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Analytical-Numerical Method for Solving a Class of Two-Point Boundary Value Problems

Ayed Hussein Al e'damat

Department of Mathematics, Faculty of Science
Al-Hussein Bin Talal University
P.O. Box 20, Ma'an-Jordan, Jordan

Abstract

In this paper, we applied analytical-numerical method to approximate solutions of two-point boundary value problems of fourth-order Volterra integrodifferential equations based on the reproducing kernel theory . The solution is represented in the form of convergent series with easily computable components. The solution methodology is based on generating the orthogonal basis from the obtained kernel function in the space $W_2^5[a,b]$. The n-term approximation is obtained and proved to converge to the analytical solution. Moreover, the proposed method has an advantage that it is possible to pick any point in the interval of integration and as well the approximate solutions and its all derivatives will be applicable. Numerical examples are given to demonstrate the computation efficiency of the presented method. Results obtained by the method indicate the method is simple and effective.

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1 Introduction

The integrodifferential equations (IDEs) occur in many areas of mathematical physics that play an important role in modeling of much physical phenomena such as electrodynamics, elasticity, electromagnetic and fluid dynamics as well as biology and engineering. However, in many branches of physics, mathematics, and engineering, solving a problem means solving a set of ordinary, partial, integral or either IDEs [1-7, 14-18, 27-37]. In fact, these equations, which have attracted considerable attention over the last two decades, are usually difficult to solve analytically, so it is required to obtain an efficient analytical-numerical solution. Therefore, many techniques arose in the studies existence and constructive approximation of solutions of such problems, Especially, those techniques that based on the reproducing kernel Hilbert space theory, for instance, the first-order Fredholm-Volterra IDEs [21, 30], the second-order IDEs Fredholm or Volterra or of mixed type [16, 19], and fourth-order IDEs [22, 23]. On the other hand, the numerical solvability of other version of differential problems can be found in [8-13, 19, 20, 24-26].

Let us consider the following nonlinear fourth-order, IDE of Volterra type in the reproducing kernel space

$$u^{(iv)}(x) + \gamma u(x) + \int_0^x [g(x)u(x) + h(x)G(u(x))]dx = f(x), \tag{1}$$

subject to the boundary conditions

$$u(a) = \alpha_0,\ u''(a) = \alpha_2,\ u(b) = \beta_0,\ u''(b) = \beta_2, \tag{2}$$

where $x \in (a,b)$, γ, α_0, α_1, β_1 and β_2 are real finite constants, G is a real nonlinear continuous function, f, g and h are given and can be approximated by Taylor polynomials, and $u(x)$ is an unknown analytic function to be determined. We suppose that the boundary value problems (1) and (2) have a unique smooth solution.

The reminder of the paper is organized as follows: several reproducing kernel spaces are described in Section 2. In Section 3, a linear operator, a complete normal orthogonal system and some essential results are introduced. Also, a method for the existence of solutions for boundary value problems (1) and (2) based on reproducing kernel space is described. The numerical example is presented in Section 4. This article ends in Section 4 with some concluding remarks and future recommendations.

2 Several reproducing kernel spaces

Definition 2.1 *Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H iff*

1. $\forall t \in E$, $K(\cdot, t) \in H$.

2. $\forall t \in E$, $\forall \varphi \in H$, $(\varphi, K(\cdot, t)) = \varphi(t)$.

Definition 2.2 *A Hilbert spaces H of functions on a set Ω is called a reproducing kernel Hilbert spaces if there exists a reproducing kernel K of H.*

The inner product space $W_2^5[a,b]$ is defined as $\{u(x) \mid u^{(i)}, i = 0,1,2,3,4$ are absolutely continuous real-valued functions on $[a,b]$, $u^{(5)} \in L^2[a,b]$, and $u(a) = u''(a) = u(b) = u''(b) = 0\}$. The inner product and norm in $W_2^5[a,b]$ are given by

$$\langle u, v\rangle_{W_2^5} = \sum_{i=0}^{2} u^{(i)}(a)\, v^{(i)}(a) + \sum_{i=0}^{1} u^{(i)}(b)\, v^{(i)}(b) + \int_a^b u^{(5)}(y) v^{(5)}(y) dy \qquad (3)$$

and $||u||_{W_2^5} = \sqrt{\langle u, u\rangle_{W_2^5}}$, respectively, where $u, v \in W_2^5[a,b]$.

