Approximate Solution for 2 Dimensional Fuzzy Parabolic Equations in QSAGE Iterative Method

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Abstract
In this study, iterative methods particularly families of Alternating Group Explicit (AGE) methods are used to solve system of linear equations generated from the discretization of two-dimensional fuzzy diffusion problems. The formulation and implementation of the Full-Sweep AGE (FSAGE), Half-Sweep AGE (HSAGE) and Quarter-Sweep AGE (QSAGE) methods were also presented. Then numerical experiments are carried out onto two example problems to verify the effectiveness of the methods.

Keywords: two-stage iteration, implicit scheme, fuzzy heat equation

1 Introduction
Abdullah [1] discovered the new concept known as half-sweep iteration via Explicit Decoupled Group (EDG) iterative method in solving two-dimensional Poisson equations. Following to this concept, further investigations have been extensively conducted in [3, 4, 10-12, 16, 18] for demonstrating the capability of the half-sweep iteration. In 2000, Othman and Abdullah [14] expanded this approach to initiate the Modified Explicit Group (MEG) method based on the quarter-sweep approach. Later, many studies have been conducted to demonstrate
the capability of the quarter-sweep iteration [17,19,20]. The QSAGE method is used as linear solver to solve fuzzy linear systems generated from the discretization of the two-dimensional fuzzy diffusion problems, whereas Gauss-Seidel (GS), Full-Sweep AGE (FSAGE) and Hall-Sweep AGE (HSAGE) methods are used as control solvers.

The outline of the paper is organized as follows. Section 2 will discuss the finite difference method based on the implicit scheme in discretizing two-dimensional fuzzy diffusion problems. While section 3 presents the formulation and implementation of the family of AGE methods in solving linear systems generated from the finite difference scheme. Section 4 shows some numerical examples and conclusions are given in section 5.

2 Finite Difference Approximation Equations

Consequently, in this paper, the general form of diffusion equation can be defined as follows

\[
\frac{\partial^2 \tilde{U}}{\partial t^2} = V \left( \frac{\partial^2 \tilde{U}}{\partial x^2} + \frac{\partial^2 \tilde{U}}{\partial y^2} \right), \quad R = [0 \leq x \leq n] \times [0 \leq y \leq m]
\] (1)

with boundary conditions

\[
\tilde{U}(x, y, 0) = \tilde{f}(x, y), \quad (x, y) \in R
\]

and initial conditions

\[
\tilde{U}(x, y, t) = g(x, y, t), \quad (x, y, t) \in \partial R \times [0 \leq t \leq T]
\]

where \( \partial R \) was an boundary of \( R \).

Now, let \( \bar{x} \) and \( \bar{y} \) be two fuzzy subset of real numbers. They are characterized by a membership function evaluated at \( t \), written \( \bar{x}(t) \) and \( \bar{y}(t) \) respectively as a number in \([0,1]\). Fuzzy numbers can be identified via the membership function. The \( \alpha \)-cut of \( \bar{x} \) and \( \bar{y} \), which \( \alpha \) is denote as a crisp number can be written as \( \bar{x}(\alpha) = \{ x \mid \bar{x}(t) \geq \alpha \} \) and \( \bar{y}(\alpha) = \{ y \mid \bar{y}(t) \geq \alpha \} \), for \( 0 < \alpha \leq 1 \). Since they are always closed and bounded interval, the \( \alpha \)-cut of fuzzy numbers can be written as \( \bar{x}(\alpha) = [\underline{x}(\alpha), \overline{x}(\alpha)] \) and \( \bar{y}(\alpha) = [\underline{y}(\alpha), \overline{y}(\alpha)] \) for all \( \alpha \) [2]. Suppose \( (\bar{a}, \bar{b}) \) and \( (\bar{c}, \bar{d}) \) be parametric form of fuzzy function \( x \) and \( y \) respectively, now for arbitrary positive integer \( n \) and \( m \) subdivided the interval \( a \leq t \leq b \) whereas \( x_i = a + ih \ (i = 0, 1, 2, \ldots, n) \) and \( y_j = a + jl \ (i = 0, 1, 2, \ldots, m) \) for \( i \) and \( j \) respectively and define the step size \( h \) and \( l \) by \( h = \frac{b-a}{n} \) and \( l = \frac{b-a}{m} \).

