Some Identities of Symmetry for Carlitz’s Twisted $q$-Euler Polynomials Associated with $p$-Adic $q$-Integral on $\mathbb{Z}_p$

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Abstract
In this paper, we establish some interesting symmetric identities for Carlitz’s twisted $q$-Bernoulli polynomials in $p$-adic field. Some interesting results and relationships are obtained.

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1 Introduction

The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Many mathematicians have studied in the area of the $q$-Euler numbers and polynomials(see [1-10]). Recently, Y. Hu studied several identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim et al.[3] derived some identities of symmetry for Carlitz’s $q$-Euler numbers and polynomials in complex field. In this paper, we establish some interesting symmetric identities for Carlitz’s twisted $q$-Euler polynomials in $p$-adic field. Throughout this
paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (\text{cf. [1-5]}).$$

Hence, $\lim_{q \to 1}[x] = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. For $g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$.

For $g \in UD(\mathbb{Z}_p)$ the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see } [2, 3]. \quad (1.1)$$

Let

$$T_p = \bigcup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},$$

where $C_{p^m} = \{ \zeta | \zeta^{p^m} = 1 \}$ is the cyclic group of order $p^m$. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \zeta^x$ (see [4]).

## 2 Symmetric identities for Carlitz’s twisted $q$-Euler numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz’s twisted $q$-Euler numbers $E_{n,q,\zeta}$ and polynomials $E_{n,q,\zeta}(x)$. For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$, twisted $q$-Euler numbers $E_{n,q,\zeta}$ are defined by

$$E_{n,q,\zeta} = \int_{\mathbb{Z}_p} \phi_\zeta(x)[x]_q^n d\mu_{-q}(x). \quad (2.1)$$

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we have

$$\int_{\mathbb{Z}_p} \phi_\zeta(x)[x]_q^n d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} \zeta^x[x]_q^n (-q)^x$$

$$= [2]_q \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + \zeta q^{1+l}} \quad (2.2)$$
By (2.1), we have

\[ E_{n,q,\zeta} = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + \zeta q^{1+l}}. \]

We set

\[ F_{q,\zeta}(t) = \sum_{n=0}^{\infty} E_{n,q,\zeta} \frac{t^n}{n!}. \]

By using above equation and (2.2), we have

\[ F_{q,\zeta}(t) = \sum_{n=0}^{\infty} E_{n,q,\zeta} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \zeta^x e^{[x]_q t} d\mu_{-q}(x) \quad (2.3) \]

Next, we introduce twisted $q$-Euler polynomials $E_{n,q,\zeta}(x)$. The twisted $q$-Euler polynomials $E_{n,q,\zeta}(x)$ are defined by

\[ E_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)[x+y]_q^n d\mu_{-q}(y). \]

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain

\[ E_{n,q,\zeta}(x) = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1 + \zeta q^{1+l}}. \quad (2.4) \]

We set

\[ F_{q,\zeta}(t, x) = \sum_{n=0}^{\infty} E_{n,q,\zeta}(x) \frac{t^n}{n!}. \]

Since $[x+y]_q = [x]_q + q^x [y]_q$, we easily obtain that

\[ E_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)[x+y]_q^n d\mu_{-q}(y) \]

\[ = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} E_{l,q,\zeta} \]

\[ = ([x]_q + q^x E_{q,\zeta})^n \]
Let \( w_1 \) and \( w_2 \) be odd numbers. Then we have

\[
\int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]q^t} d\mu_{-q^{w_1}}(y)
= \lim_{N \to \infty} \frac{1}{[p^N]_{-q^{w_1}}} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 y]q^t} \zeta^{w_1 y} (-q)^{w_1 y}
= \lim_{N \to \infty} \frac{1}{[p^N]_{-q^{w_1}}} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 y]q^t} \zeta^{w_1 y} q^{w_1 y} (-1)^{w_1 y}
= \lim_{N \to \infty} \frac{[2]_{q^{w_1}}}{2} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 i + w_1 w_2 y]q^t} \times \zeta^{w_2 j} \zeta^{w_1 i} q^{w_1 y} (-1)^{i+j} (-1)^y
\]

From (2.6), we can derive the following equation (2.7):

\[
\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]q^t} d\mu_{-q^{w_1}}(y) =
\]

\[
\lim_{N \to \infty} \frac{[2]_{q^{w_1}}}{2} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 i + w_1 w_2 y]q^t} \times \zeta^{w_2 j} \zeta^{w_1 i} q^{w_1 y} (-1)^{i+j} (-1)^y
\]

By the same method as (2.7), we have

\[
\sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]q^t} d\mu_{-q^{w_2}}(y) =
\]

\[
\lim_{N \to \infty} \frac{[2]_{q^{w_2}}}{2} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_1 j + w_1 i + w_1 w_2 y]q^t} \times \zeta^{w_1 j} \zeta^{w_2 i} q^{w_1 y} (-1)^{i+j} (-1)^y
\]

Therefore, by (2.7) and (2.8), we have the following theorem.

