Extended Goursat’s Hypergeometric Function

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Abstract

In this article, we define an extended form of the Goursat’s hypergeometric function and derive several results pertaining to it. We show that this function occurs naturally in statistical distribution theory.

Mathematics Subject Classification: 33E99, 60E05

Keywords: Beta distribution; extended beta function; extended confluent hypergeometric function; extended Gauss hypergeometric function; gamma distribution; quotient; Gauss hypergeometric function

1 Introduction

The classical beta function, denoted by $B(a,b)$, is defined (see Luke [6]) by the Euler’s integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt,$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \text{ Re}(a) > 0, \text{ Re}(b) > 0. \quad (1)$$
Based on the beta function, the Gauss hypergeometric function, denoted by $F(a, b; c; z)$, and the confluent hypergeometric function, denoted by $\Phi(b; c; z)$, for $\text{Re}(c) > \text{Re}(b) > 0$, are defined as (see Luke [6]),

$$F(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \, dt, \quad |\arg(1-z)| < \pi, \quad (2)$$

and

$$\Phi(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp(zt) \, dt. \quad (3)$$

It is also well known that, under certain conditions, the Goursat’s function (Goursat [3, p. 286]) or generalized hypergeometric function $2F_2(a, b; c, d; z) \equiv G(a, b; c, d; z)$, is defined by

$$G(a, b; c, d; z) = \frac{1}{B(a, d - a)} \int_0^1 v^{a-1}(1-v)^{d-a-1}\Phi(b; c; zv) \, dv, \quad (4)$$

where $\text{Re}(d) > \text{Re}(a) > 0$. Using the series expansions of $(1-zt)^{-a}$ and $\exp(zt)$ in (2) and (3), respectively, the following series representations of hypergeometric functions can be obtained:

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!}, \quad |z| < 1, \, \text{Re}(c) > \text{Re}(b) > 0, \quad (5)$$

$$\Phi(b; c; z) = \sum_{n=0}^{\infty} \frac{B(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!}, \quad \text{Re}(c) > \text{Re}(b) > 0. \quad (6)$$

In 1997, Chaudhry et al. [1] extended the classical beta function to the whole complex plane by introducing in the integrand of (1) the exponential factor $\exp \left[ -\sigma/t(1-t) \right]$ with $\text{Re}(\sigma) > 0$. Thus, the extended beta function is defined as

$$B(a, b; \sigma) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp \left[ -\frac{\sigma}{t(1-t)} \right] \, dt, \quad \text{Re}(\sigma) > 0. \quad (7)$$

If we take $\sigma = 0$ in (7), then for $\text{Re}(a) > 0$ and $\text{Re}(b) > 0$ we have $B(a, b; 0) = B(a, b)$. Further, replacing $t$ by $1 - t$ in (7), one can see that $B(a, b; \sigma) = B(b, a; \sigma)$. The rationale and justification for introducing this function are given in Chaudhry et al. [1] where several properties and a statistical application have also been studied. Miller [7] further studied this function and has given several additional results.

In 2004, Chaudhry et al. [2] gave definitions of the extended Gauss hypergeometric function and the extended confluent hypergeometric function,
denoted by $F_\sigma(a, b; c; z)$ and $\Phi_\sigma(b; c; z)$, respectively. These definitions were developed by considering the extended beta function (7) instead of beta function (1) that appears in the general term of the series (5) and (6). They defined, for $\Re(c) > \Re(b) > 0$, the extended Gauss hypergeometric function and the extended confluent hypergeometric function as

\begin{equation}
F_\sigma(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b + n, c - b; \sigma)}{B(b, c - b) n!} \frac{z^n}{n!}, \sigma \geq 0, |z| < 1, \quad (8)
\end{equation}

\begin{equation}
\Phi_\sigma(b; c; z) = \sum_{n=0}^{\infty} \frac{B(b + n, c - b; \sigma)}{B(b, c - b) n!} \frac{z^n}{n!}, \sigma \geq 0. \quad (9)
\end{equation}

