Statistical Type Lebesgue and Riesz Theorems

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Abstract

In the paper the concept of $\mathcal{F}$-convergence, generated by some filter is introduced. The concept of a monotone close and a right filter are also defined. Based on these concepts an analogues of classical theorems of real analysis as Lebesgue, Egorov, Riesz and Fatou, with respect to $\mathcal{F}$-convergence are established. Examples of non-close or non-monotone right filters are given. It is proven that previously known filters (e.g. generated by statistical convergence and etc.) are monotone close and right.

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1 Introduction

The idea of statistical convergence (stat-convergence) was first proposed by A. Zigmund in his famous monograph [30] where he talked about "almost convergence". The first definition of it was given by H. Fast [13] and H.
Steinhaus [29]. Later, this concept has been generalized in many directions. It should be noted that it is impossible to list all the relating papers. More details on this matter and on applications of this concept can be found in [15, 28, 27, 10, 16, 22, 24, 12, 25, 7, 23, 1, 9, 3, 4, 5, 6]. It should be noted that the methods of non-convergent sequences have long been known and they include e.g. Cesaro method, Abel method and etc. These methods are used in different areas of mathematics. For the applicability of these methods is very important that the considered space has a linear structure. Therefore, the study of statistical convergence in metric spaces is of special scientific interest. The works [20, 21] are devoted to different aspects of this problem in metric spaces. Statistical convergence is currently actively used in many areas of mathematics such as summation theory [10, 14, 11], number theory [12, 7], trigonometric series [30], probability theory [16], measure theory [25], optimization [26], approximation theory [17, 18], fuzzy theory [2] and etc.

Statistical convergence was generalized by different mathematicians (see [24, 1, 19, 8]). The concepts of $I$-convergence and $I^*$-convergence were introduced in [19, 9]. These types of convergence include many previously known convergences, including the statistical convergence. In the present paper, we introduce the concept of $\mathcal{F}$-convergence generated by some filter $\mathcal{F} \subset 2^\mathbb{N}$. In connection with this concept the classical theorems of real analysis as Lebesgue, Egorov, Riesz, Fatou theorems are generalized to this case. The concepts of monotone close and right filter are also introduced. Examples for a non-monotone close and a right filter are established. We proved that the previously known standard filters are a monotone close and a right.

2 Needful information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers. $\mathbb{R}$ will be a set of real numbers; $\chi_M(\cdot)$ is the characteristic function of $M$; $(X; \rho)$ will be a metric space. $O_\varepsilon(a)$ will be an open ball centered at $a$ and with radius $\varepsilon$, i.e. $O_\varepsilon(a) \equiv \{ x \in X : \rho(x; a) < \varepsilon \}$. $2^M$ will be a set of all subsets $M$. $|A| = \text{card} A$— will be a number of elements of $A$. $A^C \equiv \mathbb{N} \setminus A$. ⇒ will be a quantifier which means “follows”; ∧ will be a quantifier which means “and”.

Let us recall the definition of the asymptotic (or statistical) density of the set $A \subset \mathbb{N}$. Assume $\delta_n(A) = \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$, and let $\delta_*(A) = \liminf_{n \to \infty} \delta_n(A)$, $\delta^*(A) = \limsup_{n \to \infty} \delta_n(A)$. $\delta_*(A)$ and $\delta^*(A)$ are called lower and upper asymptotic density of $A$, respectively. If $\delta_*(A) = \delta^*(A) = \delta(A)$, then $\delta(A)$ is called asymptotic (or statistical) density of $A$. It should be noted that the statistical convergence is determined by means of this concept, namely, the sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is called statistical convergent to $x$, if $\delta(A_\varepsilon) = 0$, for $\forall \varepsilon > 0$, where $A_\varepsilon \equiv \{ n \in \mathbb{N} : \rho(x_n; x) \geq \varepsilon \}$. 

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Let us also recall the definition of the ideal and the filter.

A family of sets $I \subset 2^N$ is called an ideal if: (a) $\emptyset \in I$; (b) $A; B \in I \Rightarrow A \cup B \in I$; (c) $(A \in I \land B \subset A) \Rightarrow B \in I$.

A family $\mathcal{F} \subset 2^N$ is called a filter on $X$, if:

i) $\emptyset \notin \mathcal{F}$; ii) $A; B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$; iii) $A \in \mathcal{F} \land (A \subset B) \Rightarrow B \in \mathcal{F}$.

Filter, satisfying the condition

iv) If $A_1 \supset A_2 \supset \ldots \supset A_n \in \mathcal{F}, \forall n \in N \Rightarrow \exists \{n_m\}_{m \in N} \subset N; n_1 < n_2 < \ldots : \bigcup_{m=1}^{\infty} (\{n_m, n_{m+1}\} \cap A_m) \in \mathcal{F}$ is called a monotone close filter.