Theorem 2.1 *The space $W_2^5[a,b]$ is a complete reproducing kernel space. That is, for each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2^5[a,b]$ such that $\langle u(y), R_x(y)\rangle_{W_2^5} = u(x)$ for any $u(y) \in W_2^5[0,1]$ and $y \in [0,1]$. The reproducing kernel function $R_x(y)$ can be denoted by*

$$K_x(y) = \begin{cases} \sum_{i=0}^{9} p_i(x) y^i, & y \le x; \\ \sum_{i=0}^{9} q_i(x) y^i, & y > x. \end{cases} \qquad (4)$$

Proof. Through several integrations by parts for Equation (3), we obtain

$$\begin{aligned} \int_a^b u^{(5)}(y)\, K_x^{(5)}(y)\, dy &= \sum_{i=0}^{4} (-1)^{4-i} u^{(i)}(y)\, K_x^{(9-i)}(y)\Big|_{y=a}^{y=b} \\ &\quad + (-1)^3 \int_a^b u(y)\, K_x^{(10)}(y)\, dy. \end{aligned}$$

Hence, we have $\langle u(y), K_x(y)\rangle_{W_2^5} = \sum_{i=0}^{2} u^{(i)}(a)\, K_x^{(i)}(a) + \sum_{i=0}^{1} u^{(i)}(b)\, K_x^{(i)}(b) + \sum_{i=0}^{4} (-1)^{4-i} u^{(i)}(y)\, K_x^{(9-i)}(y)\Big|_{y=a}^{y=b} - \int_a^b u(y)\, K_x^{(10)}(y)\, dy$.

Since $K_x(y) \in W_2^5[a,b]$, it follows that $K_x(a) = K_x''(a) = K_x(b) = K_x''(b) = 0$. Further, since $u(x) \in W_2^5[a,b]$, one obtains $u(a) = u''(a) = u(b) = u''(b) = 0$. Thus, if $K_x^{(i)}(a) = K_x^{(i)}(b) = 0$, $i = 5, 6$, $K_x'(a) + K_x^{(8)}(a) = 0$, and $K_x'(b) - K_x^{(8)}(b) = 0$, then $\langle u(y), K_x(y)\rangle_{W_2^5} = \int_a^b u(y) \left(-K_x^{(10)}(y)\right) dy$. Now, for each $x \in [a,b]$, if $K_x(y)$ also satisfies $-K_x^{(10)}(y) = \delta(x-y)$, where δ

is the dirac-delta function, then $\langle u(y), K_x(y)\rangle_{W_2^5} = u(x)$. Obviously, $K_x(y)$ is the reproducing kernel of the space $W_2^5[a,b]$.

The characteristic equation of $-K_x^{(10)}(y) = \delta(y-x)$ is $\lambda^{10} = 0$, and their characteristic values are $\lambda = 0$ with 10 multiple roots. So, let

$$K_x(y) = \begin{cases} \sum\limits_{i=0}^{9} p_i(x)y^i, & y \le x; \\ \sum\limits_{i=0}^{9} q_i(x)y^i, & y > x. \end{cases}$$

On the other hand, let $K_x(y)$ satisfies $K_x^{(m)}(x+0) = K_x^{(m)}(x-0)$, $m = 0, 1, ..., 8$. Integrating $-K_x^{(10)}(y) = \delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $K_x^{(9)}(y)$ at $y = x$ given by $K_x^{(9)}(x-0) - K_x^{(9)}(x+0) = 1$. Through the last descriptions the unknown coefficients of Equation (4) can be obtained. This completes the proof.

Without loss of generality, The coefficients of the reproducing kernel $K_x(y)$ in Equation (4) are obtained at $a = 0$ and $b = 1$ in the boundary value problems (1) and (2) and are given as:

$$\begin{aligned}
p_0(x) &= p_2(x) = p_5(x) = p_6(x) = 0; \\
p_1(x) &= (362884x - 725782x^3 + 362903x^4 - 12x^7 + 9x^8 - 2x^9)/725764; \\
p_3(x) &= (-43895295360x + 87800025610x^3 - 43910173465x^4 + 10160696x^5 \\
&\quad -5806148x^7 + 1088673x^8 - 6x^9)/43894206720; \\
p_4(x) &= x(43896746880 - 87820346930x^2 + 43950816165x^3 - 30482088x^4 \\
&\quad +4354668x^6 - 1088709x^7 + 14x^8)/87788413440; \\
p_7(x) &= -x(362880 - 2177292x + 2903074x^2 - 1088667x^3 + 12x^6 - 9x^7 \\
&\quad +2x^8)/21947103360; \\
p_8(x) &= x(-362884 + 725782x^2 - 362903x^3 + 12x^6 - 9x^7 + 2x^8)/29262804480; \\
p_9(x) &= (362882 - 362880x - 18x^3 + 21x^4 - 12x^7 + 9x^8 - 2x^9)/131682620160;
\end{aligned}$$