Further, using the integral representation of the extended beta function (7) in (8) and (9), Chaudhry et al. [2] obtained integral representations, for $\sigma \geq 0$ and $\Re(c) > \Re(b) > 0$, of extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF) as

\begin{equation}
F_\sigma(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - zt)^a} \exp \left[ -\frac{\sigma}{t(1-t)} \right] \exp \left[ -\frac{\sigma}{t(1-t)} \right] \frac{dt}{\arg(1-z)} < \pi, \quad (10)
\end{equation}

and

\begin{equation}
\Phi_\sigma(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - zt)^a} \exp \left[ -\frac{\sigma}{t(1-t)} \right] \exp \left[ -\frac{\sigma}{t(1-t)} \right] \frac{dt}{\arg(1-z)} < \pi, \quad (11)
\end{equation}

respectively.

For $\sigma = 0$ in (10), we have $F_0(a, b; c; z) = F(a, b; c; z)$. That is, the classical Gauss hypergeometric function is a special case of the extended Gauss hypergeometric function. Likewise, taking $\sigma = 0$ in (11) yields $\Phi_0(b; c; z) = \Phi(b; c; z)$, which means that the classical confluent hypergeometric function is a special case of the extended confluent hypergeometric function. Chaudhry et al. [2] and Miller [7] found that extended forms of beta and hypergeometric functions are related to the extended beta function, Bessel and Whittaker functions, and also gave several alternative integral representations.

In this article, we define and study the extended Goursat’s hypergeometric function and derive several results pertaining to it. We also show that this function occurs in a natural way in statistical distribution theory. An extended form of the Goursat’s hypergeometric function has been defined in Section 2 and applications of the extended Goursat’s hypergeometric function are discussed in Section 3.
2 Extended Goursat’s Hypergeometric Function

In this section, we define an extended form of the Goursat’s hypergeometric function by replacing the confluent hypergeometric by its extended form in (4).

Following Chaudhry et al. [2], the extended Goursat’s hypergeometric function $G_{\sigma}(a, b; c, d; z)$ is defined by

$$G_{\sigma}(a, b; c, d; z) = \frac{1}{B(a, d - a)} \int_{0}^{1} v^{a-1} (1 - v)^{d-a-1} \Phi_{\sigma}(b; c; zv) \, dv,$$  \hspace{1em} (12)

where $\text{Re}(c) > \text{Re}(b) > 0$, and $\text{Re}(d) > \text{Re}(a) > 0$. By taking $\sigma = 0$ in (12) one gets $G_{0}(a, b; c, d; z) = {}_{2}F_{2}(a, b; c, d; z)$, which means that the classical generalized hypergeometric function is a special case of the extended Goursat’s function.

Replacing $\Phi_{\sigma}(b; c; z)$ by its equivalent integral representation given in (11) and changing the order of integration, the integral in (12) is re-written as

$$G_{\sigma}(a, b; c, d; z) = \frac{1}{B(b, c - b)} \int_{0}^{1} t^{b-1} (1 - t)^{c-b-1} \exp \left[-\frac{\sigma}{t(1-t)}\right] \Phi(a; d; zt) \, dt,$$  \hspace{1em} (13)

where the last line has been obtained by using (3). For $a = d$, we have $\Phi(a; d; zt) = \exp(zt)$, and the above expression gives $G_{\sigma}(a, b; c, a; z) = \Phi_{\sigma}(b; c; z)$. Taking $z = 0$ in (13) and applying the definition of the extended beta function, we arrive at

$$G_{\sigma}(a, b; c, d; 0) = \frac{B(b, c - b; \sigma)}{B(b, c - b)}.$$

In the integral representation given in (13), by substituting $t = (1 + u)^{-1}u$, with the Jacobian $J(t \rightarrow u) = (1 + u)^{-2}$, an alternative integral representation is obtained as

$$G_{\sigma}(a, b; c, d; z) = \frac{\exp(-2\sigma)}{B(b, c - b)} \int_{0}^{\infty} \frac{u^{b-1}}{(1 + u)^c} \exp \left[-\sigma \left(u + \frac{1}{u}\right)\right] \Phi \left(a; d; \frac{zu}{1 + u}\right) \, du.$$

