and (26) with the aid of the integral (11) is seen to give the following expression:

\[ C_{2n} = \frac{2^{\mu+n}}{(2n)!\sqrt{\pi}} \sum_{k=0}^{\infty} \mathcal{M}(k) \mathcal{N}_{p+1, q+1, 1, r} \left[ z 2^\nu \left( \frac{1}{2} - \mu - \rho, \nu \right), (\nu, a_j, A_j)_{1, m}, \left[ r_j(a_j, A_j) \right]_{n+1, p+1} \right] . \] (28)

Similarly, combining (25) and (27) with the integral (12) yields

\[ C_{2n+1} = \frac{2^{\mu+n+1}}{(2n + 1)!\sqrt{\pi}} \sum_{k=0}^{\infty} \mathcal{M}(k) \mathcal{N}_{p+2, q+1, 1, r} \left[ z 2^\nu \left( \frac{1}{2} - \mu - \rho, \nu \right), (\nu, a_j, A_j)_{1, m}, \left[ r_j(b_j, B_j) \right]_{n+1, p+1} \right] . \] (29)

Finally, setting \( C_{2n} \) in (28) and \( C_{2n+1} \) in (29) into (22) and (24), respectively, is seen to yield the desired results in Theorem 3.1.

\[ \square \]

4 Fourier Hermite Expansion

We obtain the following Fourier Hermite expansion by setting \( t = 0 \) in (20) and (21):

\[ x^{2\mu} m_q^{\gamma,\eta} \left[ g_p; h_q; cx^{2\rho} \right] \mathcal{N}_p^{m,n,r} \left[ z x^{2\nu} \right] = \frac{2^\mu}{\sqrt{\pi}} \sum_{k,n=0}^{\infty} \left( \frac{2^n}{(2n)!} \right) \mathcal{M}(k) \times \mathcal{N}_{p+1, q+1, 1, r} \left[ z 2^\nu \left( \frac{1}{2} - \mu - \rho, \nu \right), (\nu, a_j, A_j)_{1, m}, \left[ r_j(a_j, A_j) \right]_{n+1, p+1} \right] H_{2n}(x) , \] (30)

and

\[ x^{2\mu+1} m_q^{\gamma,\eta} \left[ g_p; h_q; cx^{2\rho} \right] \mathcal{N}_p^{m,n,r} \left[ z x^{2\nu} \right] = \frac{2^{\mu+1}}{\sqrt{\pi}} \sum_{k,n=0}^{\infty} \left( \frac{2^n}{(2n + 1)!} \right) \mathcal{M}(k) \times \mathcal{N}_{p+2, q+1, 1, r} \left[ z 2^\nu \left( \frac{1}{2} - \mu - \rho, \nu \right), (\nu, a_j, A_j)_{1, m}, \left[ r_j(b_j, B_j) \right]_{n+1, p+1} \right] H_{2n+1}(x) . \] (31)
5 Special Cases

Here we consider some special cases of our results.

(i) In view of Remark 1.1, setting \( \tau_i = 1 \) \((i \in 1, r)\) in the results in Theorem 2.1 yields some integral formulas whose integrands are \( I \)-function, Chebyshev Hermite polynomials and the generalized \( M \)-series. Further, setting \( \eta = 1 \) is easily seen to yield the results derived by Srivastava and Mishra [10].

(ii) If we use the relation given by (8), we get the solution of (19) in the product of Chebyshev Hermite polynomials, the generalized Mittag-Leffler function and the \( \mathbb{H} \)-function.

(iii) On taking the generalized \( M \)-series into unity and taking into account the relation (4), then we get the solution of (19) in terms of \( H \)-function, which is given by Bajpai [1].

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