$$
\begin{aligned}
q_0(x) &= x^9/362880;\\
q_1(x) &= -x(-3657870720+7315882560x^2-3658062240x^3+120960x^6\\
&\quad +90721x^7+20160x^8)/7315701120;\\
q_2(x) &= x^7/10080;\\
q_3(x) &= -x(43895295360-87800025610x^2+43910173465x^3+5806148x^6\\
&\quad -1088673x^7+6x^8)/43894206720;\\
q_4(x) &= x(43896746880-87820346930x^2+43950816165x^3+4354668x^6\\
&\quad -1088709x^7+14x^8)/87788413440;\\
q_5(x) &= -x^4/2880;\\
q_6(x) &= x^3/4320;\\
q_7(x) &= -x(362880+2903074x^2-1088667x^3+12x^6-9x^7+2x^8)/21947103360;\\
q_8(x) &= x(362880+725782x^2-362903x^3+12x^6-9x^7+2x^8)/29262804480;\\
q_9(x) &= -x(362880+18x^2-21x^3+12x^6-9x^7+2x^8)/131682620160.
\end{aligned}
$$

The following corollary summarized some important properties of the reproducing kernel $K_x(y)$ and are easily obtained.

Corollary 2.1 The reproducing kernel $K_x(y)$ is symmetric, unique, and $K_x(x) \geq 0$ for any fixed $x \in [a,b]$.

Proof. By the reproducing property, we have $K_x(y) = \langle K_x(\xi), K_y(\xi)\rangle = \langle K_y(\xi), K_x(\xi)\rangle = K_y(x)$ for each x and y. Now, let $K_x^1(y)$ and $K_x^2(y)$ be all the reproducing kernels of the space $W_2^5[a,b]$, then $K_x^1(y) = \left\langle K_x^1(\xi), K_y^2(\xi)\right\rangle = \left\langle K_y^2(\xi), K_x^1(\xi)\right\rangle = K_y^2(x) = K_x^2(y)$.
Finally, we note that $K_x(x) = \langle K_x(\xi), K_x(\xi)\rangle = \|K_x(\xi)\|^2 \geq 0$.

The inner product space $W_2^1[a,b]$ is defined as $W_2^1[a,b] = \{u(x) : u$ is absolutely continuous real valued function on $[a,b]$ and $u' \in L^2[a,b]\}$. The inner product in $W_2^1[a,b]$ is given by

$$\langle u,v\rangle_{W_2^1} = \int_a^b \left(u'(y)\,v'(y) + u(y)\,v(y)\right) dy,$$

and the norm $\|u\|_{W_2^1}$ is denoted by $\|u\|_{W_2^1} = \sqrt{\langle u,u\rangle_{W_2^1}}$, where $u,v \in W_2^1[a,b]$.

In [21], the authors have proved that the space $W_2^1[a,b]$ is a RKHS and its reproducing kernel is

$$R_x(y) = \left[\cosh(x+y-b-a) + \cosh(|x+y|-b+a)\right]/2\sinh(b-a).$$

3 Analysis of the method

Define a differential operator $L : W_2^5[a,b] \to W_2^1[a,b]$ such that $Lu(x) = u^{(iv)}(x) - \gamma u(x)$. After homogenization of the initial conditions, then boundary value problems (1) and (2) can be converted into the following form

$$\begin{aligned} &Lu\,(x) = F(x, u(x), Tu(x)),\ x \in (a, b); \\ &u\,(a) = u''(a) = 0 = u\,(b) = u''(b) = 0, \end{aligned} \tag{5}$$

where $Tu(x) = \int_0^x [g(x)u\,(x) + h(x)F(u\,(x))dx$, $u\,(x) \in W_2^5\,[a, b]$ and $F\,(x, y_1, y_2) \in W_2^1\,[a, b]$ for $y_1 = y_1(x), y_2 = y_2(x) \in W_2^5\,[a, b]$. It is clear that L is a bounded linear operator.