To simplify the formulation of the full-, half- and quarter-sweep central difference approximation equations, the finite grid network will be used as shown in Fig. 1. Implementations of these proposed iterative methods are executed onto the
interior solid nodal points until convergence test is found. Meanwhile, the approximation solutions for the remaining points can be computed by using direct method [1,14].

![Graphical representation](image)

Fig. 1: (a), (b) and (c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively.

Then, by using finite difference implicit scheme, problem (1) can be developed as

$$
\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1,k+1} - U_{i,j,k}}{\Delta t},
$$

(2a)

$$
\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1,k+1} - U_{i,j,k}}{\Delta t}.
$$

(2b)

with

$$
\Delta t = t_{j+1} - t_j
$$

and

$$
\frac{\partial^2 U}{\partial x^2} \approx \left[ \frac{U_{i-p,j,k+1} - 2U_{i,j,k+1} + U_{i+p,j,k+1}}{(ph)^2} \right].
$$

(3a)
\[
\frac{\partial^2 U}{\partial x^2} = \frac{U_{i-p,j,k+1} - 2U_{i,j,k+1} + U_{i+p,j,k+1}}{(ph)^2},
\]
(3b)

\[
\frac{\partial^2 U}{\partial y^2} = \frac{U_{i,j-p,k+1} - 2U_{i,j,k+1} + U_{i,j+p,k+1}}{(ph)^2},
\]
(4a)

\[
\frac{\partial^2 U}{\partial y^2} = \frac{U_{i,j-p,k+1} - 2U_{i,j,k+1} + U_{i,j+p,k+1}}{(ph)^2},
\]
(4b)

By using parametric form of fuzzy functions, problem (1) can be written as

\[
\frac{\partial U}{\partial t} = V\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right),
\]
(5a)

\[
\frac{\partial U}{\partial t} = V\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right).
\]
(5b)

By using (2a), (3a) and (4a), equation (5a) can be reduced to

\[
U_{i,j,k+1} - U_{i,j,k} = \frac{V\Delta t}{(ph)^2}\left(U_{i-p,j,k+1} + U_{i+p,j,k+1} + U_{i,j-p,k+1} + U_{i,j+p,k+1} - 4U_{i,j,k+1}\right)
\]
(6a)

for \( i=1p,2p,...,n-p \) and \( j=1p,2p,...,m-p \). Meanwhile, by substituting (2b), (3b) and (4b) into (5b), it can be shown

\[
U_{i,j,k+1} - U_{i,j,k} = \frac{V\Delta t}{(ph)^2}\left(U_{i-p,j,k+1} + U_{i+p,j,k+1} + U_{i,j-p,k+1} + U_{i,j+p,k+1} - 4U_{i,j,k+1}\right).
\]
(6b)

As the value of \( p \) corresponds to 1 and 2, it represents the full- and quarter-sweep cases respectively. Since both (6a) and (6b) have the same form in terms of equation, except, based on the interval of the \( \alpha \)-cuts, the differences identified only in the upper and lower boundary, thus it can be rewritten as

\[
U_{i,j,k+1} - U_{i,j,k} = \beta(U_{i-p,j,k+1} + U_{i+p,j,k+1} + U_{i,j-p,k+1} + U_{i,j+p,k+1} - 4U_{i,j,k+1})
\]
(7)

with \( \beta = \left(\frac{V\Delta t}{ph^2}\right) \). Whereas discretization scheme for half-sweep cases can be shown as

\[
U_{i,j,k+1} - U_{i,j,k} = \beta(U_{i-1,j-k+1} + U_{i+1,j+1,k+1} + U_{i+1,j-1,k+1} + U_{i+1,j+1,k+1} - 4U_{i,j,k+1})
\]
(8)
with $\beta = \frac{V \Delta t}{2h^2}$. Moreover, (7) and (8) can be represented in matrix form as follows

$$AU \approx f.$$  \hspace{1cm} (9)

