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Fractional Hermite-Hadamard-like Type Inequalities for Convex Functions

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Abstract

In this article, by setting up a generalized integral identity for differentiable functions via Riemann-Liouville fractional integrals, new estimates on generalization of Hermite-Hadamard-like types inequalities for functions whose derivatives in the absolute value at certain powers are convex are given.

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1 Introduction

Let $f : I \subseteq R_+ \to R$ be a function defined on the interval I of real numbers. Then f is called to be convex on I if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$, $t \in [0, 1]$.

There are many results associated with convex functions in the area of inequalities, but some of those is the classical Hermite-Hadamard type inequality[10, 11, 14]: Let $f : I \subseteq R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then following double inequality holds:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In recent paper[18], Tseng et. al established the following result which gives a refinement of (1):

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\leq \frac{f\big(\frac{3a+b}{4}\big)+f\big(\frac{a+3b}{4}\big)}{2} \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\leq \frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big] \leq \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $f : [a, b] \to R$ is a convex function.

Definition 1. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, x, y > 0,$$

and

$$\beta(a, x, y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, 0 < a < 1, x, y > 0,$$

is incomplete Beta function.

Definition 2. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, x-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t$$

for $c > b > 0$ and $\mid z \mid< 1$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. Let $f \in L([a, b])$. The symbols $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ denote the left-side and right-side Riemann-Liouville integrals of the order $\alpha > 0$ and are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t \quad (0 \leq a < x),$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \quad (0 < x < b),$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$ and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

In the caes of $\alpha = 1$, the fractional integrals reduces to the classical integral. Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, you may see [1-10,13-20] and the references cited therein.

Especially, in [3, 12], Imdat Işcan, Noor, and Awan proved a variant of Hermite-Hadamard-like type and Ostrowski-like type inequalities which hold for the convex functions via Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : I \subseteq [0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I^0$ with $a < b$. If $| f' |^q$ is s-convex in the second sense on $[a, b]$ for some fixed $q \geq 1$, $x \in [a, b]$, $\mu \in [0, 1]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|(1-\mu)\Big\{\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\Big\} f(x) \\
&\quad + \mu\Big\{\frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a}\Big\} \\
&\quad - \frac{\Gamma(\alpha+1)}{b-a}\Big\{J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)\Big\}\Big| \\
&\leq A_1^{1-\frac{1}{q}}(\alpha, \mu) \\
&\quad \times \Big[\frac{(x-a)^{\alpha+1}}{b-a}\Big\{\big|f'(x)\big|^q A_2(\alpha,\mu,s) + \big|f'(a)\big|^q A_3(\alpha,\mu,s)\Big\}^{\frac{1}{q}} \\
&\qquad + \frac{(b-x)^{\alpha+1}}{b-a}\Big\{\big|f'(x)\big|^q A_2(\alpha,\mu,s) + \big|f'(b)\big|^q A_3(\alpha,\mu,s)\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha,\mu) &= \frac{2\alpha\mu^{1+\frac{1}{\alpha}} + 1}{\alpha+1} - \mu, \\
A_2(\alpha,\mu,s) &= \frac{2\alpha\mu^{1+\frac{s+1}{\alpha}} + s + 1}{(s+1)(\alpha+s+1)} - \frac{\mu}{s+1}, \\
A_3(\alpha,\mu,s) &= \mu\Big(\frac{1-2(1-\mu^{\frac{1}{\alpha}})^{s+1}}{s+1}\Big) + \beta(\alpha+1, s+1) \\
&\quad - 2\beta(\mu^{\frac{1}{\alpha}}, \alpha+1, s+1).
\end{aligned}$$

In this paper, by setting up a generalized integral identity for differentiable functions via Riemann-Liouville fractional integrals, we derive new estimates on generalization of Hermite-Hadamard-like types inequalities for functions whose derivatives in the absolute value at certain powers are convex.