Now, we construct an orthogonal function system of $W_2^5\,[a, b]$. For a fixed dense set $\{x_i\}_{i=1}^{\infty}$ of $[a, b]$, let $\varphi_i\,(x) = R_{x_i}\,(x)$. So, from the properties of $R_x\,(y)$, for every $u\,(x) \in W_2^1\,[a, b]$, it follows that $\langle u\,(x), \varphi_i\,(x)\rangle_{W_2^1} = \langle u\,(x), R_{x_i}\,(x)\rangle_{W_2^1} = u\,(x_i)$. Additionally, let $\psi_i\,(x) = L^*\varphi_i\,(x)$, where L^* is the adjoint operator of L. Obviously, $\psi_i\,(x) \in W_2^5\,[a, b]$. In terms of the properties of $K_x\,(y)$, one obtains $\langle u\,(x), \psi_i\,(x)\rangle_{W_2^5} = \langle u\,(x), L^*\varphi_i\,(x)\rangle_{W_2^5} = \langle Lu\,(x), \varphi_i\,(x)\rangle_{W_2^1} = Lu\,(x_i)$, $i = 1, 2, ...$

Lemma 3.1 $\psi_i\,(x)$ can be expressed in the form $\psi_i\,(x) = L_y K_x\,(y)\big|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y.

Proof. From the above assumption, it is clear that $\psi_i\,(x) = L^*\varphi_i\,(x) = \langle L^*\varphi_i\,(x), K_x\,(y)\rangle_{W_2^5} = \langle \varphi_i\,(x), L_y K_x\,(y)\rangle_{W_2^1} = L_y K_x\,(y)\big|_{y=x_i}$.

Theorem 3.1 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, then $\{\psi_i\,(x)\}_{i=1}^{\infty}$ is the complete function system of $W_2^5\,[a, b]$.

Proof. For each fixed $u\,(x) \in W_2^5\,[a, b]$, let $\langle u\,(x), \psi_i\,(x)\rangle = 0$, $i = 1, 2, ...$, that is $\langle u\,(x), \psi_i\,(x)\rangle = \langle u\,(x), L^*\varphi_i\,(x)\rangle = \langle Lu\,(x), \varphi_i\,(x)\rangle = Lu\,(x_i) = 0$. Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, therefore, $Lu\,(x) = 0$. It follows that $u\,(x) = 0$ from the existence of L^{-1}. So, the proof of the Theorem is complete.

The orthonormal function system $\left\{\overline{\psi}_i\,(x)\right\}_{i=1}^{\infty}$ of $W_2^5\,[a, b]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i\,(x)\}_{i=1}^{\infty}$ as follows:

$$\bar{\psi}_i\,(x) = \textstyle\sum_{k=1}^{i} \beta_{ik}\psi_k\,(x), \tag{6}$$

where β_{ik} are orthogonalization coefficients given as: $\beta_{11} = 1/\left\|\psi_1\right\|$, $\beta_{ii} = 1/\sqrt{\left\|\psi_i\right\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$, $i = 1, 2, ...$, and $\beta_{ij} = -\sum_{k=j}^{i-1} c_{ik}\beta_{kj}/\sqrt{\left\|\psi_i\right\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$ for $j < i$ in which $c_{ik} = \left\langle \psi_i, \overline{\psi}_k \right\rangle_{W_2^5}$.

The structure of the next two theorems are as follows: Firstly, we will give the representation of the exact solution of boundary value problems (1) and (2) in the space $W_2^5\,[a, b]$. After that, the convergence of approximate solution $u_n\,(x)$ to the analytic solution will be proved.

Theorem 3.2 For each $u(x) \in W_2^5[a,b]$, the series $\sum_{i=1}^{\infty} \left\langle u(x), \bar{\psi}_i(x) \right\rangle \bar{\psi}_i(x)$ is convergent in the norm of $W_2^5[a,b]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of the boundary value problems (1) and (2) is unique, then this solution satisfies the form:

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \bar{\psi}_i(x). \tag{7}$$

Proof. Applying Theorem 3.1, it is easy to see that $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^5[a,b]$. Thus, $u(x)$ can be expanded in the Fourier series as $u(x) = \sum_{i=1}^{\infty} \left\langle u(x), \bar{\psi}_i(x) \right\rangle \bar{\psi}_i(x)$ about normal orthogonal system $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$. Moreover, $W_2^5[a,b]$ is a Hilbert space, then the series $\sum_{i=1}^{\infty} \left\langle u(x), \bar{\psi}_i(x) \right\rangle \bar{\psi}_i(x)$ is convergent in the norm of $W_2^5[a,b]$.