Filter $\mathcal{F}$ satisfying the following condition is called a right filter

v) $F^c \in \mathcal{F}$, for any finite subset $F \subset N$.

An ideal $I$ is called non-trivial if $I \neq \emptyset \land I \neq N$. $I \subset 2^N$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{N \setminus A : A \in I\}$ is a filter on $N$. A non-trivial ideal $I \subset 2^N$ is called admissible if and only if $I \supset \{\{n\} : n \in N\}$.

Let $(X; \rho)$ be some metric space with a metric $\rho$ and $I \subset 2^N$ be some non-trivial ideal.

**Definition 2.1.** [19]. Sequence $\{x_n\}_{n \in N} \subset X$ is called $I$-convergent to $x \in X$ ($I$-lim $x_n = x$), if $A_\varepsilon \in I, \forall \varepsilon > 0$, where $A_\varepsilon = \{n \in N : \rho(x_n; x) \geq \varepsilon\}$.

Let $I_d = \{A \subset N : d(A) = 0\}$. $I_d$ is an ideal on $N$. $I_d$- convergence means the statistical convergence.

It should be noted that if $I$ is an admissible ideal, then the usual convergence in $X$ implies $I$-convergence in $X$.

**Definition 2.2.** Sequence $\{x_n\}_{n \in N} \subset X$ is called $I^*$-convergent to $x \in X$, if $\exists M \in \mathcal{F}(I)$ (i.e. $N \setminus M \in I$), $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} : \lim_{k \to \infty} \rho(x_{m_k}; x) = 0$.

The following interesting results are proved in [19].

**Theorem 2.3.** [19] Let $I$ be an admissible ideal. If $I^*$-lim $x_n = x$ $\Rightarrow$ $I$-lim $x_n = x$.

The converse is not always true, it depends on the structure of space $(X; \rho)$, namely, we have

**Theorem 2.4.** [19] Let $(X; \rho)$ be a metric space. (i) If $X$ has no accumulation point, then $I^*$ and $I^*$-convergence coincide for each admissible ideal $I \subset 2^N$; (ii) If $X$ has an accumulation point $\xi$, then there exists an admissible ideal $I \subset 2^N$ and a sequence $\{y_n\}_{n \in N} \subset X$: $I$-lim $y_n = \xi$, but $I^*$-lim $y_n$ does not exist.

Accept the following
Definition 2.5. Sequence \( \{x_n\}_{n \in \mathbb{N}} \subset X \) is called \( \mathcal{F} \)-convergent to \( x \in X \), if \( A_\varepsilon \in \mathcal{F}, \forall \varepsilon > 0, \) where \( A_\varepsilon \equiv \{n \in \mathbb{N} : \rho(x_n, x) < \varepsilon\} \).

Let \( I \subset 2^\mathbb{N} \) be some non-trivial ideal. Then it is clear that \( \mathcal{F}(I) \equiv \{A \subset \mathbb{N} : A \in I\} \) is a filter. Conversely, if \( \mathcal{F} \subset 2^\mathbb{N} \) is a filter, then \( I(\mathcal{F}) \equiv \{A \subset \mathbb{N} : A \setminus A \in \mathcal{F}\} \) is an ideal. Therefore, the concept of \( I \)-convergence and \( \mathcal{F} \)-convergence are dual. For this reason, in sequel we consider only \( \mathcal{F} \)-convergence.

3 Main results

Let \( (M; \mathcal{M}; \mu) \) be a measurable space with a finite measure \( \mu \) and \( \mathcal{F} \subset 2^\mathbb{N} \) be some filter. Let the sequence \( f_n : M \to \mathbb{R}, \forall n \in \mathbb{N}, \) be given and \( f : M \to \mathbb{R}. \)

Definition 3.1. We say that \( f_n \xrightarrow{\mathcal{F}} f \mu \)-almost everywhere ( in short \( f_n \xrightarrow{\mathcal{F}} f \) \( a.e. \)), if \( \exists E \in \mathcal{M} : \mu(E) = 0 \land f_n(x) \xrightarrow{\mathcal{F}} f, \forall x \in M \setminus E \equiv E^C. \)

It is obvious that if \( f_n \to f \) \( a.e. \), then \( f_n \xrightarrow{\mathcal{F}} f \) \( a.e. \), but the conversely is not true. Indeed, let \( \mathcal{F} \equiv \mathcal{F}_\delta \) ( \( \mathcal{F}_\delta \) means a statistical convergence). Put \( e_0 \equiv \{n^2 : n \in \mathbb{N}\}. \) Let \( f_n(x) = x^n + \chi_{e_0}(n), \forall n \in \mathbb{N}, x \in M, \) where \( M = [0, 1]. \) It is clear that \( \{f_n(x)\}_{n \in \mathbb{N}} \) doesn’t converge for \( \forall x \in M, \) but \( f_n(x) \xrightarrow{\mathcal{F}} 0, \) \( a.e. \).

Also define the concept of \( \mathcal{F} \)-convergence in measure.