### 3 Family of Alternating Group Explicit Iterative Methods

Consider a class of methods be mentioned by Evans [5] which is based on the splitting of the matrix $A$ into the sum of its constituent symmetric and positive definite matrices, as follows

$$A = G_1 + G_2 + G_3 + G_4$$  \hspace{1cm} (10)

where $G_1$ and $G_2$ are the forward and backward differences in the $x$-plane and $G_3$ and $G_4$ are similar difference in $y$-plane. Then

$$\text{diag}(G_1) = \text{diag}(G_2) = \frac{1}{4} \text{diag}(A)$$

with

$$G_1 = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix}_{n \times n^2}$$

and

$$G_2 = \begin{bmatrix}
1 & -1 & 0 & 0 & \ldots & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & \ldots & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\
\end{bmatrix}_{n \times n^2}.$$
By reordering the points column-wise along $y$-direction, $G_3$ and $G_4$ literally have the same structure as $G_1$ and $G_2$ respectively,

$$
\begin{align*}
\bar{G}_3 &= G_1 = \\
&= \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}.
\end{align*}
$$

and

$$
\begin{align*}
\bar{G}_4 &= G_2 = \\
&= \begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}.
\end{align*}
$$

Then (9) becomes

$$
(G_1 + G_2 + G_3 + G_4)U_{\sim j+1} = f_{\sim j}.
$$

Thus, the explicit form of AGE method can be written as

$$
U_{\sim}^{(k+\frac{1}{4})} = (r_1 I + G_1)^{-1} \left[ 2f_{\sim} + (r_1 I + G_1 - 2A) \right],
$$

$$
U_{\sim}^{(k+\frac{3}{4})} = (r_1 I + G_2)^{-1} \left[ G_2 U^{(k)}_{\sim} + r_1 U^{(k+\frac{1}{4})}_{\sim} \right],
$$

$$
U_{\sim}^{(k+1)} = (r_1 I + G_3)^{-1} \left[ G_3 U^{(k)}_{\sim} + r_2 U^{(k+\frac{1}{2})}_{\sim} \right],
$$

and

$$
U_{\sim}^{(k+\frac{3}{4})} = (r_1 I + G_4)^{-1} \left[ G_4 U^{(k)}_{\sim} + r_2 U^{(k+\frac{3}{4})}_{\sim} \right].
$$
Approximate solution for 2 dimensional fuzzy parabolic equations

From (12) to (15), therefore, the implementation of the families of AGE methods is presented in Algorithm 1.

**Algorithm 1: Families of AGE methods**

i. Initialize $U^{(0)} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

ii. First sweep

Compute

$$U^{(k+\frac{1}{2})} = (r_1 I + G_1)^{-1} \left[ 2f + (r_1 I + G_1 - 2A) \right]$$

iii. Second sweep

Compute

$$U^{(k+\frac{3}{2})} = (r_1 I + G_2)^{-1} \left[ G_2 U^{(k)} + r_1 U^{(k+\frac{1}{2})} \right]$$

iv. Third sweep

Compute

$$U^{(k+1)} = (r_2 I + G_3)^{-1} \left[ G_3 U^{(k)} + r_2 U^{(k+\frac{3}{2})} \right]$$

v. Fourth sweep

Compute

$$U^{(k+2)} = (r_2 I + G_4)^{-1} \left[ G_4 U^{(k)} + r_2 U^{(k+1)} \right]$$

vi. Convergence test. If the convergence criterion i.e. $\left\| U^{(k+1)} - U^{(k)} \right\|_{\infty} \leq \varepsilon$ is satisfied, go to Step (vii). Otherwise go back to Step (ii).

vii. Display approximate solutions.

**4 Numerical Experiments**

Two examples of two-dimensional fuzzy parabolic problems are considered to verify the effectiveness of the proposed iterative methods. During implementation, the value of the tolerance error, considered, $\varepsilon = 10^{-10}$.

