Travelling Waves for a Generalized Symmetric Regularized-Long-Wave Model

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1 Introduction

In this paper we study the existence of travelling wave solutions for the generalized nonlinear model

\[
\begin{align*}
    u_t - u_{xxt} + u^p u_x - v_x &= 0, \\
    v_t - u_x &= 0,
\end{align*}
\]  

(1)
when \( p = \frac{p_1}{p_2} \geq 1 \), \((p_1, p_2) = 1\) and \( p_2 \) odd. Here, \( u, v \) are real-valued functions and the subscripts denote the derivative with respect to the spatial variable \( x \) and the time \( t \). If \( p = 1 \) the Symmetric Regularized-Long-Wave system (1) is a model for the weakly nonlinear ion acoustic and space-charge waves, where \( u \) and \( v \) are the dimensionless fluid velocity and electron charge density, respectively. This system was introduced by C. Seyler and D. Fenstermacher in [6], where a weakly nonlinear analysis of the cold-electron fluid equation is made.

For equations that model the evolution of nonlinear waves, it is very important to determine the existence and uniqueness of solution for the associated initial value problem, and the existence of special solutions as the travelling waves. For instance, travelling wave solutions are important in the study of dynamics of wave propagation in many applied models such as fluid dynamics, acoustic, oceanography, and weather forecasting. An important application is the use of solitons (travelling wave of finite energy) as an efficient means of long-distance communication.

In the work [3], using semigroup theory, L. Chen showed that the initial value problem associated to the system (1) is global well-posed in the Sobolev type space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \). Later C. Banquet in [1] improved this result, establishing the global well-posedness in the space \( H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) for \( s \geq 0 \). The more important ingredient here is a bilinear estimate obtained by J. L. Bona and N. Tzvetkov in [2]. On the periodic case, C. Banquet in [2], using an estimate obtained by D. Roumégoux in [9], proved that system (1) with initial data \((u_0, v_0) \in H^s_{\text{per}}([-L, L]) \times H^{s-1}_{\text{per}}([-L, L])\) is globally well-posed if \( s \geq 0 \).

In terms of travelling wave solutions for the model (1). This is, solutions of the form

\[
  u(x, t) = \phi(x - ct), \quad v(x, t) = \psi(x - ct).
\]

It is well known the existence for \( p = 1 \) and wave speed \( c > 1 \). The stability for these solitary waves was established by L. Chen in [3], using the approach of Grillakis, Shatah and Strauss. In addition, C. Banquet showed the existence of periodic travelling waves. For \( L > 2\pi \) and \( c > \frac{L}{\sqrt{L^2-4\pi^2}} \), he proved the existence of a smooth curve of \( L \)-periodic travelling wave solutions for (1), where \( L \) is fixed ahead on time.

In this paper, using a variational approach, we show the existence of travelling wave solutions for the model (1), when \( p \geq 1 \) and \( |c| > 1 \). We consider a suitable minimization problem and characterize travelling wave for the equation (1) as multiples of the minimizers to this minimization problem. Using Lions’ Concentration-Compactness Principle, we prove that any minimizing sequence converges strongly, after an appropriate translation, to a minimizer.

It is worth to note that various methods have been used to obtain travelling
wave solutions (periodic and continuous) for the Symmetric Regularized-Long-Wave system and its generalizations, see for instance X. Fei [5], Y. Chen and B. Li [4] and W. Zhang [10]. Of course the methods used for these authors to construct the solutions, for the SRLW model, are completely different than ours.

We recall that one of the main features of wave models is that they come equipped with a Hamiltonian structure. As a principle, existence results of solutions for these models follow from the existence of conserved quantities and the use of energy estimates. It is also a fact that the natural space in which to consider the existence results of travelling wave solutions is dictated by the definition of either the Hamiltonian or the energy. For our particular problem we distinguish the following conservation laws

\[ E(u,v) = \frac{1}{2} \int_{\mathbb{R}} \left( u^2 + u_x^2 + v^2 \right) dx, \quad V(u,v) = \int_{\mathbb{R}} \left[ uv - \frac{1}{(p+2)(p+1)} u^{p+1} \right] dx. \]

Then the energy \( E \) and the Hamiltonian \( H = E + V \) are conserved quantities. In addition, we see directly that the these functionals are well defined for \( u \in H^1(\mathbb{R}) \) and \( v \in L^2(\mathbb{R}) \). So, we study the existence of travelling waves in the energy space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \). The following theorem is our main result.

**Theorem 1.1.** Let \( |c| > 1 \). Then the generalized model \((1)\) admits non-trivial traveling wave solutions, \((u(x,t), v(x,t) = (\phi(x - ct), \psi(x - ct)))\), in the energy space \( X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \).

This paper is organized as follows. In Section 2, we characterize travelling wave solutions for the model \((1)\) as multiples of the minimizers to a variational problem. In Section 3, we prove the existence of such minimizers by using the Concentration-Compactness Theorem. Throughout this work \( C \) denotes a generic constant whose value may change from instance to instance.