2 Lemmas

Now we turn our attention to establish integral inequalities of Hermite-Hadamard-like type inequality for convex functions via Riemann-Liouville fractional integrals, we need the following lemmas:

Lemma 1. *Let $f : I \to R$ be a differentiable function on the interior I^0 of an interval I in R_+ such that $f' \in L([a,b])$, where $a, b \in I$ with $0 \le a < b$. Then, for any $\lambda, \mu \in [0,1]$ and $\alpha > 0$, the following identity holds:*

$$\begin{aligned} &I_f(a,b;x;\lambda,\mu) \\ &\equiv \frac{(x-a)^\alpha}{b-a}\Big\{\lambda f(a) + (1-\lambda) f\big(x\big)\Big\} + \frac{(b-x)^\alpha}{b-a}\Big\{\mu f\big(x\big) + (1-\mu) f(b)\Big\} \\ &\quad - \frac{\Gamma(\alpha+1)}{b-a}\Big\{J^\alpha_{a^+} f(x) + J^\alpha_{b^-} f(x)\Big\} \\ &= \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 (1-\lambda-t^\alpha) f'\big(ta+(1-t)x\big)\mathrm{d}t \\ &\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 (t^\alpha-\mu) f'\big(tb+(1-t)x\big)\mathrm{d}t, \end{aligned} \tag{2}$$

where $x = ra + (1-r)b$ for $r \in [0,1]$.

Proof. Integrating by parts and changing variable of definite integral, we have:

$$\begin{aligned} &\int_0^1 (1-\lambda-t^\alpha) f'\big(ta+(1-t)x\big)\mathrm{d}t \\ &= \frac{1}{x-a}\Big\{\lambda f(a) + \big(1-\lambda\big) f\big(x\big) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} J^\alpha_{a^+} f\big(x\big)\Big\}. \end{aligned} \tag{3}$$

Similarly, we have

$$\begin{aligned} &\int_0^1 (t^\alpha-\mu) f'\big(tb+(1-t)x\big)\mathrm{d}t \\ &= \frac{1}{b-x}\Big\{(1-\mu) f(b) + \mu f\big(x\big) - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha} J^\alpha_{b^-} f\big(x\big)\Big\}. \end{aligned} \tag{4}$$

By the simple computations, we obtain the desired result.

Note that, (a) if we choose $x = \frac{a+b}{2}$ in Lemma 1, then we have

$$\begin{aligned}
&\frac{1}{2}\Big[\Big\{\lambda f(a) + (1-\lambda) f\big(\frac{a+b}{2}\big)\Big\} + \Big\{\mu f\big(\frac{a+b}{2}\big) + (1-\mu) f(b)\Big\}\Big] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big\{J^\alpha_{a^+} f\big(\frac{a+b}{2}\big) + J^\alpha_{b^-} f\big(\frac{a+b}{2}\big)\Big\} \\
&= \frac{b-a}{4}\Big[\int_0^1 (1-\lambda-t^\alpha) f'\big(ta + (1-t)\frac{a+b}{2}\big)\mathrm{d}t \\
&\quad + \int_0^1 (t^\alpha - \mu) f'\big(tb + (1-t)\frac{a+b}{2}\big)\mathrm{d}t\Big], \qquad (5)
\end{aligned}$$

and, (b) if we choose $\lambda = \mu = \frac{1}{2}$ and $x = \frac{a+b}{2}$ in Lemma 1, then we have

$$\begin{aligned}
&\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2} + f\big(\frac{a+b}{2}\big)\Big\} \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big\{J^\alpha_{a^+} f\big(\frac{a+b}{2}\big) + J^\alpha_{b^-} f\big(\frac{a+b}{2}\big)\Big\} \\
&= \frac{b-a}{4}\Big[\int_0^1 \big(\frac{1}{2} - t^\alpha\big) f'\big(ta + (1-t)\frac{a+b}{2}\big)\mathrm{d}t \\
&\quad + \int_0^1 \big(t^\alpha - \frac{1}{2}\big) f'\big(tb + (1-t)\frac{a+b}{2}\big)\mathrm{d}t\Big]. \qquad (6)
\end{aligned}$$

For the simplicities of notations, let

$$\begin{aligned}
\delta_1(\alpha,\xi) &= \int_0^1 \big|t^\alpha - \xi\big|\mathrm{d}t, \qquad \delta_2(\alpha,\xi) = \int_0^1 t\big|t^\alpha - \xi\big|\mathrm{d}t, \\
\varrho_1(\alpha,\xi,p) &= \int_0^1 \big|t^\alpha - \xi\big|^p\mathrm{d}t, \quad \varrho_2(\alpha,\xi,p) = \int_0^1 t\big|t^\alpha - \xi\big|^p\mathrm{d}t. \qquad (7)
\end{aligned}$$