Definition 3.2. We say that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) \( \mathcal{F} \)-converges in measure to \( f(x) \) on \( M, \) if \( \mu(\{\{f_n - f\} \geq \sigma\}) \xrightarrow{\mathcal{F}} 0, \forall \sigma > 0, \) and this kind of convergence we will denote as \( f_n \xrightarrow{\mathcal{F}} f \) on \( M. \)

However, if \( f_n \xrightarrow{\mu} f, \) then \( f_n \xrightarrow{\mathcal{F}} f, \) but the vice versa is not true. The previous example is suitable in this case.

Let \( f_n \xrightarrow{\mathcal{F}} f \) \( a.e. \). Put

\[
F_x \equiv \{\{n^2\}_{k \in \mathbb{N}} \in \mathcal{F} : (n_1^2 < n_2^2 < \ldots) \land (f_{n_{k}^2}(x) \to f(x), k \to \infty)\}.
\]

Let \( E \subset M : \mu(E) = 0 \land f_n(x) \xrightarrow{\mathcal{F}} f(x), \forall x \in E^C. \) It is clear that \( F_x \neq \emptyset, \forall \alpha \in E^C. \) Assume \( \bigcap_{x \in E^C} F_x \bigcap \mathcal{F} \neq \emptyset, \) i.e. \( \exists e \equiv \{n_{k}^2\}_{k \in \mathbb{N}} \subset \mathbb{N} : (e \in \bigcap_{x \in E^C} F_x) \land (e \in \mathcal{F}). \) Consider the sequence \( g_k(x) = f_{n_k^2}(x), \forall k \in \mathbb{N}, \forall x \in M. \) It is clear that \( g_k \to f \) \( a.e. \). Then by classical Lebesgue theorem we obtain that \( g_k \xrightarrow{\mu} f, \) i.e. \( f_{n_k^2} \xrightarrow{\mu} f. \) Since, \( e \in \mathcal{F}, \) hence it follows that \( f_n \xrightarrow{\mathcal{F}} f. \) Consequently, the following theorem is true.
Theorem 3.3. Let \( f_n \overset{\mathcal{F}}{\rightarrow} f \) a.e. and \( \bigcap_{x \in E^C} F_x \subset \mathcal{F} \), where \( \mu(E) = 0 \) \( \land f_n(x) \overset{\mathcal{F}}{\rightarrow} f(x) \), \( \forall x \in E^C \). Then \( f_n \overset{\mathcal{F}}{\rightarrow} f \).

It is absolutely clear that the converse statement is generally not true.

Let \( f_n \overset{\mathcal{F}}{\rightarrow} f \) on \( M \). Take \( \forall \sigma > 0 \). Put

\[
e_k(\sigma) \equiv \left\{ n \in N : \mu(\{|f_n - f| \geq \sigma\}) < \frac{1}{k} \right\}, \quad k \in N.
\]

From \( f_n \overset{\mathcal{F}}{\rightarrow} f \Rightarrow e_k(\sigma) \in \mathcal{F}, \forall k \in N \). It is clear that \( e_1(\sigma) \supset e_2(\sigma) \supset \ldots \).

Suppose that \( \mathcal{F} \) is a monotone close filter. Then \( \exists \{n_k^\sigma\}_{k \in N} \subset N, n_1^\sigma < n_2^\sigma < \ldots : \bigcup_{k=1}^{\infty} (n_k^\sigma, n_{k+1}^\sigma) \cap e_k(\sigma) \in \mathcal{F} \). Let

\[
N_0(\sigma) \equiv [1, n_1^\sigma] \bigcup \left\{ k \in N : k \in (n_m^\sigma, n_{m+1}^\sigma] \cap e_m(\sigma), m \in N \right\},
\]

where \( e_m(\sigma) \equiv N \setminus e_m(\sigma) \). Put

\[
g_k^\sigma \equiv \left\{ f, \text{ if } k \in N_0(\sigma), \quad f_k, \text{ if otherwise.} \right\}
\]

Let us show that \( \mu(\{|g_k^\sigma - f| \geq \sigma\}) \rightarrow 0, \quad k \rightarrow \infty \). Take \( \forall \varepsilon > 0 \). Let \( \sigma > 0 \) be an arbitrary number. If \( k \in N_0(\sigma) \Rightarrow g_k^\sigma \equiv f \Rightarrow \mu(\{|g_k^\sigma - f| \geq \sigma\}) = 0 < \varepsilon \). If \( k \notin N_0(\sigma) \), then \( \exists m \in N : n_m^\sigma < k \leq n_{m+1}^\sigma \land k \notin e_m(\sigma) \Rightarrow k \in e_m(\sigma) \Rightarrow \mu(\{|g_k^\sigma - f| \geq \sigma\}) < \frac{1}{m} \) (so \( g_k^\sigma = f_k \Rightarrow \mu(\{|g_k^\sigma - f| \geq \sigma\}) < \varepsilon \), for \( \frac{1}{m} \leq \varepsilon \). Consequently, \( \mu(\{|g_k^\sigma - f| \geq \sigma\}) \rightarrow 0, \quad k \rightarrow \infty \). Let us show that \( g_k^\sigma \overset{\mathcal{F}}{\sim} f_k \), i.e. \( K_\sigma \equiv \{ k \in N : g_k^\sigma = f_k \} \in \mathcal{F} \). Indeed, it is easy to see that