**Theorem 2.1** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 0 \) (mod 2), \( w_2 \equiv 0 \) (mod 2), we have

\[
\frac{2}{[2]_{q^{w_1}}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]q^t} d\mu_{-q^{w_1}}(y)
= \frac{2}{[2]_{q^{w_2}}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]q^t} d\mu_{-q^{w_2}}(y).
\]
By substituting Taylor series of $e^{xt}$ into (2.9) and after elementary calculations, we have the following corollary.

**Corollary 2.2** For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 0 \pmod{2}$, $w_2 \equiv 0 \pmod{2}$, we have

$$
[2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} j \int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]_{q^{w_1}}^n d\mu_{-q^{w_1}}(y) = [2]_{q^{w_2}}[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} j \int_{\mathbb{Z}_p} \zeta^{w_2 y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]_{q^{w_2}}^n d\mu_{-q^{w_2}}(y).
$$

By (2.5) and Corollary 2.2, we have the following theorem.

**Theorem 2.3** For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 0 \pmod{2}$, $w_2 \equiv 0 \pmod{2}$, we have

$$
[2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} E_{n, q^{w_1}, \zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right) = [2]_{q^{w_2}}[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} E_{n, q^{w_2}, \zeta^{w_2}} \left( w_1 x + \frac{w_1}{w_2} j \right).
$$

By (2.5), we can derive the following equation (2.10):

$$
\int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]_{q^{w_1}}^n d\mu_{-q^{w_1}}(y) = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_2]_q}{[w_1]_q} \right)^i \left[ j \right]_{q^{w_2(n-i)j}} \int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + y \right]_{q^{w_1}}^{n-i} d\mu_{-q^{w_1}}(y) \quad (2.10)
$$

$$
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_2]_q}{[w_1]_q} \right)^i \left[ j \right]_{q^{w_2(n-i)j}} E_{n-i, q^{w_1}, \zeta^{w_1}} (w_2 x).
$$

By (2.10) and Theorem 2.3, we have

$$
[2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} j \int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]_{q^{w_1}}^n d\mu_{-q^{w_1}}(y) = [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} \left[ w_2 \right]_{q^{w_2}} \left[ w_1 \right]_{q^{w_2}}^{n-i} E_{n-i, q^{w_1}, \zeta^{w_1}} (w_2 x) \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2(n-i)j} \left[ j \right]_{q^{w_2}}^{i}
$$

$$
= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} \left[ w_2 \right]_{q^{w_2}} \left[ w_1 \right]_{q^{w_2}}^{n-i} E_{n-i, q^{w_1}, \zeta^{w_1}} (w_2 x) S_{n,i}(w_1, \zeta^{w_2}, q^{w_2}).
$$

(2.11)
where
\[ S_{n,i}(w_1, \zeta, q) = \sum_{j=0}^{w_1-1} (-1)^j \zeta^j q^{(n-i+1)j} [j]_q^i. \]

By the same method as (2.11), we get
\[
[2]_{q=1} [w_2]_q^{w_2-1} \sum_{j=0}^{w_2-1} (-1)^j \zeta^w_1 q^{w_1j} \int_{\mathbb{Z}_p} \zeta^{w_2y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]_q^n d\mu_{q^{w_2}}(y)
= [2]_{q=1} [w_1]_q^n [w_2]_q^{n-i} E_{n-i,q^{w_2},\zeta^{-2w}} (w_1 x) S_{n,i}(w_2, \zeta^{w_1}, q^{w_1}).
\]

By (2.11) and (2.12), we have the following theorem.

**Theorem 2.4** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 0 \pmod{2} \), \( w_2 \equiv 0 \pmod{2} \), we have
\[
[2]_{q=2} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]_q^i [w_1]_q^{n-i} E_{n-i,q^{w_1},\zeta^{w_1}} (w_2 x) S_{n,i}(w_1, \zeta^{w_2}, q^{w_2})
= [2]_{q=1} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_1},\zeta^{w_2}} (w_1 x) S_{n,i}(w_2, \zeta^{w_1}, q^{w_1}).
\]

By (2.5) and Theorem 2.4, we have the following corollary.

**Corollary 2.5** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 0 \pmod{2} \), \( w_2 \equiv 0 \pmod{2} \), we have
\[
\sum_{i=0}^{n} \sum_{l=0}^{n-i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} n-i-l \\ l \end{array} \right) [2]_{q=2} [w_2]_q^i [w_1]_{q^{w_1}}^{n-i-l} [w_1]_q^{n-i} E_{l,q^{w_1},\zeta^{w_1}}
\times S_{n,i}(w_1, \zeta^{w_2}, q^{w_2}) q^{w_1 w_2 x} [x]_{q^{w_1 w_2}}^{n-i-l}
= \sum_{i=0}^{n} \sum_{l=0}^{n-i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} n-i-l \\ l \end{array} \right) [2]_{q=1} [w_1]_q^i [w_1]_{q^{w_2}}^{n-i-l} [w_2]_q^{n-i} E_{l,q^{w_2},\zeta^{w_2}}
\times S_{n,i}(w_2, \zeta^{w_1}, q^{w_1}) q^{w_1 w_2 x} [x]_{q^{w_1 w_2}}^{n-i-l}.
\]

**References**


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