Lemma 2. *For $0 \le \xi \le 1$, one has*

$$(a)\delta_1(\alpha,\xi) = \begin{cases} \frac{1}{\alpha+1}, & \xi = 0 \\ \frac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \xi, & 0 < \xi < 1 \\ \frac{\alpha}{\alpha+1}, & \xi = 1, \end{cases}$$

$$(b)\delta_2(\alpha,\xi) = \begin{cases} \frac{1}{\alpha+2}, & \xi = 0 \\ \frac{\alpha\xi^{1+\frac{2}{\alpha}}+1}{\alpha+2} - \frac{\xi}{2}, & 0 < \xi < 1 \\ \frac{\alpha}{2(\alpha+2)}, & \xi = 1, \end{cases}$$

$$(c)\varrho_1(\alpha,\xi,p)=\begin{cases}\frac{1}{\alpha p+1}, & \xi=0\\ \frac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta(\frac{1}{\alpha},p+1)+\frac{(1-\xi)^{p+1}}{\alpha(p+1)} & \\ \times{}_2F_1(1-\frac{1}{\alpha},1;p+2;1-\xi), & 0<\xi<1\\ \frac{1}{\alpha}\beta(p+1,\frac{1}{\alpha}), & \xi=1,\end{cases}$$

$$(d)\varrho_2(\alpha,\xi,p)=\begin{cases}\frac{1}{\alpha p+2}, & \xi=0\\ \frac{\xi^{p+\frac{2}{\alpha}}}{\alpha}\beta(\frac{2}{\alpha},p+1)+\frac{(1-\xi)^{p+1}}{\alpha(p+1)} & \\ \times{}_2F_1(1-\frac{2}{\alpha},1;p+2;1-\xi), & 0<\xi<1\\ \frac{1}{\alpha}\beta(p+1,\frac{2}{\alpha}), & \xi=1.\end{cases}$$

Proof. These equalities follows from a straightfoward computation of definite integrals.

3 Main results

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard and Ostrowski type for convex functions via fractional integrals.

Theorem 3.1. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a, b \in I$ with $0 \le a < b$. If $|f'|$ is convex on $[a,b]$ for all $\lambda, \mu \in [0,1]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Big|I_f(a,b;x;\lambda,\mu)\Big|\\ &\le \frac{(x-a)^{\alpha+1}}{b-a}\Big[\delta_2(\alpha,1-\lambda)\big|f'(a)\big|\\ &\qquad +\big\{\delta_1(\alpha,1-\lambda)-\delta_2(\alpha,1-\lambda)\big|f'(x)\big|\Big]\\ &\quad +\frac{(b-x)^{\alpha+1}}{b-a}\Big[\delta_2(\alpha,\mu)\big|f'(b)\big|\\ &\qquad +\big\{\delta_1(\alpha,\mu)-\delta_2(\alpha,\mu)\big|f'(x)\big|\Big],\end{aligned}$$

where

$$\delta_1(\alpha,\xi)=\frac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\xi,\quad \delta_2(\alpha,\xi)=\frac{\alpha\xi^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\xi}{2}$$

for $\xi \in [0,1]$.