\[
\bigcup_{m=1}^{\infty} (n_m^\sigma, n_{m+1}^\sigma] \cap e_m(\sigma) \subset K_\sigma,
\]

holds. Then from the condition iii) of filter it follows that \( K_\sigma \in \mathcal{F} \). Thus, if \( f_k \overset{\mathcal{F}}{\rightarrow} f \), then for \( \forall \sigma > 0 \), \( \exists \{g_k^\sigma\}_{k \in N} : g_k^\sigma \overset{\mathcal{F}}{\sim} f_k \) and it is true

\[
\lim_{k \to 0} \mu(\{|g_k^\sigma - f| \geq \sigma\}) = 0.
\]  \hspace{1cm} (1)

If \( \mathcal{F} \) is a right filter, then the contrary is also true. So, let \( \mathcal{F} \) be a right filter and the relation (1) holds. Let \( \sigma > 0 \) be an arbitrary number. Consider the sequence \( \{g_k^\sigma\}_{k \in N} \) from the relation (1). Take \( \forall \varepsilon > 0 \). Then

\[
\exists n_\varepsilon \in N : \mu(\{|g_k^\sigma - f| \geq \sigma\}) < \varepsilon, \forall n \geq n_\varepsilon.
\]

We have

\[
\{ n \in N : n \geq n_\varepsilon \} \cap K_\sigma \subset \{ n \in N : \mu(\{|f_n - f| \geq \sigma\}) < \varepsilon \}.
\]
It is clear that \( \{n \in N : n \geq n_\varepsilon \} \cap K_\sigma \in \mathcal{F} \). Then from the condition iii) it follows
\[
\{n \in N : \mu (\{ |f_n - f| \geq \sigma \}) < \varepsilon \} \in \mathcal{F}.
\]
Thus, the following theorem is true.

**Theorem 3.4.** Let \((M; \mathcal{M}; \mu)\) be a measurable space with a finite measure \(\mu\), and \(\mathcal{F} \subset 2^N\) be a monotone close filter. Then the following statements are equivalent to each other.

\[
\alpha) \ f_n \mathcal{F} \rightarrow f; \quad \beta) \ \forall \sigma > 0, \exists \{g_k^\sigma\}_{k \in N} : g_k^\sigma \mathcal{F} f_k \wedge \lim_{k \to \infty} \mu \left(\{ |g_k^\sigma - f| \geq \sigma \} \right) = 0.
\]

An analogue of the Riesz theorem is also true with respect to \(\mathcal{F}\)-convergence. Put \(f_n \mathcal{F} \rightarrow f\). Assume that \(\mathcal{F}\) is a right filter. Take \(\sigma_1 > \sigma_2 > \ldots : \sigma_k \to 0, \ k \to \infty\). Let \(\sum_{k=1}^{\infty} \eta_k < +\infty, \eta_k > 0, \forall k \in N\). So
\[
\{n : \mu (\{ |f_n - f| \geq \sigma_1 \}) < \eta_1 \} \in \mathcal{F},
\]
then \(\exists n_1 \in N : \mu (\{ |f_{n_1} - f| \geq \sigma_1 \}) < \eta_1\).

Since \(\mathcal{F}\) is a right filter, then it is clear that
\[
[1, n_1] \setminus \bigcap \{n \in N : \mu (\{ |f_n - f| \geq \sigma_2 \}) < \eta_2 \} \neq \emptyset.
\]
So, \(\exists n_2 > n_1 : \mu (\{ |f_{n_2} - f| \geq \sigma_2 \}) < \eta_2\).

Continuing in the same way, we obtain a sequence \(\{n_k\}_{k \in N} \subset N : n_k > n_{k-1}\) and
\[
\mu (\{ |f_{n_k} - f| \geq \sigma_k \}) < \eta_k.
\]
The validity of \(f_{n_k} \to f\) a.e. can be proven similarly to the classical Riesz theorem. Thus, we have proved the following theorem.

**Theorem 3.5.** Let \((M; \mathcal{M}; \mu)\) be a measurable space with a finite measure \(\mu\), and \(\mathcal{F} \subset 2^N\) be a right filter. If \(f_n \mathcal{F} \rightarrow f\), then \(\exists \{n_k\} \subset N : f_{n_k} \to f\) a.e. on \(M\).

Egorov theorem is also valid.