If we take $\sigma = 0$ in the above expression, we arrive at

$$G(a, b; c, d; z) = \frac{1}{B(b, c - b)} \int_{0}^{\infty} \frac{u^{b-1}}{(1 + u)^c} \Phi \left(a; d; \frac{zu}{1 + u}\right) \, du.$$
Replacing \( \exp(-\sigma/t) \) and \( \exp[-\sigma/(1 - t)] \) by their respective series expansions involving Laguerre polynomials \( L_n(\sigma) \equiv L_n^{(0)}(\sigma) \ (n = 0, 1, 2, \ldots) \) given in Miller [7, Eq. 3.4a, 3.4b], namely,

\[
\exp \left( -\frac{\sigma}{t} \right) = \exp(-\sigma) t \sum_{n=0}^{\infty} L_n(\sigma)(1 - t)^n, \ |t| < 1,
\]

and

\[
\exp \left( -\frac{\sigma}{1 - t} \right) = \exp(-\sigma) (1 - t) \sum_{m=0}^{\infty} L_m(\sigma)t^m, \ |t| < 1,
\]

in (13), an alternative representation for \( G_{\sigma}(a, b; c, d; z) \) is given by

\[
G_{\sigma}(a, b; c, d; z) = \exp(-2\sigma) \sum_{m,n=0}^{\infty} B(b + m + 1, c + n + 1 - b)L_m(\sigma)L_n(\sigma) \\
\times G(a, b + m + 1; c + m + n + 2, d; z).
\]

The Mellin transform of \( G_{\sigma}(a, b; c, d; z) \), using (13), is given by

\[
\int_0^{\infty} \sigma^{s-1} G_{\sigma}(a, b; c, d; z) \, d\sigma = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}\Phi(a; d; zt) \int_0^\infty \exp \left[ -\frac{\sigma}{t(1-t)} \right] \sigma^{s-1} \, d\sigma \, dt
\]

\[
= \frac{\Gamma(s)}{B(b, c - b)} \int_0^1 t^{b+s-1}(1 - t)^{c-b+s-1}\Phi(a; d; zt) \, dt,
\]

for \( \text{Re}(s) > 0 \). Now, evaluation of the above integral using (4) yields

\[
\int_0^{\infty} \sigma^{s-1} G_{\sigma}(a, b; c, d; z) \, d\sigma = \frac{\Gamma(s)B(b + s, c - b + s)}{B(b, c - b)} G(a, b + s; c + 2s, d; z),
\]

where \( \text{Re}(s + b) > 0 \) and \( \text{Re}(s + c - b) > 0 \).

**Theorem 2.1.** If \( \sigma > 0, c > b > 0 \) and \( d > a > 0 \), then

\[
|G_{\sigma}(a, b; c, d; z)| \leq \exp(-4\sigma)G(a, b; c, d; z) \leq \frac{\exp(-1)}{4\sigma}G(a, b; c, d; z).
\]

**Proof.** Recently, Nagar, Morán-Vásquez and Gupta [8] have shown that

\[
|\Phi_{\sigma}(b; c; z)| \leq \exp(-4\sigma)\Phi(b; c; z) \leq \frac{\exp(-1)}{4\sigma}\Phi(b; c; z),
\]

where \( \sigma > 0 \) and \( c > b > 0 \). Now, using the above inequality in (12) and writing the resulting inequality using (4), we get the desired result. \( \square \)
Theorem 2.2. If $\sigma > 0$, $\Re(a) > \alpha > 0$ and $\beta > 0$, then
\[
\int_0^\infty x^{\alpha-1} G_\sigma(a, b; c, d; -\beta x) \, dx = \frac{\beta^{-\alpha} \Gamma(\alpha) B(a - \alpha, d - a) B(b - \alpha, c - b; \sigma)}{B(b, c - b) B(a, d - a)}.
\]

Proof. Using (13) and changing the order of integration, we have
\[
\int_0^\infty x^{\alpha-1} G_\sigma(a, b; c, d; -\beta x) \, dx = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \exp \left[ -\frac{\sigma}{t(1-t)} \right] \times \int_0^\infty x^{\alpha-1} \Phi(a; d; -\beta xt) \, dx \, dt.
\]
Now, first integrating $x$ using the result (Nagar, Morán-Vásquez and Gupta [8])
\[
\int_0^\infty x^{\alpha-1} \Phi(b; c; -\alpha x) \, dx = \frac{\Gamma(a) B(b - a, c - b)}{B(b, c - b)} \alpha^{-a}
\]
and then $t$ using Euler’s beta integral, we get the desired result. \qed

3 Statistical Distributions

In this section, we give applications of the Goursat’s hypergeometric function to statistical distribution theory.