Proof. From Lemma 1 and the convexity of $| f' |$ on $[a, b]$, we have

$$\begin{aligned}
&\Big|I_f(a,b;x;\lambda,\mu)\Big| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \big|1-\lambda-t^\alpha\big|\big|f'\big(ta+(1-t)x\big)\big|\mathrm{d}t \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \big|t^\alpha-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|\mathrm{d}t\Big] \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \big|1-\lambda-t^\alpha\big|\Big\{t\big|f'\big(a\big)\big|+(1-t)\big|f'\big(x\big)\big|\Big\}\mathrm{d}t \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \big|t^\alpha-\mu\big|\Big\{t\big|f'\big(b\big)\big|+(1-t)\big|f'\big(x\big)\big|\Big\}\mathrm{d}t \\
&= \frac{(x-a)^{\alpha+1}}{b-a}\Big[\Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|t\mathrm{d}t\Big\}\big|f'\big(a\big)\big| \\
&\qquad + \Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|(1-t)\mathrm{d}t\Big\}\big|f'(x)\big|\Big] \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\Big[\Big\{\int_0^1 \big|t^\alpha-\mu\big|t\mathrm{d}t\Big\}\big|f'(b)\big| \\
&\qquad + \Big\{\int_0^1 \big|t^\alpha-\mu\big|(1-t)\mathrm{d}t\mathrm{d}t\Big\}\big|f'(x)\big|\Big\}\Big] \\
&= \frac{(x-a)^{\alpha+1}}{b-a}\Big[\delta_2(\alpha,1-\lambda)\big|f'(a)\big| \\
&\qquad + \Big\{\delta_1(\alpha,1-\lambda)-\delta_2(\alpha,1-\lambda)\Big\}\big|f'(x)\big|\Big] \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\Big[\delta_2(\alpha,\mu)\big|f'(b)\big| \\
&\qquad + \Big\{\delta_1(\alpha,\mu)-\delta_2(\alpha,\mu)\Big\}\big|f'(x)\big|\Big].
\end{aligned}$$

Corollary 3.1. *In Theorem 3.1, if we choose* $\lambda = \mu = \frac{1}{2}$ *and* $x = \frac{a+b}{2}$*, we have*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2}+f\big(\frac{a+b}{2}\big)\Big\} \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big\{J^\alpha_{a^+}f(\frac{a+b}{2})+J^\alpha_{b^-}f(\frac{a+b}{2})\Big\}\Big| \\
&\le \frac{b-a}{4}\Big[\delta_2(\alpha,\frac{1}{2})\Big\{\big|f'(a)\big|+\big|f'(b)\big|\Big\} \\
&\quad + 2\Big\{\delta_1(\alpha,\frac{1}{2})-\delta_2(\alpha,\frac{1}{2})\Big\}\big|f'\big(\frac{a+b}{2}\big)\big|\Big].
\end{aligned}$$

Theorem 3.2. *Let* $f : I \subseteq R \to R$ *be a differentiable function on the interior* I^0 *of an interval* I *and* $f' \in L([a, b])$*, where* $a, b \in I$ *with* $0 \le a < b$*. If* $|f'|^q$ *is* *convex on $[a,b]$ for a fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for all $\lambda,\mu\in[0,1]$ and $\alpha>0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|I_f(a,b;x;\lambda,\mu)\Big| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\varrho_1^{\frac{1}{p}}(\alpha,1-\lambda,p)\Big\{\frac{\big|f'(a)\big|^q+\big|f'(x)\big|^q}{2}\Big\}^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\varrho_1^{\frac{1}{p}}(\alpha,\mu,p)\Big\{\frac{\big|f'(b)\big|^q+\big|f'(x)\big|^q}{2}\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{8}$$

Proof. From Lemma 1, by using the well-known Hölder integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;x;\lambda,\mu)\Big| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \big|1-\lambda-t^\alpha\big|\big|f'\big(ta+(1-t)x\big)\big|\mathrm{d}t \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \big|t^\alpha-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|\mathrm{d}t \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|^p\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f'\big(ta+(1-t)x\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\Big\{\int_0^1 \big|t^\alpha-\mu\big|^p\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f'\big(tb+(1-t)x\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&= \frac{(x-a)^{\alpha+1}}{b-a}\varrho_1^{\frac{1}{p}}(\alpha,1-\lambda,p)\Big\{\int_0^1 \big|f'\big(ta+(1-t)x\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\varrho_1^{\frac{1}{p}}(\alpha,\mu,p)\Big\{\int_0^1 \big|f'\big(tb+(1-t)x\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{9}$$

Since $\mid f'\mid^q$ is convex on $[a,b]$, we have

$$\int_0^1 \mid f'\big(ta+(1-t)x\big)\mid^q \mathrm{d}t \le \frac{\big|f'(a)\big|^q+\big|f'(x)\big|^q}{2}, \tag{10}$$

$$\int_0^1 \mid f'\big(tb+(1-t)x\big)\mid^q \mathrm{d}t \le \frac{\big|f'(b)\big|^q+\big|f'(x)\big|^q}{2}. \tag{11}$$

By substituting (10) and (11) in (9), we get the desired result.