**Theorem 3.6.** Let \((M; \mathcal{M}; \mu)\) be a measurable space with a finite measure \(\mu\), and \(\mathcal{F} \subset 2^N\) be some filter. If \(f_n \mathcal{F} f\) a.e. on \(M\) and \(\bigcap_{x \in E_C} F_x \cap \mathcal{F} \neq \emptyset\) (\(E\) is from Definition 3.1), then for \(\forall \delta > 0, \exists M_\delta \in \mathcal{M} : 1) \mu (M \setminus M_\delta) < \delta; 2) \sup_{x \in M_\delta} |f_n(x) - f(x)| \mathcal{F} 0\).
Indeed, let $f_n \overset{\mathcal{F}}{\to} f$ a.e. on $M$ and $\bigcap_{x \in E^C} F_x \cap \mathcal{F} \neq \emptyset$. Put $e \in \bigcap_{x \in E^C} F_x \cap \mathcal{F}$: $e \equiv \{n_1 < n_2 < \ldots \}$. So, $f_{n_k} \to f$, $k \to \infty$, a.e. on $M$. Further reasoning directly follows from Egorov’s theorem.

Accept the following

**Definition 3.7.** Let $(M; \mathcal{M}; \mu)$ be a measurable space with a finite measure $\mu$. We say that the sequence $\{f_n\}_{n \in N}$ is $\mathcal{F}$-bounded by the function $g(x)$ a.e. on $M$, if $\exists E \in \mathcal{M}$, $\mu(E) = 0$:

$$\{n \in N : |f_n(x)| \leq g(x), \forall x \in E^C\} \in \mathcal{F}.$$ 

Function $g$ is called $\mathcal{F}$-majorant of the sequence $\{f_n\}_{n \in N}$. Let $f_n \overset{\mathcal{F}}{\to} \mu f$ and integrable function $g$ be $\mathcal{F}$-majorant of $\{f_n\}_{n \in N}$, where $\mathcal{F} \subset 2^N$ be some right filter.

Let

$$e_1 \equiv \{n \in N : |f_n(x)| \leq g(x), \forall x \in E^C\},$$

where $E \in \mathcal{M}$ is a set from the Definition 3.7. So, $e_1 \in \mathcal{F}$. Let $e_1 \equiv \{n_1 < n_2 < \ldots \}$. Consider $\tilde{f}_k = f_{n_k}$, $\forall k \in N$. Consequently

$$|\tilde{f}_n(x)| \leq g(x), \forall x \in E^C, \forall n \in N. \quad (2)$$

It is not difficult to see that $\tilde{f}_n \overset{\mathcal{F}}{\to} \mu f$. Indeed, take $\forall \sigma > 0$, and let

$$e_k(\sigma) \equiv \{n \in N : \mu(\{|f_n - f| \geq \sigma\}) < \frac{1}{k}\}.$$ 

It is clear that $e_k(\sigma) \in \mathcal{F}$, $\forall k \in N$. Therefore $(e_1 \cap e_k(\sigma)) \in \mathcal{F}$, $\forall k \in N$. As a result, $\tilde{e}_k(\sigma) \in \mathcal{F}$, $\forall k \in N$, where

$$\tilde{e}_k(\sigma) \equiv \{n \in N : \mu(\{|\tilde{f}_n - f| \geq \sigma\}) < \frac{1}{k}\}.$$ 

Then by Theorem 3.5 $\exists \{\tilde{n}_k\}_{k \in N} \subset N : \tilde{f}_{\tilde{n}_k} \to f$, $k \to \infty$, a.e. on $M$. As a result, from (2) we obtain $|f(x)| \leq g(x)$ a.e. on $M$, and as a result, $f$ is an integrable function on $M$. Take $\forall \sigma > 0$, and denote

$$E_n(\sigma) \equiv \{|\tilde{f}_n - f| \geq \sigma\}, \quad F_n(\sigma) \equiv \{|\tilde{f}_n - f| < \sigma\}.$$ 

We have

$$\delta_n = \left| \int_M \tilde{f}_n d\mu - \int_M f d\mu \right| \leq \int_{E_n(\sigma)} |\tilde{f}_n - f| d\mu + \int_{F_n(\sigma)} |\tilde{f}_n - f| d\mu, \forall n \in N.$$
Taking into account (2), we have
\[ \delta_n \leq 2 \int_{E_n(\sigma)} g \, d\mu + \sigma \mu(M), \forall n \in \mathbb{N}. \]