**Problem 1** [8]

$$\frac{\partial \tilde{U}}{\partial t} (x, y, t) = \frac{\partial^2 \tilde{U}}{\partial x^2} (x, y, t) + \frac{\partial^2 \tilde{U}}{\partial y^2} (x, y, t), \quad 0 \leq x, \ y \leq 1, \ t \geq 0 \tag{16}$$

where $\tilde{k}[\alpha] = [k(\alpha), \tilde{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ with the initial condition $\tilde{U}(x, y, 0) = \sin(\pi y) \sin(\pi x)$. The boundary conditions are $\tilde{U}(x, 0, t) = \tilde{U}(x, 1, t) = 0$ and $\tilde{U}(0, y, t) = \tilde{U}(1, y, t) = 0$. The exact solution for
\[
\frac{\partial U}{\partial t}(x, y; t; \alpha) = \frac{\partial^2 U}{\partial x^2}(x, y; t; \alpha) + \frac{\partial^2 U}{\partial y^2}(x, y; t; \alpha)
\] (17a)

and
\[
\frac{\partial \overline{U}}{\partial t}(x, y; t; \alpha) = \frac{\partial^2 \overline{U}}{\partial x^2}(x, y; t; \alpha) + \frac{\partial^2 \overline{U}}{\partial y^2}(x, y; t; \alpha)
\] (17b)

are
\[
U(x, y; t; \alpha) = k(\alpha) \sin(\pi y) \sin(\pi x) e^{-\pi^2 t}
\] (18a)

and
\[
\overline{U}(x, y; t; \alpha) = \overline{k}(\alpha) \sin(\pi y) \sin(\pi x) e^{-\pi^2 t}
\] (18b)

respectively.

**Problem 2**
\[
\frac{\partial \overline{U}}{\partial t}(x, y; t) = \frac{\partial^2 \overline{U}}{\partial x^2}(x, y; t) + \frac{\partial^2 \overline{U}}{\partial y^2}(x, y; t), \quad 0 \leq x, \ y \leq 1, \ t \geq 0
\] (19)

where \( \overline{k}[\alpha] = [k(\alpha), \overline{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha] \) with the initial condition
\( \overline{U}(x, y, 0) = \sin(\pi y) \sin(\pi x) \). The boundary conditions \( \overline{U}(0, t) = \overline{U}(1, t) = 0 \) and
\( \overline{U}(0, y, t) = \overline{U}(1, y, t) = 0 \). The exact solution for
\[
\frac{\partial \overline{U}}{\partial t}(x, y; t; \alpha) = \frac{\partial^2 \overline{U}}{\partial x^2}(x, y; t; \alpha) + \frac{\partial^2 \overline{U}}{\partial y^2}(x, y; t; \alpha)
\] (20a)

and
\[
\frac{\partial \overline{U}}{\partial t}(x, y; t; \alpha) = \frac{\partial^2 \overline{U}}{\partial x^2}(x, y; t; \alpha) + \frac{\partial^2 \overline{U}}{\partial y^2}(x, y; t; \alpha)
\] (20b)

are
\[
\overline{U}(x, y; t; \alpha) = \overline{k}(\alpha) \sin\left(\frac{1}{2} \pi y\right) \sin\left(\frac{1}{2} \pi x\right) e^{-\left(\frac{\pi^2}{2}\right) t}
\] (21a)

and
\[
\overline{U}(x, y; t; \alpha) = \overline{k}(\alpha) \sin\left(\frac{1}{2} \pi y\right) \sin\left(\frac{1}{2} \pi x\right) e^{-\left(\frac{\pi^2}{2}\right) t}
\] (21b)

respectively.