Corollary 3.2. *In Theorem 3.2, if we choose $\lambda=\mu=\frac{1}{2}$ and $x=\frac{a+b}{2}$, we* *have*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2}+f\big(\frac{a+b}{2}\big)\Big\} \\
&\quad -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big\{J_{a^+}^{\alpha}f(\frac{a+b}{2})+J_{b^-}^{\alpha}f(\frac{a+b}{2})\Big\}\Big| \\
&\leq \frac{b-a}{4}\varrho_1^{\frac{1}{p}}(\alpha,\frac{1}{2},p)\Big[\Big\{\frac{\mid f'(a)\mid^q+\mid f'(\frac{a+b}{2})\mid^q}{2}\Big\}^{\frac{1}{q}} \\
&\quad +\Big\{\frac{\mid f'(b)\mid^q+\mid f'(\frac{a+b}{2})\mid^q}{2}\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

By using the double inequality (1), we have the following corollary:

Corollary 3.3. *In Theorem 3.2, if we choose $\lambda=\mu=\frac{1}{2}$ and $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2}+f\big(\frac{a+b}{2}\big)\Big\} \\
&\quad -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big\{J_{a^+}^{\alpha}f(\frac{a+b}{2})+J_{b^-}^{\alpha}f(\frac{a+b}{2})\Big\}\Big| \\
&\leq \frac{b-a}{4}\Big[\varrho_1^{\frac{1}{p}}(\alpha,1-\lambda,p)^{\frac{1}{p}}\Big\{\frac{3\mid f'\big(a\big)\mid^q+\mid f'\big(b\big)\mid^q}{4}\Big\}^{\frac{1}{q}} \\
&\quad +\varrho_1^{\frac{1}{p}}(\alpha,\mu,p)^{\frac{1}{p}}\Big\{\frac{\mid f'\big(b\big)\mid^q+3\mid f'\big(b\big)\mid^q}{4}\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Theorem 3.3. *Let $f: I\subseteq R\to R$ be a differentiable function on the interior I^0 of an interval I and $f'\in L([a,b])$, where $a,b\in I$ with $0\leq a<b$. If $|f'|^q$ is convex on $[a,b]$ for a fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for all $\lambda,\mu\in[0,1]$ and $\alpha>0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\lambda,\mu)\Big| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\Big[\varrho_2(\alpha,1-\lambda,q)\big|f'(a)\big|^q \\
&\qquad +\big\{\varrho_1(\alpha,1-\lambda,q)-\varrho_2(\alpha,1-\lambda,q)\big\}\big|f(x)\big|^q\Big]^{\frac{1}{q}} \\
&\quad +\frac{(b-x)^{\alpha+1}}{b-a}\Big[\varrho_2(\alpha,\mu,q)\big|f'(b)\big|^q \\
&\qquad +\big\{\varrho_1(\alpha,\mu,q)-\varrho_2(\alpha,\mu,q)\big\}\big|f(x)\big|^q\Big]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1, by using the well-known Hölder integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\lambda,\mu)\Big| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|^q\big|f'\big(ta+(1-t)\frac{a+b}{2}\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\Big\{\int_0^1 \big|t^\alpha-\mu\big|^q\big|f'\big(tb+(1-t)\frac{a+b}{2}\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (12)
\end{aligned}$$

Note that

$$\begin{aligned}
&(i) \int_0^1 \big|1-\lambda-t^\alpha\big|^q\big|f'\big(ta+(1-t)x\big)\big|^q \mathrm{d}t \\
&\quad \le \int_0^1 \big|1-\lambda-t^\alpha\big|^q\Big\{t\big|f'(a)\big|^q+(1-t)\big|f(x)\big|^q\Big\}\mathrm{d}t \\
&\quad = \varrho_2(\alpha,1-\lambda,q)\big|f'(a)\big|^q \\
&\qquad + \big\{\varrho_1(\alpha,1-\lambda,q)-\varrho_2(\alpha,1-\lambda,q)\big\}\big|f'(x)\big|^q, \qquad (13) \\
&(ii) \int_0^1 \big|t^\alpha-\mu\big|^q\big|f'\big(tb+(1-t)x\big)\big|^q \mathrm{d}t \\
&\quad \le \int_0^1 \big|t^\alpha-\mu\big|^q\Big\{t\big|f'(b)\big|^q+(1-t)\big|f(x)\big|^q\Big\}\mathrm{d}t \\
&\quad = \varrho_2(\alpha,\mu,q)\big|f'(b)\big|^q+\big\{\varrho_1(\alpha,\mu,q)-\varrho_2(\alpha,\mu,q)\big\}\big|f'(x)\big|^q. \qquad (14)
\end{aligned}$$