Let \( \varepsilon > 0 \) be an arbitrary number. Let \( \delta > 0 \) such that from \( \mu(\tilde{E}) < \delta \Rightarrow \int_{E} g \, d\mu < \frac{\varepsilon}{4} \). The choice of \( \delta \) is always possible (by virtue of the absolutely continuity of the Lebesgue integral. As \( \sigma \) we take \( \sigma = \frac{\varepsilon}{2\mu(M)} \). Let \( k_0 \in \mathbb{N} : \frac{1}{k_0} < \delta \). If \( n \in e_{k_0}(\sigma) \), then it is clear that \( \mu(E_n(\sigma)) < \frac{1}{k_0} < \delta \Rightarrow \int_{E_n(\sigma)} g \, d\mu < \frac{\varepsilon}{4} \). Consequently
\[ \delta_n \leq \varepsilon, \forall n \in e_{k_0}(\sigma) \in \mathscr{F}, \ldots \delta_n \mathscr{F} \rightarrow 0. \]
So, it is valid

**Theorem 3.8.** Let \((M; \mathcal{M}; \mu)\) be a measurable space with a finite measure \( \mu \), and \( \mathcal{F} \subset 2^{\mathbb{N}} \) be some right filter. Let the sequence of integrable functions \( \{f_n(x)\}_{n \in \mathbb{N}} \) have the integrable \( \mathcal{F} \)-mojarant \( g(x) \) and \( f_n \mathscr{F} \rightarrow_{\mu} f \). Then the function \( f \) is also integrable and
\[ \int_M f_n(x) \, dx \mathscr{F} \rightarrow \int_M f(x) \, dx, \]
holds.

This theorem is an analogue of the Lebesgue theorem on passage to the limit. Note that Fatou’s lemma also has an analogue with respect to the filter. Namely, we have

**Theorem 3.9.** Let \( f_n \mathscr{F} \rightarrow_{\mu} f \), where \( \mathcal{F} \subset 2^{\mathbb{N}} \) be some right filter. Then the inequality
\[ \int_M f \, d\mu \leq \sup_n \int_M f_n \, d\mu, \]
holds.

Indeed, let \( f_n \mathscr{F} \rightarrow_{\mu} f \), and \( \mathcal{F} \) be a right filter. Then by Theorem 3.5 \( \exists \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} : f_n \rightarrow f, k \rightarrow \infty, \) a.e. on \( M \). Rest of the proof is established similarly to the classical case.

### 4 On filters

**I. Non-right filter.** Let \( e_0 \subset N \land e_0 \neq N \). Assume
\[ \mathcal{F}_{e_0} \equiv \{e \subset N : e_0 \subset e\}. \]
It is not difficult to see that $\mathcal{F}_{e_0}$ is a non-right filter.

**II. Non-monotone close filter.** Let $A_k \equiv \{ n 2^k : n \in N \}$, $\forall k \in N$. Assume

$$\mathcal{F} \equiv \{ M \subset 2^N : \exists k \in N \Rightarrow A_k \subset M \}.$$ 

It is clear that $\emptyset \notin \mathcal{F}$. Put $A; B \in \mathcal{F} \Rightarrow \exists k_1; k_2 \in N : (A_{k_1} \subset A) \land (A_{k_2} \subset B)$. Let $k_0 = \max \{ k_1; k_2 \}$. It is obvious that $(A \cap B) \supset A_{k_0}$, i.e. the condition ii) of the filter satisfies. Let $(A \in \mathcal{F}) \land (A \subset B)$. Consequently, $\exists k_0 \in N : A_{k_0} \subset A \Rightarrow A_{k_0} \subset B \Rightarrow B \in \mathcal{F}$. Thus, $\mathcal{F}$ is a filter. Let us show that $\mathcal{F}$ is a non-monotone close filter. It is clear that $A_1 \supset A_2 \supset \ldots$. Let $\exists n_k; n_1 < n_2 < \ldots : \bigcup_{m=1}^{\infty} ((n_m, n_{m+1}] \cap A_m) \in \mathcal{F}$. Consequently, $\exists p \in N : A_p \subset \bigcup_{m=1}^{\infty} ((n_m, n_{m+1}] \cap A_m)$. Put $k_0 \in N : 2k_0 + 1 > n_{p+1}$. It is easy to see that $(2k_0 + 1) 2^p \notin A_k$, $\forall k > p \Rightarrow (2k_0 + 1) 2^p \notin \bigcup_{m=1}^{\infty} ((n_m, n_{m+1}] \cap A_m)$. The obtained contradiction shows that $\mathcal{F}$ is a non-monotone close filter.

**III. An ordinary convergence.** $\mathcal{F} \equiv \{ M \subset N : M^c \equiv N \setminus M \text{ is a finite set} \}$. $\mathcal{F}$-convergence, generated by this filter, coincides with the ordinary convergence.