Since $\langle v(x), \varphi_i(x) \rangle = v(x_i)$ for each $v(x) \in W_2^1[a,b]$. Hence, we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \bar{\psi}_i(x) \right\rangle \bar{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle \bar{\psi}_i(x) \qquad (8) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), L^* \varphi_k(x) \rangle \bar{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle F(x, u(x), Tu(x)), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \bar{\psi}_i(x). \end{aligned}$$

The proof of the theorem is complete.

Remark 3.1 If Equation (1) is linear, then the analytical solution can be obtained directly from Equation (7). In the case of Equation (1) is nonlinear, the approximate solution can be obtained using the following iterative method: according to Equation (7), the representation of the solution of IDE (1) can be denoted by $u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$, where

$$A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)).$$

In fact, A_i, $i = 1, 2, ..., n$.are unknown, we will approximate A_i using known B_i. For a numerical computations, we define initial function $u_0(x_1)$, put $u_0(x_1) = u(x_1)$, and define the n-term approximation to $u(x)$ by:

$$u_n(x) = \sum_{i=1}^{n} B_i \bar{\psi}_i(x), \tag{9}$$

where the coefficients B_i, $i = 1, ..., n$ are given by

$$B_i = \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{k-1}\left(x_k\right), Tu_{k-1}\left(x_k\right)\right). \tag{10}$$

In the iteration process of Equation (9), we can guarantee that the approximation $u_n\left(x\right)$ satisfies the boundary conditions of Equations (1) and (2).

Next, we will proof $u_n\left(x\right)$ in the iterative formula (9) is converge to the exact solution $u\left(x\right)$ of Equation (1). The Lemma 3.2 through Lemma 3.4 are collected for future use.

Lemma 3.2 If $u\left(x\right) \in W_2^5\left[a, b\right]$, then there exists $M > 0$ such that $||u||_C \leq M\,||u||_{W_2^5}$, where $||u||_C = \max_{a \leq x \leq b}\{|u(x)| + |u'(x)| + |u''(x)| + \left|u^{(3)}(x)\right| + \left|u^{(4)}(x)\right|\}$.

Proof. For any $x \in (a, b)$, we have $u^{(i)}\left(x\right) = \left\langle u(y), K_x^{(i)}(y)\right\rangle_{W_2^5}$. using the expression of $K_x(y)$, one gets $\left\|K_x^{(i)}(y)\right\|_{W_2^5} \leq M_i$, $i = 0, 1, ..., 4$. Thus, we have $\left|u^{(i)}\left(x\right)\right| = \left|\left\langle u(y), K_x^{(i)}(y)\right\rangle_{W_2^5}\right| \leq \left\|K_x^{(i)}(y)\right\|_{W_2^5} \left\|u\left(x\right)\right\|_{W_2^5} \leq M_i \left\|u\left(x\right)\right\|_{W_2^5}$, $i = 0, 1, ..., 4$. Hence, $||u\left(x\right)||_C \leq M\,||u\left(x\right)||_{W_2^5}$, where $M = \max_{i=0,1,..,4} M_i$.

Now, by using the above lemma together with Theorem 3.2, for each $x \in [a, b]$, it is clear that

$$|u\left(x\right) - u_n\left(x\right)| \leq ||u\left(x\right) - u_n\left(x\right)||_C \leq M\,||u\left(x\right) - u_n\left(x\right)||_{W_2^5} \to 0 \text{ as } n \to \infty.$$

i.e. $u_n\left(x\right)$ convergent uniformly to $u\left(x\right)$. Furthermore, it is obviously that $u_n\left(x_n\right) \to u\left(y\right)$ as $x_n \to y$, $n \to \infty$. Thus, by means of the continuation of $Tu\left(\cdot\right)$, it is obtained that $Tu_n\left(x_n\right) \to Tu\left(y\right)$ as $n \to \infty$. Hence, by the continuity of F, we have $F(x_n, u_{n-1}\left(x_n\right), Tu_{n-1}\left(x_n\right)) \to F(y, u\left(y\right), Tu\left(y\right))$ as $n \to \infty$.