## 2 Variational Preliminaries

By a travelling wave solution we shall mean a solution \((u, v)\) for the system \((1)\) of the form

\[ u(x,t) = \phi(x - ct), \quad v(x,t) = \psi(x - ct). \]

Then, one sees that \((\phi, \psi)\) must satisfy

\[ \begin{cases} \vphantom{\frac{1}{2}} & c(\phi - \phi'')' - \frac{1}{p+1}(\phi^{p+1})' + \psi' = 0, \\ & c\psi' + \phi' = 0 = 0, \end{cases} \]
which, upon integration, yields

\[
\begin{cases}
  c(\phi' - \phi'') - \frac{1}{p+1} \phi^{p+1} + \psi = A_1, \\
  c\psi + \phi = 0 = A_2.
\end{cases}
\]

where \( A_1, A_2 \) are integration constants. Among all the travelling wave solutions we shall focus on solutions which have the additional property that the waves are localized and that \((\phi, \psi)\) and its derivatives decay at infinity, that is,

\[
\phi^{(n)}(y) \to 0, \quad \psi^{(n)}(y) \to 0 \quad \text{as} \quad |y| \to \infty.
\]

Under this decay assumption the integration constants vanish and then the travelling wave equation take the form

\[
\begin{cases}
  c(\phi' - \phi'') - \frac{1}{p+1} \phi^{p+1} + \psi = 0, \\
  c\psi + \phi = 0 = 0.
\end{cases}
\]

We will establish the existence of a solution of (1) in the weak sense by using a variational approach in which weak solutions correspond to critical points of an energy under a special constrain. We begin by defining the appropriate functional spaces. Set \( U \subset \mathbb{R} \), then \( L^p(U) \) denotes the usual Lebesgue space. In addition, the usual Sobolev space \( H^k(U), k \in \mathbb{Z}^+ \), is the Hilbert space defined as the closure of \( C^\infty(U) \) with respect to inner product

\[
(\phi, \psi)_{H^k} = \sum_{n=0}^k \int_U \phi^{(n)} \cdot \psi^{(n)} \, dx.
\]

Also we consider the Hilbert space \( \mathcal{X} = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) with respect to the norm

\[
\|(\phi, \psi)\|_{\mathcal{X}}^2 = \|\phi\|^2_{H^1(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left[ \phi^2 + (\phi')^2 + \psi^2 \right] \, dx.
\]

If we multiply the travelling wave system (1) with a test couple \((w, z) \in \mathcal{X}\), then, after integrating by parts, a travelling wave solution \((\phi, \psi) \in \mathcal{X}\) satisfies the system

\[
\int_{\mathbb{R}} \left( c(\phi w + \phi' w') - \frac{1}{p+1} \phi^{p+1} w + \psi w \right) c\psi z + \phi z \, dx = 0,
\]

**Definition 2.1.** We say that \((\phi, \psi)\) is a weak solution of (1) if for all \((w, z) \in \mathcal{X}\), the system (2) holds.
Our strategy to prove the existence of a solution of (2) is to consider the following minimization problem

$$I_c := \inf \{ I_c(\phi, \psi) : v \in \mathcal{X} \text{ with } G(\phi) = 1 \}, \quad (3)$$

where the energy $I_c$ and the constrain $G$ are functionals defined in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ given by

$$I_c(\phi, \psi) = \frac{1}{2} \int_{\mathbb{R}} \left[ |c| (\phi^2 + (\phi')^2 + \psi^2) + 2 \text{sgn}(c) \phi \psi \right] dx, \quad (4)$$

$$G(\phi, \psi) = G(\phi) = \frac{\text{sgn}(c)}{(p+1)(p+2)} \int_{\mathbb{R}} \phi^{p+2} dx. \quad (5)$$

where $\text{sgn}$ denotes the signum function. We start by showing some properties of $I_c$ and $G$.

**Lemma 2.2.** Let $|c| > 1$. Then

i) The functionals $I_c$ and $G$ are well defined in $\mathcal{X} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and smooth.

ii) The functional $I_c$ is nonnegative. Moreover, there are $C_1(c) < C_2(c)$ such that

$$C_1 \| (\phi, \psi) \|^2_{\mathcal{X}} \leq I_c(\phi, \psi) \leq C_2 \| (\phi, \psi) \|^2_{\mathcal{X}}. \quad (6)$$

iii) $I_c$ is finite and positive.

**Proof.** i) $I_c$ is clearly well defined for $(\phi, \psi) \in \mathcal{X}$. Moreover, note that if $\phi \in H^1(\mathbb{R})$ then, using the Hölder inequality and the fact that the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous for $q \in [2, \infty]$, we see that there is a constant $C = C(p) > 0$ such that

$$|G(\phi)| \leq C \| \phi \|_{L^2} \| \phi \|_{L^2(\mathbb{R}^{p+1})}^{p+1} \leq C \| \phi \|_{H^1}^{p+2} \leq C \| (\phi, \psi) \|_{\mathcal{X}}^{p+2}. \quad (7)$$

So, $G$ is well defined.