First, we define the gamma and the beta type 1 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [5].

Definition 3.1. A random variable $X$ is said to have a gamma distribution with parameters $\theta (> 0), \kappa (> 0)$, denoted by $X \sim \text{Ga}(\kappa, \theta)$, if its probability density function (pdf) is given by
\[
\{\theta^\kappa \Gamma(\kappa)\}^{-1} x^{\kappa-1} \exp \left( -\frac{x}{\theta} \right), \quad x > 0.
\]

Note that for $\theta = 1$, the above distribution reduces to a standard gamma distribution and in this case we write $X \sim \text{Ga}(\kappa)$.

Definition 3.2. A random variable $X$ is said to have a beta type 1 distribution with parameters $(a, b)$, $a > 0$, $b > 0$, denoted as $X \sim \text{B1}(a, b)$, if its pdf is given by
\[
\{B(a, b)\}^{-1} x^{a-1} (1 - x)^{b-1}, \quad 0 < x < 1,
\]
where $B(a, b)$ is the beta function.
Next, we define the extended confluent hypergeometric function distribution.

**Definition 3.3.** A random variable $X$ is said to have an extended confluent hypergeometric function distribution with parameters $(\nu, \alpha, \beta, \sigma)$, denoted by $X \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$, if its pdf is given by

$$
\frac{B(\alpha, \beta - \alpha)x^{\nu-1}\Phi_\sigma(\alpha; \beta; -x)}{\Gamma(\nu)B(\alpha - \nu, \beta - \alpha; \sigma)}, \quad x > 0,
$$

where $\nu > 0$, $\beta > \alpha > 0$ if $\sigma > 0$ and $\beta > \alpha > \nu > 0$ if $\sigma = 0$.

**Definition 3.4.** A random variable $Y$ is said to have a generalized extended confluent hypergeometric function distribution with parameters $(\nu, \alpha, \beta, \lambda, \sigma)$, denoted by $Y \sim \text{ECH}(\nu, \alpha, \beta; \lambda; \sigma)$, if its pdf is given by

$$
\frac{B(\alpha, \beta - \alpha)\lambda^{\nu}y^{\nu-1}\Phi_\sigma(\alpha; \beta; -\lambda y)}{\Gamma(\nu)B(\alpha - \nu, \beta - \alpha; \sigma)}, \quad y > 0, \lambda > 0,
$$

where $\nu > 0$, $\beta > \alpha > 0$ if $\sigma > 0$ and $\beta > \alpha > \nu > 0$ if $\sigma = 0$.

For $\sigma = 0$, (16) and (17) reduce to the confluent hypergeometric function pdf and generalized confluent hypergeometric function pdf (see Gupta and Nagar [4]). Note that the pdf (16) can be obtained from (17) by taking $\lambda = 1$. Conversely, (17) can be obtained from (16) by the transformation $X = \lambda Y$, $\lambda > 0$, with the Jacobian $J(x \to y) = \lambda$.

The extended confluent hypergeometric function distribution can be derived as the distribution of the quotient of independent gamma and extended beta type 1 variables (Nagar, Morán-Vásquez and Gupta [8]).

The next theorem derives the cumulative distribution function (cdf) of $X$ where $X \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$.

**Theorem 3.1.** If $X \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$, then

$$
P(X \leq x) = \frac{B(\alpha, \beta - \alpha)x^\nu G_\sigma(\nu, \alpha; \beta, \nu + 1; -x)}{(\nu + 1)B(\alpha - \nu, \beta - \alpha; \sigma)},
$$

where $G_\sigma(a, b; c, d; z)$ is the extended Goursat’s function defined by (12).