Lemma 3.3 $Lu_n\left(x_j\right) = F\left(x_j, u_{j-1}\left(x_j\right), Tu_{j-1}\left(x_j\right)\right)$, $j \leq n$.

Proof. The proof will be obtained by induction as follows. If $j \leq n$, then

$$\begin{aligned} Lu_n\left(x_j\right) &= \sum_{i=1}^{n} B_i L\bar{\psi}_i\left(x_j\right) = \sum_{i=1}^{n} B_i \left\langle L\bar{\psi}_i\left(x\right), \phi_j\left(x\right)\right\rangle_{W_2^1} \\ &= \sum_{i=1}^{n} B_i \left\langle \bar{\psi}_i\left(x\right), L^*\phi_j\left(x\right)\right\rangle_{W_2^5} = \sum_{i=1}^{n} B_i \left\langle \bar{\psi}_i\left(x\right), \psi_j\left(x\right)\right\rangle_{W_2^5}. \end{aligned}$$

The orthogonality of $\left\{\bar{\psi}_i\left(x\right)\right\}_{i=1}^{\infty}$ yields that

$$\begin{aligned} \sum_{l=1}^{j} \beta_{jl} Lu_n\left(x_l\right) &= \sum_{i=1}^{n} B_i \left\langle\, \bar{\psi}_i\left(x\right), \sum_{l=1}^{j} \beta_{jl}\psi_l\left(x\right)\,\right\rangle_{W_2^5} = \sum_{i=1}^{n} B_i \left\langle \bar{\psi}_i\left(x\right), \bar{\psi}_j\left(x\right)\right\rangle_{W_2^5} \\ &= B_j = \sum_{l=1}^{j} \beta_{jl} F\left(x_l, u_{l-1}\left(x_l\right), Tu_{l-1}\left(x_l\right)\right). \end{aligned}$$

Now, if $j=1$, then $Lu_n(x_1)=F(x_1,u_0(x_1),Tu_0(x_1))$ and if $j=2$, then $\beta_{21}Lu_n(x_1)+\beta_{22}Lu_n(x_2)=\beta_{21}F(x_1,u_0(x_1),Tu_0(x_1))+\beta_{22}F(x_2,u_1(x_2),Tu_1(x_2))$. So, $Lu_n(x_2)=F(x_2,u_1(x_2),Tu_1(x_2))$. Moreover, it is easy to see by induction that $Lu_n(x_j)=F(x_j,u_{j-1}(x_j),Tu_{j-1}(x_j))$.

Lemma 3.4 For $j\le n$, we have $Lu_n(x_j)=Lu(x_j)$.

Proof. We know $u_n(x)$ converge uniformly to $u(x)$. It follows that, by taking the limits on both sides of Equation (9); $u(x)=\sum\limits_{i=1}^{\infty}B_i\bar{\psi}_i(x)$. Therefore $u_n(x)=P_nu(x)$, where P_n is an orthogonal projector from $W_2^5[a,b]$ to Span$\{\psi_1,\psi_2,...,\psi_n\}$. Also,

$$\begin{aligned}Lu_n(x_j) &= \langle Lu_n(x),\phi_j(x)\rangle_{W_2^1}=\langle u_n(x),L^*\phi_j(x)\rangle_{W_2^5}\\ &= \langle P_nu(x),\psi_j(x)\rangle_{W_2^5}=\langle u(x),P_n\psi_j(x)\rangle_{W_2^5}\\ &= \langle u(x),\psi_j(x)\rangle_{W_2^5}=\langle Lu(x),\phi_j(x)\rangle_{W_2^1}=Lu(x_j).\end{aligned}$$

Theorem 3.2 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and $||u_n||_{W_2^5}$ is bounded, then $u_n(x)$ in the iterative formula (9) convergent to the exact solution $u(x)$ of boundary value problems (1) and (2) in the space $W_2^5[a,b]$ and $u(x)=\sum\limits_{i=1}^{\infty}B_i\bar{\psi}_i(x)$, where B_i is given by Equation (10).

Proof. First of all, we will prove the convergence of $u_n(x)$. By Equation (9), we have $u_{n+1}(x)=u_n(x)+B_{n+1}\bar{\psi}_{n+1}(x)$. From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned}||u_{n+1}||_{W_2^5}^2 &= ||u_n||_{W_2^5}^2+(B_{n+1})^2=||u_{n-1}||_{W_2^5}^2+(B_n)^2+(B_{n+1})^2\\ &= ...=||u_0||_{W_2^5}^2+\sum_{i=1}^{n+1}(B_i)^2.\end{aligned}$$

Again from boundedness of $||u_n||_{W_2^5}$, we have $\sum\limits_{i=1}^{\infty}(B_i)^2<\infty$, that is, $\{B_i\}_{i=1}^{\infty}\in l^2$.