ii) We define

$$C_1 = \frac{|c| - 1}{2}, \quad C_2 = \frac{|c| + 1}{2}.$$
Then, using Young inequality, we see that

\[ C_1 \|(\phi, \psi)\|_X^2 = C_1 \int_{\mathbb{R}} \left[ \phi^2 + (\phi')^2 + \psi^2 \right] dx \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}} \left[ (|c| - 1)\phi^2 + |c|(\phi')^2 + (|c| - 1)\psi^2 \right] dx \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}} \left[ |c| (\phi^2 + (\phi')^2 + \psi^2) + 2\phi \psi \right] dx \]

\[ = I_c(\phi, \psi) \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}} \left[ (|c| + 1)\phi^2 + |c|(\phi')^2 + (|c| + 1)\psi^2 \right] dx \]

\[ \leq C_2 \|(\phi, \psi)\|_X^2. \]

iii) Note that there exists \( \phi_0 \in H^1(\mathbb{R}) \) such that \( G(\phi_0) \neq 0 \). Then there exists \( t_0 \in \mathbb{R} \) such that for \( \phi = t_0 \phi_0 \) we obtain

\[ G(\phi) = t_0^{p+2} G(\phi_0) = 1. \]

Thus, we notice that the infimum in (4) is being taken in a nonempty class. On the other hand, the inequalities (6)-(7) imply that there is \( C > 0 \) such that for any \( (\phi, \psi) \in X \) with \( G(\phi) = 1 \),

\[ C (I_c(\phi, \psi))^{\frac{p+2}{2}} \geq C \|(\phi, \psi)\|_X^{p+2} \geq G(\phi, \psi) = 1. \]

Meaning that

\[ I_c(\phi, \psi) \geq C^{\frac{p+2}{2}} > 0. \]

Hence, the infimum \( I_c \) is finite and positive.

**Theorem 2.3.** If \((\phi_0, \psi_0) \in X\) is a minimizer for the problem (3), then \((\phi, \psi) = \alpha(\phi_0, \psi_0)\) is a nontrivial weak solution of (1) with \( \alpha = \frac{2}{p+2} I_c \).

**Proof.** To simplify the notation, we observe that

\[ I_c(\phi, \psi) = I_1(\phi) + I_2(\psi) + I_3(\phi, \psi), \]

where

\[ I_1(\phi) = \frac{|c|}{2} \int_{\mathbb{R}} \left[ \phi^2 + (\phi')^2 \right] dx, \quad I_2(\psi) = \frac{|c|}{2} \int_{\mathbb{R}} \psi^2 dx, \quad I_3(\phi, \psi) = \text{sgn}(c) \int_{\mathbb{R}} \phi \psi dx. \]

First we assume \( c > 0 \). By the Lagrange Theorem there is a multiplier \( \alpha \) such that for any \( w \in H^1(\mathbb{R}) \),

\[ \langle I'_c(\phi_0, \psi_0), (w, z) \rangle - \alpha \langle G'(\phi_0, \psi_0), (w, z) \rangle = 0. \]
But, a direct calculation shows that
\[
\langle I_c'(\phi_0, \psi_0), (w, z) \rangle = \int_{\mathbb{R}} \left( c(\phi w + \phi' w') + \psi w \right) dx,
\]
\[
\langle G'(\phi_0, \psi_0), (w, z) \rangle = \frac{1}{p+1} \int_{\mathbb{R}} \left( \phi^{p+1} w \right) dx.
\]
In particular, putting \((w, z) = (\phi_0, \psi_0)\) in (8), we have that
\[
2I_1(\phi_0) + I_3(\phi_0, \psi_0) - \alpha (p + 2)G(\phi_0) = 0 \quad (9)
\]
\[
2I_2(\psi_0) + I_3(\phi_0, \psi_0) = 0. \quad (10)
\]
Combining (9)-(10) we obtain,
\[
2I_c(\phi_0, \psi_0) - \alpha (p + 2)G(\phi_0) = 0.
\]
But, by using \(G(\phi_0) = 1\) we see that \(\alpha = \frac{2}{p+2} I_c\). Moreover, from (8) we see that \(\alpha(\phi_0, \psi_0)\) is a nontrivial solution of the integral equation (2). In a similar fashion we obtain the result when \(c < 0\).

**Remark 2.4.** From the Theorem 2.3 we note that the existence of travelling wave solutions for the model (1) is based on the existence of a minimizer for the problem (3). In the next section we will show the existence of such minimizer, for this we will use an important result by P. L. Lions, known as the Concentration-Compactness Theorem (see [7], [8]).

**Theorem 2.5.** (P. L. Lions, [7], [8]) Suppose \(\{\nu_m\}\) is a sequence of nonnegative measures on \(\mathbb{R}\) such that
\[
\lim_{m \to \infty} \int_{\mathbb{R}} d\nu_m = \mathcal{I}.
\]
Then there is a subsequence of \(\{\nu_m\}\) (which is denoted in the same way) that satisfies only one of the following properties.

**Vanishing.** For any \(R > 0\),
\[
\lim_{m \to \infty} \left( \sup_{x \in \mathbb{R}} \int_{B_R(x)} d\nu_m \right) = 0, \quad (11)
\]
where \(B_R(x)\) is the ball in \(\mathbb{R}\) of radius \(R\) centered at \(x\).