**Proof.** From (16), we have

$$
P(X \leq x) = \frac{B(\alpha, \beta - \alpha)}{\Gamma(\nu)B(\alpha - \nu, \beta - \alpha; \sigma)} \int_0^x t^{\nu-1}\Phi_\sigma(\alpha, \beta; -t) \, dt.
$$

Now, taking $y = t/x$ in the above integral and using the integral representation (12) in the resulting expression, we obtain the desired result. \qed
Theorem 3.2. Let the random variables $X$ and $Y$ be independent, $X \sim \text{Ga}(\lambda)$ and $Y \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$. Then, the pdf of $S = Y + X$ is given by

$$B(\alpha, \beta - \alpha) \frac{\Gamma(\nu + \lambda)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)\Gamma(\lambda)B(\alpha - \nu, \beta - \alpha; \sigma)} s^{\nu + \lambda - 1} \exp(-s)G_\sigma(\nu, \beta - \alpha; \beta, \nu + \lambda; s), \ s > 0,$$

where $G_\sigma(a, b; c, d; z)$ is the extended Goursat’s hypergeometric function defined by (12).

Proof. Since $X$ and $Y$ are independent, from (14) and (16), we get the joint pdf of $X$ and $Y$ as

$$B(\alpha, \beta - \alpha) \frac{\Gamma(\nu + \lambda)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)\Gamma(\lambda)B(\alpha - \nu, \beta - \alpha; \sigma)} x^{\lambda - 1}y^{\nu - 1} \exp(-x)\Phi_\sigma(\alpha; \beta; -y), \ x > 0, \ y > 0.$$

Making the transformation $S = Y + X$ and $R = Y/(Y + X)$ with the Jacobian $J(x, y \to s, r) = s$, we obtain the joint pdf of $S$ and $R$ as

$$B(\alpha, \beta - \alpha) \frac{\Gamma(\nu + \lambda)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)\Gamma(\lambda)B(\alpha - \nu, \beta - \alpha; \sigma)} s^{\nu - 1}(1 - r)^{\lambda - 1} \Phi_\sigma(\beta - \alpha; \beta; rs),$$

where $s > 0$ and $0 < r < 1$. Clearly, $R$ and $S$ are not independent. Integration of $r$ in the joint pdf of $S$ and $R$ by using (12) yields the marginal pdf of $S$. □

Theorem 3.3. Let the random variables $U$ and $V$ be independent, $U \sim \text{B1}(\lambda, \gamma)$ and $V \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$. Then $X = V/U$ has the pdf

$$B(\alpha, \beta - \alpha)B(\lambda + \nu, \gamma) \frac{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)} x^{\nu - 1}G_\sigma(\lambda + \nu, \alpha; \beta, \lambda + \gamma + \nu; -x), \ x > 0,$$

where $G_\sigma(a, b; c, d; z)$ is the extended Goursat’s hypergeometric function defined by (12).

Proof. Using (15), (16) and the independence of $U$ and $V$, the joint pdf of $U$ and $V$ is given by

$$B(\alpha, \beta - \alpha)u^{\lambda - 1}(1 - u)^{\gamma - 1}v^{\nu - 1}\Phi_\sigma(\alpha; \beta; -v) \frac{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}, \ 0 < u < 1, \ v > 0.$$

Using the transformation $X = V/U$ with the Jacobian $J(v \to x) = u$, we obtain the joint pdf of $U$ and $X$ as

$$B(\alpha, \beta - \alpha)x^{\nu - 1}u^{\lambda + \nu - 1}(1 - u)^{\gamma - 1}\Phi_\sigma(\alpha; \beta; -xu) \frac{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}, \ 0 < u < 1, \ x > 0.$$

Integrating the above expression with respect to $u$, we find the marginal pdf of $X$ as

$$B(\alpha, \beta - \alpha)x^{\nu - 1} \frac{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)}{\Gamma(\nu)B(\lambda, \gamma)B(\alpha - \nu, \beta - \alpha; \sigma)} \int_0^1 u^{\lambda + \nu - 1}(1 - u)^{\gamma - 1}\Phi_\sigma(\alpha; \beta; -xu) du.$$

Finally, evaluation of the above integral by using (12) yield the desired result. □
Acknowledgments. The research work of DKN was supported by the Sistema Universitario de Investigación, Universidad de Antioquia under the project no. IN10182CE.

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Received: April 10, 2015; Published: May 14, 2015