Since $(u_m-u_{m-1})\perp(u_{m-1}-u_{m-2})\perp...\perp(u_{n+1}-u_n)$ it follows that if $m>n$, then

$$\begin{aligned}||u_m(x)-u_n(x)||_{W_2^5}^2 &= ||u_m(x)-u_{m-1}(x)+u_{m-1}(x)...+u_{n+1}(x)-u_n(x)||_{W_2^5}^2\\ &= ||u_m(x)-u_{m-1}(x)||_{W_2^5}^2+...+||u_{n+1}(x)-u_n(x)||_{W_2^5}^2\\ &= \sum_{i=n+1}^{m}(B_i)^2.\end{aligned}$$

Consequently, as $n,m\to\infty$ we have $||u_m(x)-u_n(x)||_{W_2^5}^2\to 0$ as $\sum\limits_{i=n+1}^{m}(B_i)^2\to 0$. Considering the completeness of $W_2^5[a,b]$, there exists a $u(x)\in W_2^5[a,b]$ such that $u_n(x)\to u(x)$ as $n\to\infty$ in the sense of the norm of $W_2^5[a,b]$.

Second, we will prove that $u(x)$ is the solutions of boundary value problems (1) and (2). From Lemmas 3.3 and 3.4, since $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, for any $x \in [a,b]$, there exists subsequence $\left\{x_{n_j}\right\}$, such that $x_{n_j} \to x$ as $j \to \infty$. It is clear that $Lu\left(x_{n_j}\right) = F(x_{n_j}, u_{n_j-1}(x_k), Tu_{n_j-1}(x_k))$. Hence, let $j \to \infty$, by the continuity of F, we have $Lu(x) = F(x, u(x), Tu(x))$. That is, $u(x)$ satisfies IDE (1). Since $\bar{\psi}_i(x) \in W_2^5[a,b]$, clearly, $u(x)$ satisfies the boundary conditions (2). In other words $u(x)$ is the solution of boundary value problems (1) and (2), where $u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x)$ and B_i is given by Equation (10).

Theorem 3.3 *Assume that $u(x) \in W_2^5[a,b]$ is the solution of boundary value problems (1) and (2) and $r_n(x)$ is the difference between the approximate solution $u_n(x)$ and the exact solution $u(x)$. Then, $r_n(x)$ is monotone decreasing in the sense of the norm of $W_2^5[a,b]$. i.e. $r_n \to 0$ as $n \to \infty$.*

Proof. It is obvious that

$$\begin{aligned} ||r_n(x)||_{W_2^5}^2 &= ||u(x) - u_n(x)||_{W_2^5}^2 \\ &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{n} \beta_{nk} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)) \bar{\psi}_i(x) \right\|_{W_2^5}^2 \\ &= \left\| \sum_{i=n+1}^{\infty} A_i \bar{\psi}_i(x) \right\|_{W_2^5}^2 = \sum_{i=n+1}^{\infty} (A_i)^2, \end{aligned}$$

and $||r_{n-1}(x)||_{W_2^5}^2 = \sum_{i=n}^{\infty} (A_i)^2$. Thus, $||r_n(x)||_{W_2^5} \leq ||r_{n-1}(x)||_{W_2^5}$. Consequently, the difference $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^5}$. So, the proof of the theorem is complete.

4 Numerical Examples

In order to have a clear overview of our method, two examples with known exact solutions are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other. Through this paper the numerical computation performed by using Mathematica 7.0 software package.

Example 4.1 Consider the following linear Volterra integrodifferential equation

$$\begin{aligned} &u^{(iv)}(x) = x(1+e^x) + 3e^x + u(x) - \int_0^x u(t)dt,\ 0 \leq x \leq 1; \\ &u(0) = 1,\ u''(0) = 2,\ u(1) = 1 + e,\ u''(1) = 3e. \end{aligned} \tag{11}$$

The exact solution is $u(x) = 1 + xe^x$. Using RKHS method, taking $N = 10$ and $n = 1$ with $x_i = i/N$, $i = 0, 1, ..., N$; the numerical results at some selected gird points are given in Table 1.