**Dichotomy.** There exist \(\theta \in (0, \mathcal{I})\) such that for any \(\tau > 0\), there are \(R > 0\) and a sequence \(\{x_m\}\) in \(\mathbb{R}\) with the following property: Given \(R' > R\) there are nonnegative measures \(\nu^1_m, \nu^2_m\) such that

1. \(0 \leq \nu^1_m + \nu^2_m \leq \nu_m\),
2. \( \text{supp} (\nu^1_m) \subset B_R(x_m), \quad \text{supp} (\nu^2_m) \subset \mathbb{R} \setminus B'_R(x_m), \)

3. \( \limsup_{m \to \infty} (|\theta - \int_{\mathbb{R}} d\nu^1_m| + |(I - \theta) - \int_{\mathbb{R}} d\nu^2_m|) \leq \tau. \)

**Compactness.** There exists a sequence \( \{x_m\} \) in \( \mathbb{R} \) such that for any \( \tau > 0 \), there is \( R > 0 \) with the property that

\[
\int_{B_R(x_m)} d\nu_m \geq I - \tau, \quad \text{for all } m. \tag{12}
\]

### 3 Existence of Minimizers

In order to apply the Concentration-Compactness Theorem to our case, let us assume that \( \{(\phi_m, \psi_m)\} \) in \( X \) is a minimizing sequence for \( I_c \), then we define the positive measures \( \{\nu_m\} \) by \( d\nu_m = \rho_m dx \), where \( \rho_m \) is defined as

\[
\rho_m = C \frac{1}{2} \left( \phi_m^2 + (\phi'_m)^2 + \psi_m^2 \right) + \phi_m \psi_m, \tag{13}
\]

which correspond to the integrand of \( I_c(\phi_m, \psi_m) \). From the Theorem 2.5, there exists a subsequence of \( \{\nu_m\} \) (which we denote the same) that satisfies either vanishing, or dichotomy, or compactness. We will see that vanishing and dichotomy can be ruled out, and so using compactness we will establish that the minimizing sequence \( \{(\phi_m, \psi_m)\} \) is compact in \( X \), up to translation, as a consequence of a local compact embedding result.

We will establish some technical result. The first one is related with the characterization of “vanishing sequences”.

**Lemma 3.1.** (Vanishing sequences) Let \( R > 0 \) be given and let \( \{(\phi_m, \psi_m)\} \) be a bounded sequence in \( X \) such that

\[
\lim_{m \to \infty} \left( \sup_{x \in \mathbb{R}} \int_{B_R(x)} d\nu_m \right) = 0.
\]

Then we have that

\[
\lim_{m \to \infty} G(\phi_m) = \lim_{n \to \infty} \int_{\mathbb{R}} \phi^p_m \, dx = 0.
\]

In particular, if \( \{(\phi_m, \psi_m)\} \) is a minimizing sequence for \( I_c \), then vanishing is ruled out.

**Proof.** Let \( x \in \mathbb{R}, \; R > 0 \) and \( B_R = B_R(x) \). Using the same ideas given in the proof of (4), we have that there is \( C > 0 \) such that

\[
\|\phi_m\|_{H^1(B_R)}^2 \leq \|(\phi_m, \psi_m)\|_{H^1(B_R) \times L^2(B_R)}^2 \leq C \int_{B_R} d\nu_m.
\]
Thus, since the embedding $H^1(B_R) \hookrightarrow L^q(B_R)$ is continuous for $q \in [2, \infty]$, we obtain that

\[
\int_{B_R} |\phi_m|^{p+2} \, dx \leq \|\phi_m\|_{L^2(B_R)} \|\phi_m\|_{L^{2(p+1)}(B_R)}^{p+1} \\
\leq C \|\phi_m\|_{H^1(B_R)} \|\phi_m\|_{H^1(B_R)}^{p+1} \\
\leq C \|\phi_m\|_{H^1(B_R) \times L^2(B_R)} \left( \int_{B_R} d\nu_m \right)^{\frac{p+1}{2}} \\
\leq C \|\phi_m\|_{H^1(B_R) \times L^2(B_R)} \left( \sup_{x \in \mathbb{R}} \int_{B_R(x)} d\nu_m \right)^{\frac{p+1}{2}}.
\]

Covering $\mathbb{R}$ by balls of radius $R$ in such a way that each point of $\mathbb{R}$ is contained in at most two balls, we find that

\[
\int_{\mathbb{R}} |\phi_m|^{p+2} \, dx \leq 2C \|\phi_m\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} \left( \sup_{x \in \mathbb{R}} \int_{B_R(x)} d\nu_m \right)^\frac{p+1}{2}.
\]

Thus, under the assumptions of the lemma,

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \phi_m^{p+2} \, dx = 0.
\]

As a consequence of this we see that

\[
\lim_{m \to \infty} G(\phi_m) = 0.
\]

Now, if we assume that \{$(\phi_m, \psi_m)$\} is a minimizing sequence for $\mathcal{I}_c$, then we have that $G(\phi_m) = 1$ and that \{$(\phi_m, \psi_m)$\} is a bounded sequence, then from the previous fact we reach a contradiction.