For the purpose of observations on the feasibility of the FSAGE, HSAGE and QSAGE methods, three parameters were observed such as number of iterations, execution time (in seconds) and Hausdorff distance (as mention in Definition 1) and recorded in Table 1 to 5.
Definition 1 [13]
Given two minimum bounding rectangles P and Q, a lower bound of the Hausdorff distance from the elements confined by P to the elements confined by Q is defined as

\[ HausDistLB(P, Q) = \max \{ \text{MinDist}(f_\alpha, Q) : f_\alpha \in \text{FacesOf}(P) \} \]

5 Conclusions

In this paper, the family of AGE iterative methods was used to solve linear systems arise from the discretization of fuzzy diffusion equation using the implicit difference scheme. The results show that QSAGE method is more superior in terms of the number of iterations, execution time and Hausdorff distance compared to the HSAGE, FSAGE and FSGS methods. Since AGE is well suited for parallel computation, it can be considered as a main advantage because this method has groups of independent task which can be implemented simultaneously. It is hoped that the capability of the proposed method will be helpful for the further investigation in solving any multi-dimensional fuzzy partial differential equations [7]. Other family of AGE methods as in [6,9,15,21] also can be used as linear solvers in solving the same problem.

References


http://dx.doi.org/10.1007/978-3-540-30497-5_10


Table 1: Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha = 0.00$.

<table>
<thead>
<tr>
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<th>Methods</th>
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<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
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<td>60</td>
<td>181</td>
<td>546</td>
<td>1500</td>
<td>2134</td>
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<td>35</td>
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<td>328</td>
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<td>QSAGE</td>
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<td>16</td>
<td>32</td>
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<td>Execution time</td>
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<td>0.80</td>
<td>4.41</td>
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Problem 2

| Number of iterations | FSGS | 168 | 561 | 1884 | 6186 | 19449 |
| HSAGE | 31 | 95 | 317 | 1077 | 3596 |
| QSAGE | 20 | 34 | 77 | 209 | 654 |
| Execution time | FSGS | 2.07 | 8.62 | 67.43 | 773.95 | 9764.77 |
| HSAGE | 0.81 | 3.42 | 36.09 | 396.16 | 5524.52 |
Table 1: (Continued): Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha = 0.00$.

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Table 2: Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha = 0.25$.

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Table 3: Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha = 0.50$.

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</thead>
<tbody>
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<td></td>
<td>FSGS</td>
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<td>183</td>
<td>560</td>
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<td>583</td>
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<td>107</td>
<td>335</td>
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<td></td>
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<td>79</td>
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<tr>
<td>Execution</td>
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<td>0.83</td>
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Table 3: (Continued): Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha=0.50$.

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
<th>QSAGE</th>
</tr>
</thead>
<tbody>
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<td>Hausdorff</td>
<td>5.7630e-05</td>
<td>5.4145e-05</td>
<td>7.2506e-05</td>
<td>5.8154e-05</td>
</tr>
<tr>
<td>Distance</td>
<td>7.6107e-04</td>
<td>7.6111e-04</td>
<td>7.6116e-04</td>
<td>7.6129e-04</td>
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<tr>
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<td>8.62</td>
<td>68.65</td>
<td>783.15</td>
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Table 4: Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha=0.75$.

<table>
<thead>
<tr>
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<th>Methods</th>
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<th>$32$</th>
<th>$64$</th>
<th>$128$</th>
<th>$256$</th>
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</thead>
<tbody>
<tr>
<td>FSGS</td>
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<td>184</td>
<td>562</td>
<td>1585</td>
<td>2698</td>
<td></td>
</tr>
<tr>
<td>FSAGE</td>
<td>22</td>
<td>61</td>
<td>190</td>
<td>586</td>
<td>1645</td>
<td></td>
</tr>
<tr>
<td>HSAGE</td>
<td>14</td>
<td>36</td>
<td>108</td>
<td>337</td>
<td>1003</td>
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</tr>
<tr>
<td>QSAGE</td>
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<table>
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<table>
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<th>HSAGE</th>
<th>QSAGE</th>
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</thead>
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<td>7.1698e-07</td>
<td>6.9921e-07</td>
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<tr>
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</table>
Table 5: Numerical results of FSGS, FSAGE, HSAGE and QSAGE methods at $\alpha = 1.00$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Methods</th>
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<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
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<tbody>
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<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>HSAGE</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>QSAGE</td>
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Received: January 20, 2015; Published: June 26, 2015