As we mention, we used the grid nodes mentioned earlier in order to obtain approximate solutions. Moreover, it is possible to pick any point in $[a, b]$ and as well the approximate solutions and its all derivative up to order four will be applicable using the same previous partition of $[a, b]$. Next, the numerical results for Equation 11 which include the absolute error at some selected nodes in $[0, 1]$ for $u^{(m)}(x)$, $m = 0, 1, ..., 4$ are given in Tables 2.

Table 1. Numerical results of Equation 11:

x	Analytical solution	Numerical solution	Absolute error	Relative error
0.0	1	1	0	0
0.1	1.11051709	1.11053279	1.57045×10^{-5}	1.41416×10^{-5}
0.2	1.24428055	1.24430767	2.71233×10^{-5}	2.17984×10^{-5}
0.3	1.40495764	1.40499425	3.66132×10^{-5}	2.60600×10^{-5}
0.4	1.59672987	1.59677497	4.51004×10^{-5}	2.82455×10^{-5}
0.5	1.82436063	1.82441227	5.16400×10^{-5}	2.83058×10^{-5}
0.6	2.09327128	2.09332536	5.40872×10^{-5}	2.58386×10^{-5}
0.7	2.40962689	2.40967711	5.02229×10^{-5}	2.08426×10^{-5}
0.8	2.78043274	2.78047172	3.89784×10^{-5}	1.40189×10^{-5}
0.9	3.21364280	3.21366408	2.12832×10^{-5}	6.62279×10^{-6}
1	3.71828182	3.71828182	0	0

Table 2. Absolute error of Equation 11 for $u^{(m)}(x)$:

m	$x = 0.16$	$x = 0.48$	$x = 0.64$	$x = 0.96$
0	2.29071×10^{-5}	5.05821×10^{-5}	5.33973×10^{-5}	8.71294×10^{-6}
1	1.10139×10^{-4}	5.66427×10^{-5}	3.10001×10^{-5}	2.15982×10^{-4}
2	2.71006×10^{-4}	3.58422×10^{-4}	7.08834×10^{-4}	1.41202×10^{-4}
3	2.75029×10^{-3}	2.36821×10^{-3}	1.40319×10^{-3}	3.75497×10^{-3}
4	1.47678×10^{-2}	4.04658×10^{-3}	2.1177×10^{-2}	2.28301×10^{-3}

Example 4.2 Consider the nonlinear IDE

$$\begin{aligned} &u^{(iv)}(x) = 1 + \int_0^x e^{-t} u^2(t) dt, \ 0 < x < 1; \\ &u(0) = 1, \ u''(0) = 1, \ u(1) = e, \ u''(1) = e. \end{aligned} \tag{12}$$

The exact solution is $u(x) = e^x$. Using RKHS method, taking $N = 10$ and $n = 1$ with $x_i = i/N$, $i = 0, 1, ..., N$; the numerical results at some selected gird points are given in Table 3.

Table 3. Numerical results of Equation 12:

x	Analytical solution	Numerical solution	Absolute error	Relative error
0.0	1	1	0	0
0.1	1.10517091	1.10512941	4.15075×10^{-5}	3.75575×10^{-5}
0.2	1.22140275	1.22132182	8.09282×10^{-5}	6.62584×10^{-5}
0.3	1.34985880	1.34974418	1.14620×10^{-4}	8.49124×10^{-5}
0.4	1.49182469	1.49168565	1.39040×10^{-4}	9.32015×10^{-5}
0.5	1.64872127	1.64857014	1.51122×10^{-4}	9.16601×10^{-5}
0.6	1.82211880	1.82197016	1.48633×10^{-4}	8.15715×10^{-5}
0.7	2.01375270	2.01362212	1.30582×10^{-4}	6.48452×10^{-5}
0.8	2.22554092	2.2254433	9.76267×10^{-5}	4.38665×10^{-5}
0.9	2.45960311	2.45955067	5.24392×10^{-5}	2.13201×10^{-5}
1	2.71828182	2.71828182	0	0

5 Conclusion

In this paper,we construct a reproducing kernel space in which each function satisfies boundary value conditions of considered problems. In this space, a numerical algorithm is presented for solving fourth-order integrodifferential equation of Volterra type. The analytical solution is given with series form in $W_2^5[a,b]$. The approximate solution obtained by present algorithm converges to analytical solution uniformly. The numerical results are displayed to demonstrate the validity of the present algorithm.

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