In order to rule out dichotomy, we will establish a splitting result for a sequence \{$(\phi_m, \psi_m)$\} in $\mathcal{X}$. Fix a function $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R}^+)$ such that $\text{supp} \, \phi \subset B_2(0)$ and $\phi \equiv 1$ in $B_1(0)$. If $R > 0$ and $x_0 \in \mathbb{R}$, we define a split for $(\phi, \psi) \in \mathcal{X}$ given by

\[
\phi = \phi^1 + \phi^2, \quad \psi = \psi^1 + \psi^2
\]

where

\[
\phi^1 = \phi \varphi_R, \quad \phi^2 = \nu(1 - \varphi_R), \quad \psi^1 = \psi \varphi_R, \quad \psi^2 = \psi(1 - \varphi_R).
\]

with

\[
\varphi_R(x) = \varphi \left( \frac{x - x_0}{R} \right)
\]

In addition, we define $A_R(x_0)$ by

\[
A_R(x_0) = B_{2R}(x_0) \setminus B_R(x_0).
\]
Lemma 3.2. (A splitting result) Let $R_m > 0$ and $x_m \in \mathbb{R}$ be sequences. Define $A(m) = A_{R_m}(x_m)$ and $\varphi_m(x) = \varphi\left(\frac{x - x_m}{R_m}\right)$. If
\[
\limsup_{m \to \infty} \left( \int_{A(m)} d\nu_m \right) = 0. \tag{14}
\]
Then we have that
\begin{enumerate}[i)]
\item \[\lim_{m \to \infty} \left[ I_c(\phi_m, \psi_m) - I_c(\phi_1^1, \psi_1^1) - I_c(\phi_2^2, \psi_2^2) \right] = 0.\]
\item \[\lim_{m \to \infty} \left[ G(\phi_m) - G(\phi_1^1) - G(\phi_2^2) \right] = 0.\]
\end{enumerate}

Proof. We need to recall that
\[I_c(\phi, \psi) = I_1(\phi) + I_2(\psi) + I_3(\phi, \psi),\]
First, we will see that
\[I_j(z_m) = I_j(z_1^1) + I_j(z_2^2) + o(1), \text{ as } m \to \infty,\]
where $z_m = \phi_m$ for $j = 1$, $z_m = \psi_m$ for $j = 2$, and $z_m = (\phi_m, \psi_m)$ for $j = 3$. In fact, note that
\[
\delta^{(0)} \phi_m := \int_{\mathbb{R}} \left[ (\phi_m)^2 - (\phi_1^1)^2 - (\phi_2^2)^2 \right] dx = 2 \int_{A(m)} \varphi_m(1 - \varphi_m)(\phi_m)^2 dx.
\]
Then
\[|\delta^{(0)} \phi_m| \leq C \int_{A(m)} (\phi_m)^2 dx \leq C \int_{A(m)} d\nu_m \to 0.\]
Similarly we have that
\[\delta^{(0)} \psi_m := \int_{\mathbb{R}} \left[ (\psi_m)^2 - (\psi_1^1)^2 - (\psi_2^2)^2 \right] dx \to 0.\]
We again see that
\[
\delta^{(1)} \phi_m := \int_{\mathbb{R}} \left[ (\phi_m')^2 - (\phi_1^1')^2 - (\phi_2^2')^2 \right] dx
\]
\[= 2 \int_{A(m)} \left[ \varphi_m(1 - \varphi_m)(\phi_m')^2 + (1 - 2\varphi_m)(\phi_m\phi_m'\varphi_m') - \nu_m^2 (\phi_m')^2 \right] dx.
\]
Consequently,
\[|\delta^{(1)} \phi_m| \leq C \int_{A(m)} [(\phi_m)^2 + (\phi_m')^2] dx \leq C \int_{A(m)} d\nu_m \to 0.\]
It is not hard to prove that if
\[
\delta^{(0)}(\phi_m\psi_m) := \int_{\mathbb{R}} [\phi_m\psi_m - \phi_m^1\psi_m^1 - \phi_m^2\psi_m^2] \, dx,
\]
we have that \(|\delta^{(0)}(\phi_m\psi_m)| \rightarrow 0\). Then we obtain that
\[
\lim_{m \rightarrow \infty} [I_c(\phi_m,\psi_m) - I_c(\phi_m^1,\psi_m^1) - I_c(\phi_m^2,\psi_m^2)] = 0.
\]
Next, we will show the item \(ii\). We notice that
\[
\int_{\mathbb{R}} \left[ (\phi_m)^{p+2} - (\phi_m^1)^{p+2} - (\phi_m^2)^{p+2} \right] \, dx 
\leq C \int_{A(m)} (\phi_m)^{p+2} \, dx 
= \|\phi_m\|^{p+2}_{L^p(A(m))} 
\leq C \|\phi_m\|^{p+2}_{H^1(A(m))} 
\leq C \|\phi_m,\psi_m\|^{p+2}_{H^1(A(m)) \times L^2(A(m))} 
\leq C \left( \int_{A(m)} \left[ (\phi_m)^2 + (\phi_m')^2 + \psi_m^2 \right] \, dx \right)^{\frac{p+2}{2}} 
\leq C \left( \int_{A(m)} d\nu_m \right)^{\frac{p+2}{2}} \rightarrow 0.
\]
Then, from the definition of \(G\), we conclude as \(m \rightarrow \infty\) that
\[
G(\phi_m) = G(\phi_m^1) + G(\phi_m^2) + o(1).
\]

Using the previous result we have the following lemma.