**IV. Statistical convergence.** Assume $\mathcal{F}_\delta \equiv \{ M \subset N : \delta (M) = 1 \}$. $\mathcal{F}_\delta$ is a filter. It is not difficult to see that $\mathcal{F}_\delta$ is a right filter. Let us show that $\mathcal{F}_\delta$ is a monotone close filter. Let $K_1 \supset K_2 \supset \ldots \land (\delta (K_n) = 1, \forall n \in N)$. Obviously, $\delta (K_n^c) = 0, \forall n \in N$. Therefore $\exists \{ n_k \}_{k \in N} \subset N; n_1 < n_2 < \ldots$:

$$\frac{1}{n} \left| I_n \cap K_m^c \right| < \frac{1}{m}, \forall n \geq n_m.$$ 

Assume $N_0 = \tilde{N}_0 \cup I_n$, where $\tilde{N}_0 \equiv \{ k \in N : n_m < k \leq n_{m+1} \land (k \in K_m^c) \}$. It is clear that $\delta (N_0) = \delta (\tilde{N}_0)$. Take $\forall n \in N$. Then $\exists m \in N : n_m < n \leq n_{m+1}$. Without loss of generality, we will assume that $n > n_1$. Let us show that

$$\left( I_n \cap \tilde{N}_0 \right) \subset \left( I_n \cap K_m^c \right).$$

Let $k \in \left( I_n \cap \tilde{N}_0 \right) \Rightarrow \exists m_0 \leq m : n_{m_0} < k \leq n_{m_0+1} \land (k \in K_m^c) \Rightarrow k \in K_m^c$. Thus, the inclusion (3) is true. Consequently

$$\frac{1}{n} \left| I_n \cap \tilde{N}_0 \right| \leq \frac{1}{n} \left| I_n \cap K_m^c \right| < \frac{1}{m}.$$

From (4) it directly follows that $\delta (\tilde{N}_0) = 0$, as a result, $\delta (N_0) = 0 \Rightarrow \delta (N_0^c) = 1 \Rightarrow N_0^c \in \mathcal{F}_\delta$. In the sequel, it should be pointed out $N_0^c = \{ k \in N : n_m < k \leq n_{m+1} \land (k \in K_m) \}$. Thus, $\mathcal{F}_\delta$ is a monotone close filter. That the $\mathcal{F}_\delta$ satisfies the condition v) is obvious. Then, with respect to $\mathcal{F}_\delta$-convergence is true the following.
Statement 4.1. Filter $\mathcal{F}_\delta$, generated by statistical density, is a monotone close and right filter.

V. Logarithmic convergence. Let $M \subset N$. Assume

$$l_n (M) = \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_M (k)}{k},$$

where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. If $\exists \lim_{n \to \infty} l_n (M) = l (M)$, then $l (M)$ is called a logarithmic density of the set $M$. Let $\mathcal{F}_l \equiv \{ M \subset N : l (M) = 1 \}$. The following lemma is true.

Lemma 4.2. If $l (M_k) = 1$, $k = 1, 2 \Rightarrow l (M_1 \cap M_2) = 1$.

Proof. We have

$$M_1 \cap M_2 = \left( M_1 \cup M_2 \right) \setminus \left[ (M_2 \setminus M_1) \cup (M_1 \setminus M_2) \right].$$

Consequently

$$M_1 \cap M_2 \cap I_n = \left[ \left( M_1 \cup M_2 \right) \cap I_n \right] \setminus \left[ \left( (M_2 \setminus M_1) \cup (M_1 \setminus M_2) \right) \cap I_n \right]. \tag{5}$$

From $((M_2 \setminus M_1) \cap I_n) \subset (M_1^c \cap I_n)$, we get

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_2 \setminus M_1} (k) \leq \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1^c} (k). \tag{6}$$

It is absolutely clear that, if $l (M) = 1$, then $l (M^c) = 0$. Then from (6) we obtain $l (M_2 \setminus M_1) = 0$. Similarly, we have $l (M_1 \setminus M_2) = 0$. So

$$\left( (M_2 \setminus M_1) \cup (M_1 \setminus M_2) \right) \cap I_n = \left( (M_2 \setminus M_1) \cap I_n \right) \cup \left( (M_1 \setminus M_2) \cap I_n \right),$$

it is clear that

$$l \left( (M_2 \setminus M_1) \cup (M_1 \setminus M_2) \right) = 0. \tag{7}$$

It is easy to see that $l (M_1 \cup M_2) = 1$. From (5) we get

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1 \cap M_2} (k) = \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1 \cup M_2} (k) - \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{(M_2 \setminus M_1) \cup (M_1 \setminus M_2)} (k).$$

Taking into account (7) we get $l (M_1 \cap M_2) = 1$. \qed
Hence that $F$ is a non-trivial ideal [11]. Therefore, $\mathcal{F}_t$ is a filter. Let $K_1 \supset K_2 \supset \ldots \land (l(K_n) = 1, \forall n \in N) \Rightarrow l(K_m^c) = 0, \forall n \in N$. Therefore, $\exists \{n_k\}_{k \in N} \subset N, n_1 < n_2 < \ldots : \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{K_n^c}(k)}{k} < \frac{1}{m}, \forall n \geq n_m$. Similarly to the previous example, let $N_0 = \bar{N}_0 \cup I_n$, where

$$\bar{N}_0 \equiv \{k \in N : n_m \leq k \leq n_{m+1} \land (k \in K_m^c)\}.$$  

It is clear that $l(N_0) = l(\bar{N}_0)$. Let $n \in N \Rightarrow \exists m \in N : n_m < n \leq n_{m+1}$. As before, we assume that $n > n_1$. It is clear that, (3) is true, i.e.