By substituting (13) and (14) in (12), we get the desired result.

Corollary 3.4. *In Theorem 3.3, if we choose* $\lambda=\mu=\frac{1}{2}$ *and* $x=\frac{a+b}{2}$, *we have*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2}+f\big(\frac{a+b}{2}\big)\Big\} \\
&\qquad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big\{J^\alpha_{a^+}f\big(\frac{a+b}{2}\big)+J^\alpha_{b^-}f\big(\frac{a+b}{2}\big)\Big\}\Big| \\
&\le \frac{b-a}{4}\Big[\Big\{\varrho_2(\alpha,\frac{1}{2},q))\big|f'(a)\big|^q \\
&\qquad + \big(\varrho_1(\alpha,\frac{1}{2},q)-\varrho_2(\alpha,\frac{1}{2},q)\big)\big|f'\big(\frac{a+b}{2}\big)\big|^q\Big\}^{\frac{1}{q}} \\
&\qquad + \Big\{\varrho_2(\alpha,\frac{1}{2},q))\big|f'(b)\big|^q \\
&\qquad + \big(\varrho_1(\alpha,\frac{1}{2},q)-\varrho_2(\alpha,\frac{1}{2},q)\big)\big|f'\big(\frac{a+b}{2}\big)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Corollary 3.5. *In Theorem 3.3, if we choose $\lambda = \mu = \frac{1}{2}$ and $x = \frac{a+b}{2}$, we have*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big\} \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big\{J^{\alpha}_{a^+}f(\frac{a+b}{2}) + J^{\alpha}_{b^-}f(\frac{a+b}{2})\Big\}\Big| \\
&\leq \frac{b-a}{4}\Big[\Big\{\Big(\frac{\varrho_1(\alpha,\frac{1}{2},q)+\varrho_2(\alpha,\frac{1}{2},q)}{2}\Big)\big|f'(a)\big|^q \\
&\quad + \Big(\frac{\varrho_1(\alpha,\frac{1}{2},q)-\varrho_2(\alpha,\frac{1}{2},q)}{2}\Big)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big(\frac{\varrho_1(\alpha,\frac{1}{2},q)-\varrho_2(\alpha,\frac{1}{2},q)}{2}\Big)\big|f'(a)\big|^q \\
&\quad + \Big(\frac{\varrho_1(\alpha,\frac{1}{2},q)+\varrho_2(\alpha,\frac{1}{2},q)}{2}\Big)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Theorem 3.4. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a, b \in I$ with $0 \leq a < b$. If $|f'|^q$ is convex on $[a,b]$ for a fixed $q \geq 1$, then, for all $\lambda, \mu \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|I_f(a,b;x;\lambda,\mu)\Big| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\delta_1^{1-\frac{1}{q}}(\alpha,1-\lambda)\Big\{\delta_2(\alpha,1-\lambda) \mid f'\big(a\big) \mid^q \\
&\quad + \big\{\delta_1(\alpha,1-\lambda) - \delta_2(\alpha,1-\lambda)\big\} \mid f'(x) \mid^q \Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\delta_1^{1-\frac{1}{q}}(\alpha,\mu)\Big\{\delta_2(\alpha,\mu) \mid f'\big(b\big) \mid^q \\
&\quad + \big\{\delta_1(\alpha,\mu) - \delta_2(\alpha,\mu)\big\} \mid f'(x) \mid^q \Big\}^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1, by using the well-known power-mean integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;x;\lambda,\mu)\Big| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \big|1-\lambda-t^{\alpha}\big|\big|f'\big(ta+(1-t)x\big)\big|\mathrm{d}t \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \big|t^{\alpha}-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\Big[\Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|\mathrm{d}t\Big\}^{1-\frac{1}{q}} \\
&\quad \times \Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|\big|f'\big(ta+(1-t)x\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\int_0^1 \big|t^\alpha-\mu\big|\mathrm{d}t\Big\}^{1-\frac{1}{q}}\Big\{\int_0^1 \big|t^\alpha-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big] \\
&= \frac{(x-a)^{\alpha+1}}{b-a}\delta_1^{1-\frac{1}{q}}(\alpha,1-\lambda)\Big\{\int_0^1 \big|1-\lambda-t^\alpha\big|\big|f'\big(ta+(1-t)x\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\delta_1^{1-\frac{1}{q}}(\alpha,\mu)\Big\{\int_0^1 \big|t^\alpha-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}. \qquad (15)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&(i) \int_0^1 \big|1-\lambda-t^\alpha\big|\big|f'\big(ta+(1-t)x\big)\big|^q \mathrm{d}t \\
&\quad \leq \delta_2(\alpha,1-\lambda)\big|f'(a)\big|^q + \big\{\delta_1(\alpha,1-\lambda)-\delta_2(\alpha,1-\lambda)\big\}\big|f'(x)\big|^q, \qquad (16) \\
&(ii) \int_0^1 \big|t^\alpha-\mu\big|\big|f'\big(tb+(1-t)x\big)\big|^q \mathrm{d}t \\
&\quad \leq \delta_2(\alpha,\mu)\big|f'(b)\big|^q + \big\{\delta_1(\alpha,\mu)-\delta_2(\alpha,\mu)\big\}\big|f'(x)\big|^q. \qquad (17)
\end{aligned}
$$