**Lemma 3.3.** Let \(\{(\phi_m,\psi_m)\}\) be a minimizing sequence for \(I_c\). Then dichotomy is not possible.

**Proof.** Assume that dichotomy occurs, then we can choose a sequence \(\tau_m \rightarrow 0\), \(R_m \rightarrow \infty\) such that
\[
\text{supp} (\nu_m^1) \subset B_{R_m}(x_m), \quad \text{supp}(\nu_m^2) \subset \mathbb{R} \setminus B_{2R_m}(x_m)
\]
and
\[
\limsup_{m \rightarrow \infty} \left( \left| \theta - \int_{\mathbb{R}} d\nu_m^1 \right| + \left| (I_c - \theta) - \int_{\mathbb{R}} d\nu_m^2 \right| \right) = 0.
\]
for some $\theta \in (0, I_c)$. Using these facts, we have that
\[ \limsup_{m \to \infty} \left( \int_{A(m)} d\nu_m \right) = 0. \]

In fact,
\[ \int_{A(m)} d\nu_m = \left( \int_\mathbb{R} - \int_{B_{R_m}(x_m)} - \int_{\mathbb{R} \setminus B_{2R_m}(x_m)} \right) d\nu_m \]
\[ \leq \int_\mathbb{R} d\nu_m - \int_\mathbb{R} d\nu_1^m - \int_\mathbb{R} d\nu_2^m \]
\[ \leq \left( \int_\mathbb{R} d\nu_m - I_c \right) + \left| \theta - \int_\mathbb{R} d\nu_1^m \right| + \left| (I_c - \theta) - \int_\mathbb{R} d\nu_2^m \right|. \]

Using (3) and Lemma 3.2 we conclude that
\[ \lim_{m \to \infty} [I_c(\phi_m, \psi_m) - I_c(\phi_m, \psi_m) - I_c(\phi_m, \psi_m)] = 0, \]
\[ \lim_{m \to \infty} [G(\phi_m) - G(\phi_1^m) - G(\phi_2^m)] = 0. \]

Now, let $\lambda_{m,i} = G(\phi_i^m)$, for $i = 1, 2$. Passing to a subsequence if necessary we have that $\lambda_i := \lim_{m \to \infty} \lambda_{m,i}$ exists. Now, let us prove that $\lambda_i \neq 0$. Assume that $\lim_{m \to \infty} \lambda_{m,1} = 0$, then $\lim_{m \to \infty} \lambda_{m,2} = 1$ (we proceed in a similar way in the other case). Therefore $\lambda_{m,2} > 0$, for $m$ large enough. Then we consider
\[ w_m = \lambda_{m,2}^{-\frac{1}{p+2}} \phi_m^2, \quad z_m = \lambda_{m,2}^{-\frac{1}{p+2}} \psi_m^2. \]

So that
\[ (w_m, z_m) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \quad G(w_m) = 1. \]

Using that $\varphi_m \equiv 1$ in $B_{R_m}(x_m)$ we have a contradiction since
\[ I_c = \lim_{m \to \infty} I_c(\varphi_m, \psi_m) \]
\[ = \lim_{m \to \infty} \left( I_c(\phi_m^1, \psi_m^1) + I_c(\phi_m^2, \psi_m^2) \right) \]
\[ \geq \lim_{m \to \infty} \left( \int_{B_{R_m}(x_m)} d\nu_m + \lambda_{m,2}^{\frac{2}{p+2}} I_c(w_m, z_m) \right) \]
\[ \geq \lim_{m \to \infty} \left( \int_\mathbb{R} d\nu_1^m + \lambda_{m,2}^{\frac{2}{p+2}} I_c \right) \]
\[ = \theta + I_c. \]

In other words, $|\lambda_{m,i}| > 0$ for $m$ large enough. Then we are allowed to define
\[ w_{m,i} = \lambda_{m,i}^{-\frac{1}{p+2}} \phi_m^i, \quad z_{m,i} = \lambda_{m,i}^{-\frac{1}{p+2}} \psi_m^i \quad i = 1, 2. \]
Note that \((w_{m,i}, z_{m,i}) \in \mathcal{X}\) and \(G(w_{m,i}) = 1\). Hence,

\[
\mathcal{I}_c = \lim_{m \to \infty} \left( I_c(\phi_m^1, \psi_m^1) + I_c(\phi_m^2, \psi_m^2) \right) = \lim_{m \to \infty} \left( |\lambda_{m,1}|^{\frac{2}{p+2}} I_c(w_{m,1}) + |\lambda_{m,2}|^{\frac{2}{p+2}} I_c(w_{m,2}) \right) \geq \left( |\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} \right) \mathcal{I}_c.
\]

Then

\[1 \geq |\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} \geq (|\lambda_1| + |\lambda_2|)^{\frac{2}{p+2}} \geq |\lambda_1 + \lambda_2|^{\frac{2}{p+2}} = 1.\]

Hence, \(|\lambda_1| + |\lambda_2| = 1\). Using that \(\lambda_1 + \lambda_2 = 1\) and \(\lambda_i \neq 0\), we have that \(\lambda_i > 0\) and

\[\lambda_1^{\frac{2}{p+2}} + \lambda_2^{\frac{2}{p+2}} = (\lambda_1 + \lambda_2)^{\frac{2}{p+2}}. \tag{15}\]

But (15) gives us a contradiction, because for \(t \in \mathbb{R}^+\) the function \(f(t) = t^{\frac{2}{p+2}}\) is strictly concave, meaning that

\[f(t_1 + t_2) < f(t_1) + f(t_2), \text{ for } t_1, t_2 > 0.\]

In other words, we have ruled out dichotomy.