$$\left(I_n \cap \bar{N}_0\right) \subset \left(I_n \cap K_m^c\right).$$

Hence

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{\bar{N}_0^c}(k)}{k} \leq \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{K_m^c}(k)}{k} < \frac{1}{m}, \forall n \geq n_m.$$  

Consequently, $l(\bar{N}_0) = 0 \Rightarrow l(N_0) = 0 \Rightarrow l(N_0^c) = 1 \Rightarrow N_0^c \in \mathcal{F}_t$. It is clear that

$$N_0^c \equiv \{k \in N : n_m < k \leq n_{m+1} \land (k \in K_m)\}.$$  

It directly follows that $\mathcal{F}_t$ is a right filter. Should pay attention to that, if $\exists \delta (M) \Rightarrow \exists l(M) \land l(M) = \delta (M)$. The converse is not generally true.

**Statement 4.3.** Filter $\mathcal{F}_t$, generated by logarithmic density, is a monotone close and right filter.

**VI. Uniform convergence.** Let $M \subset N \land (t \in Z_+; s \in N)$. Assume

$$M(t+1;t+s) = |n \in M : t+1 \leq n \leq t+s|.$$  

Let

$$\beta_s(M) = \lim \inf_{t \to \infty} M(t+1;t+s),$$  

$$\beta^s(M) = \lim \sup_{t \to \infty} M(t+1;t+s).$$

If $\lim_{s \to \infty} \frac{\beta_s(M)}{s} = \lim_{s \to \infty} \frac{\beta^s(M)}{s} = \beta(M)$, then the quantity $\beta(M)$ is called the uniform density of the set $M$. Put $\mathcal{F}_\beta \equiv \{M \subset N : \beta(M) = 1 \}$. Let us show that $\mathcal{F}_\beta$ is a filter. It is clear that

$$M(t+1;t+s) + M^c(t+1;t+s) = |[t+1,t+s]| = s.$$  

Hence it directly follows that $\beta(M) = 1 \Leftrightarrow \beta(M^c) = 0$. $I_\beta \equiv \{M \subset N : \beta(M) = 0\}$ is a non-trivial ideal [11]. Therefore, $\mathcal{F}_\beta$ is a filter. It is clear
that $F_\beta$ satisfies the condition v). Let us show that $F_\beta$ is a monotone close filter. Let $K_1 \supset K_2 \supset \ldots \land (\beta (K_n) = 1, \forall n \in N) \Rightarrow \beta (K_m^c) = 0, \forall n \in N \Rightarrow \exists \{n_k\}_{k \in N} \subset N, n_1 < n_2 < \ldots :

\frac{\beta^s (K_m^c)}{s} < \frac{1}{m}, \forall s \geq n_m.

As before, we set $N_0 = \tilde{N}_0 \cup I_{n_1}$, where $\tilde{N}_0 \equiv \{k \in N : n_m \leq k \leq n_{m+1} \land (k \in K_m^c)\}$. It is clear that $\beta (N_0) = \beta (\tilde{N}_0)$. Let $n > n_1$ be an arbitrary integer. Then $\exists m \in N : n_m < n \leq n_{m+1}$. It is obvious that the inclusion

$$\left( I_n \cap \tilde{N}_0 \right) \subset \left( I_n \cap K_m^c \right),$$

in this case is also true. From the arbitrariness of $n \in N$ we have

$$\left( \tilde{N}_0 \cap [t+1;t+s] \right) \subset \left( K_m^c \cap [t+1;t+s] \right).$$

Consequently

$$\tilde{N}_0 (t+1;t+s) \leq K_m^c (t+1;t+s),$$

and, as a result

$$\beta^s (\tilde{N}_0) \leq \beta^s (K_m^c).$$

Thus

$$\frac{\beta^s (\tilde{N}_0)}{s} \leq \frac{\beta^s (K_m^c)}{s} < \frac{1}{m}, \forall s \geq n_m.$$

From this relation it directly follows

$$\beta (\tilde{N}_0) = 0 \Rightarrow \beta (N_0) = 0 \Rightarrow \beta (N_0^c) = 1 \Rightarrow N_0^c \in F_\beta,$$

where

$$N_0^c \equiv \{k \in N : n_m < k \leq n_{m+1} \land (k \in K_m)\},$$

i.e. $F_\beta$ is a monotone close filter.

**Statement 4.4.** Filter $F_\beta$, generated by the uniform convergence, is a monotone close and right filter.

Following [19], number of such examples can be extended.

**Remark 4.5.** Similar results can be obtained with respect to concepts of $I$-convergence and $I^*$-convergence.

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References


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