By substituting (16) and (17) in (15), we get the desired result.

Corollary 3.6. *In Theorem 3.4, if we choose* $\lambda = \mu = \frac{1}{2}$ *and* $x = \frac{a+b}{2}$*, we have*

$$
\begin{aligned}
&\Big|\frac{1}{2}\Big\{\frac{f(a)+f(b)}{2} + f\big(\frac{a+b}{2}\big)\Big\} \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big\{J_{a^+}^\alpha f(\frac{a+b}{2}) + J_{b^-}^\alpha f(\frac{a+b}{2})\Big\}\Big| \\
&\leq \frac{b-a}{4}\delta_1^{1-\frac{1}{q}}(\alpha,\frac{1}{2})\Big[\Big\{\delta_2(\alpha,\frac{1}{2})\big|f'(a)\big|^q \\
&\quad + \big(\delta_1(\alpha,\frac{1}{2})-\delta_2(\alpha,\frac{1}{2})\big)\big|f'(\frac{a+b}{2})\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\delta_2(\alpha,\frac{1}{2})\big|f'(b)\big|^q \\
&\quad + \big(\delta_1(\alpha,\frac{1}{2})-\delta_2(\alpha,\frac{1}{2})\big)\big|f'(\frac{a+b}{2})\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}
$$

By the double inequality (1), we get the following corollary:

Corollary 3.7. *In Theorem 3.4, if we choose* $\lambda = \mu = \frac{1}{2}$ *and* $x = \frac{a+b}{2}$*, we have*

$$\begin{aligned}
&\Big|I_f(a,b;\frac{1}{2},\frac{1}{2})\Big| \\
&\leq \frac{b-a}{4}\delta_1^{1-\frac{1}{q}}(\alpha,\frac{1}{2})\Big[\Big\{\Big(\frac{\delta_1(\alpha,\frac{1}{2})+\delta_2(\alpha,\frac{1}{2})}{2}\Big)\big|f'(a)\big|^q \\
&\qquad + \Big(\frac{\delta_1(\alpha,\frac{1}{2})-\delta_2(\alpha,\frac{1}{2})}{2}\Big)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big(\frac{\delta_1(\alpha,\frac{1}{2})-\delta_2(\alpha,\frac{1}{2})}{2}\Big)\big|f'(a)\big|^q \\
&\qquad + \Big(\frac{\delta_1(\alpha,\frac{1}{2})+\delta_2(\alpha,\frac{1}{2})}{2}\Big)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

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