Now we are in position to obtain the main result in this section: the existence of a minimizer for \(\mathcal{I}_c\). Since we ruled out vanishing and dichotomy above for a minimizing sequence of \(\mathcal{I}_c\), then by P. L. Lion’s Concentration-Compactness Theorem, there exists a subsequence of \(\{\nu_m\}\) (which we denote the same) satisfying compactness. We will see as a consequence of local compact embedding that a minimizing sequence \(\{(\phi_m, \psi_m)\}\) is compact in \(H^1(\mathbb{R})\), up to translation.

**Theorem 3.4.** If \(\{(\phi_m, \psi_m)\}\) is a minimizing sequence for (3), then there is a subsequence (which we denote the same), a sequence of points \(x_m \in \mathbb{R}\), and a minimizer \((\phi_0, \psi_0) \in \mathcal{X}\) of (3), such that the translated functions

\[(\tilde{\phi}_m, \tilde{\psi}_m) = (\phi_m(\cdot + x_m), \psi_m(\cdot + x_m))\]

converge to \((\phi_0, \psi_0)\) strongly in \(\mathcal{X}\).

**Proof.** Let \(\{(\phi_m, \psi_m)\}\) be a minimizing sequence for (3). In other words,

\[
\lim_{m \to \infty} I_c(\phi_m, \psi_m) = \mathcal{I}_c, \quad G(\phi_m) = 1.
\]

By compactness, there exists a sequence \(x_m\) in \(\mathbb{R}\) such that for a given \(\tau > 0\), there exists \(R > 0\) with the following property,

\[
\int_{B_R(x_m)} d\nu_m \geq \mathcal{I}_c - \tau, \quad \text{for all } m \in \mathbb{N}. \tag{16}
\]
Using this we may localize the minimizing sequence \( \{(\phi_m, \psi_m)\} \) around the origin by defining
\[
\tilde{\rho}_m(x) = \rho_m(x + x_m), \quad \tilde{\phi}_m(x) = \phi_m(x + x_m), \quad \tilde{\psi}_m(x) = \psi_m(x + x_m).
\]
Thus, we have the following localized inequality
\[
\int_{B_R(0)} \tilde{\rho}_m \, dx = \int_{B_R(x_m)} d\nu_m \geq I_c - \tau, \quad \text{for all } m \in \mathbb{N},
\]
and also that
\[
\lim_{m \to \infty} I_c(\tilde{\phi}_m, \tilde{\psi}_m) = \lim_{m \to \infty} I_c(\phi_m, \psi_m) = I_c, \quad G(\tilde{\phi}_m) = G(\phi_m) = 1.
\]

Then by (6) we note that \( \{(\tilde{\phi}_m, \tilde{\psi}_m)\} \) is a bounded sequence in \( \mathcal{X} \). On the other hand, since \( \tilde{\phi}_m \in H^1(U) \) for any bounded open set \( U \subset \mathbb{R} \) and the embedding \( H^1(U) \hookrightarrow L^q(U) \) is compact for \( q \in [2, \infty] \), then there exist a subsequence of \( \{(\tilde{\phi}_m, \tilde{\psi}_m)\} \) (which we denote the same) and \( (\phi_0, \psi_0) \in \mathcal{X} \) such that
\[
\tilde{\phi}_m \to \phi_0 \quad \text{in } H^1(\mathbb{R}) \quad \text{and} \quad \tilde{\psi}_m \to \psi_0, \quad \tilde{\phi}_m \to \phi_0, \quad \tilde{\phi}_m' \to \phi_0' \quad \text{in } L^2(\mathbb{R}),
\]
and we also have that
\[
\tilde{\phi}_m \to \phi_0, \quad \tilde{\phi}_m' \to \phi_0', \quad \tilde{\psi}_m \to \psi_0 \quad \text{in } L^2_{loc}(\mathbb{R}).
\]
Moreover,
\[
\tilde{\phi}_m \to \phi_0, \quad \tilde{\phi}_m' \to \phi_0', \quad \tilde{\psi}_m \to \psi_0 \quad \text{a.e in } \mathbb{R}.
\]

Using these facts we will show that some subsequence of \( \{(\tilde{\phi}_m, \psi_m)\} \) (which we denote the same) converges strongly in \( \mathcal{X} \) to a nontrivial minimizer \( (\phi_0, \psi_0) \) of (3). This is,
\[
\tilde{\phi}_m \to \phi_0, \quad \tilde{\phi}_m' \to \phi_0', \quad \tilde{\psi}_m \to \psi_0 \quad \text{in } L^2(\mathbb{R}).
\]

In fact, using (17), (18) and the definition of \( I_c \) we have that for \( \tau > 0 \), there exists \( R > 0 \) such that for \( m \) large enough,
\[
\int_{B_R(0)} |\tilde{\phi}_m|^2 \, dx \geq \int_{\mathbb{R}} |\tilde{\phi}_m|^2 \, dx - 2\tau.
\]

Then we have that
\[
\int_{\mathbb{R}} |\phi_0|^2 \, dx \leq \liminf_{m \to \infty} \int_{\mathbb{R}} |\tilde{\phi}_m|^2 \, dx
\]
\[
\leq \liminf_{m \to \infty} \int_{B_R(0)} |\tilde{\phi}_m|^2 \, dx + 2\tau
\]
\[
= \int_{B_R(0)} |\phi_0|^2 \, dx + 2\tau
\]
\[
\leq \int_{\mathbb{R}} |\phi_0|^2 \, dx + 2\tau.
\]
Therefore
\[
\liminf_{m \to \infty} \int_{\mathbb{R}} |\tilde{\phi}_m|^2 \, dx = \int_{\mathbb{R}} |\phi_0|^2 \, dx.
\]
Thus, there exist a subsequence of \(\{\tilde{\phi}_m\}\) such that \(\tilde{\phi}_m \to \phi_0\) in \(L^2(\mathbb{R})\). Using a similar argument we prove the other part of (19). Now, using (19) and the fact that the inclusion \(H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\) is continuous for \(q \in [2, \infty]\), we have that
\[
G(\phi_0) = \lim_{m \to \infty} G(\tilde{\phi}_m) = 1. \tag{20}
\]
In fact,
\[
\int_{\mathbb{R}} \left[ (\tilde{\phi}_m)^{p+2} - (\phi_0)^{p+2} \right] \, dx
= \int_{\mathbb{R}} \left[ (\tilde{\phi}_m - \phi_0) (\tilde{\phi}_m)^{p+1} + \phi_0 ( (\tilde{\phi}_m)^{p+1} - (\phi_0)^{p+1} ) \right] \, dx.
\]
But we see that
\[
\int_{\mathbb{R}} (\tilde{\phi}_m - \phi_0) (\tilde{\phi}_m)^{p+1} \, dx \leq \|\tilde{\phi}_m - \phi_0\|_{L^2} \|\tilde{\phi}_m\|^{p+1}_{L^2(p+1)}
\leq C \|\tilde{\phi}_m - \phi_0\|_{H^1} \|\tilde{\phi}_m\|^{p+1}_{H^1},
\leq C \left[ I_c(\tilde{\phi}_m, \psi_m) \right]^{\frac{p+1}{2}} \|\tilde{\phi}_m - \phi_0\|_{H^1},
\leq C \|\tilde{\phi}_m - \phi_0\|_{H^1} \to 0,
\]
and also, using that the embedding \(H^1(U) \hookrightarrow L^q(U)\) is continuous for \(q \in [2, \infty]\),
\[
\int_{\mathbb{R}} \phi_0 \left[ (\tilde{\phi}_m)^{p+1} - (\phi_0)^{p+1} \right] \, dx
\leq C \|\phi_0\|_{L^\infty} \|\tilde{\phi}_m - \phi_0\|_{L^2} \sum_{j=0}^p \| (\tilde{\phi}_m)^{p-j} (\phi_0)^j \|_{L^2}
\leq C \|\phi_0\|_{H^1} \|\tilde{\phi}_m - \phi_0\|_{H^1} \sum_{j=0}^p \|\tilde{\phi}_m\|^{p-j}_{L^4} \|\phi_0\|^j_{L^4}
\leq C \|\phi_0\|_{H^1} \|\tilde{\phi}_m - \phi_0\|_{H^1} \sum_{j=0}^p \left( \|\tilde{\phi}_m\|^{2(p-j)}_{H^1} + \|\phi_0\|^{2j}_{H^1} \right).
\]
Next, there are \(r_1, r_2 > 0\) such that
\[
\|\tilde{\phi}_m\|^{2(p-j)}_{H^1} \leq \|\tilde{\phi}_m\|^{2r_1}_{H^1}, \quad \|\phi_0\|^{2j}_{H^1} \leq \|\phi_0\|^{2r_2}_{H^1}, \quad j = 1, \ldots, p.
\]
So that
\[\int_{\mathbb{R}} \phi_0 \left[ (\tilde{\phi}_m)^{p+1} - (\phi_0)^{p+1} \right] dx \leq C \|\phi_0\|_{H^1} \|\tilde{\phi}_m - \phi_0\|_{H^1} \left( \|\tilde{\phi}_m\|_{H^1}^{2r_1} + \|\phi_0\|_{H^1}^{2r_2} \right) \]
\[\leq C \left[ I(\phi_0)^{\frac{1}{2}} \|\tilde{\phi}_m - \phi_0\|_{H^1} \left( \left[I_c(\tilde{\phi}_m)\right]^{r_1} + \left[I_c(\phi_0)\right]^{r_2} \right) \right] \rightarrow 0 \]

Then from the definition of $G$ we conclude that (20) holds, which implies that $\phi_0 \neq 0$. On the other hand, from (19), we see that
\[\lim_{m \to \infty} I_c(\tilde{\phi}_m, \tilde{\psi}_m) = I_c(\phi_0, \psi_0) = \mathcal{I}_c, \quad \lim_{m \to \infty} I_c(\tilde{\phi}_m - \phi_0, \tilde{\psi}_m - \psi_0) = 0.\]

Moreover, the sequence $(\tilde{\phi}_m, \tilde{\psi}_m)$ converges to $(\phi_0, \psi_0)$ in $\mathcal{X}$, since
\[\|\tilde{\phi}_m - \phi_0\|_{H^1(\mathbb{R})} + \|\tilde{\psi}_m - \psi_0\|_{L^2(\mathbb{R})} \leq C_1 I_c(\tilde{\phi}_m - \phi_0, \tilde{\psi}_m - \psi_0).\]

Then we concluded that $\{(\tilde{\phi}_m, \tilde{\psi}_m)\}$ converges to $(\phi_0, \psi_0)$ in $\mathcal{X}$ and $(\phi_0, \psi_0)$ is a minimizer for $\mathcal{I}_c$. \hfill \Box

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