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On Basicity of the System of Solutions of the Cauchy Problem for the Sturm-Liouville Equation in the Generalized Lebesgue Space

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Abstract

We consider the Cauchy problems with spectral parameter for two types of the Sturm-Liouville operator. It‘s proved that the sequence of values for the spectral parameter can be chosen so that the corresponding system of solutions forms an isomorphic basis to the classical trigonometrical systems in the Lebesgue space with variable summability index.

Mathematics Subject Classification: 30B60, 34B24, 34L10

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1 Introduction

Consider the differential expression L_λ of the Sturm-Liouville problem

$$L_\lambda y = -y''(t) + (q(t) - \lambda)y(t), t \in (0, \pi),$$

with spectral parameter $\lambda \in C$ (C is a complex plane). The different spectral problems relatively to L_λ in the various functional spaces are sufficiently well studied. The essence is to find the spectrum and the corresponding resolution of identity. It should be noted that recently the interest to studying of spectral problems for L_λ is increased, when the spectral parameter enters into the boundary conditions. Basically, this tendency began to rapid develop after the works [17, 16, 18]. In [12] the problem is otherwise. Namely, the Cauchy problem for the homogeneous equation $L_\lambda = 0$, depending on the parameter λ is considered. Finding the conditions on the sequence $\{\lambda_n\}_{n\in N}$, when the corresponding family of solutions forms a Riesz basis in L_2.

It should be noted that recently in connection with some specific problems of mechanics and mathematical physics the interest to study of one or another questions in the generalized Lebesgue $L_{p(\cdot)}$ and Sobolev $W^m_{p(\cdot)}$ spaces is greatly increased. The interest aroused, apart from mathematical curiosity, by possible applications to models with the so called non-standard growth in fluid mechanics, elasticity theory, in differential equations, see for example, [25, 10, 19, 20] and references therein.

Applying the Fourier method to solve many problems for partial differential equations leads to the study of spectral problems for Sturm-Liouville equation. In this paper we consider the Cauchy problems $L_\lambda u = 0$ in the spaces $L_{p(\cdot)}$. We study the basicity of the system of solutions in $L_{p(\cdot)}$. The obtained results include the results of work [12]. Propose a different method of investigation of the basis (as the method of investigation [12] in this case is not applicable). Thus it uses the results with respect to the basis properties (completeness, minimality, basicity, etc.) of perturbed trigonometric systems in Lebesgue spaces. Numerous works of various authors (see. e.g. [23, 24, 26, 1, 2, 3, 4, 5, 6]) are devoted to this direction. Basicity of trigonometric systems with linear perturbation is studied in generalized Lebesgue spaces in [7, 8, 28].

2 Necessary facts

We first provide some notations and facts from the theory of $L_{p(\cdot)}$ spaces and close bases. At first we define $L_{p(\cdot)}$ space.

Let $p : [0, \pi] \to [1, +\infty)$ be a Lebesgue measurable function. Denote by L_0 the set of all measurable (with respect to Lebesgue measure) on $[0, \pi]$ functions. We put $I_p(f) \equiv \int_0^\pi |f(t)|^{p(t)}\, dt$. Let $L \equiv \{f \in L_0 : I_p(f) < +\infty\}$. L becomes a linear space with the usual linear operations. For $1 \le p^- \le p^+ < +\infty$ the space L is Banach with a norm $\|\cdot\|_{p(\cdot)}$:

$$\|f\|_{p(\cdot)} \equiv \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

where $p^\pm = \sup\limits_{(0,\pi)} vrai\, p^{\pm 1}(t)$ (see, e.f., [29, 21, 11]). Denote this space by $L_{p(\cdot)}$.

Consider the following weak Lipschitz class WL_π:

$$WL_\pi \equiv \Big\{ p:\ p(-\pi) = p(\pi)\,,\ \exists c>0;\ \forall t_1,t_2 \in [0,\pi]\,,\ |t_1-t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1)-p(t_2)| \le \tfrac{c}{-\ln|t_1-t_2|} \Big\}.$$

Denote by $q(t)$ the conjugate of $p(t)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let us give some facts from this theory. The generalized Holder inequality

$$\int_0^\pi |f(t)\,g(t)|\,dt \le c\left(p^-;p^+\right)\ \|f\|_{p(\cdot)}\,\|g\|_{q(\cdot)}\,,$$

holds, where $c(p^-;p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. The following property is true.

Property A. *If $|f(t)| \le |g(t)|$ a.e. in $(0,\pi)$, then $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.*

It is easy to see that there holds

Proposition 2.1. *Let $p \in WL_\pi$, $p^- > 0$. A function $\rho(t) = \prod_{k=1}^m |t-t_k|^{\alpha_k}$ ($\{t_k\} \subset [0,\pi]$, $t_i \ne t_j$, for $i \ne j$; $\{\alpha_k\} \subset R$) belongs to $L_{p(\cdot)}$ if $\alpha_k > -\frac{1}{p(t_k)}$, $k = \overline{1,m}$.*

We also need some facts from the theory of close bases. We accept the following standard notations.

B-space is a Banach space; X^* is a space conjugated to X ; $f(x)$ means value of the functional f in x for $(f;x) \in X^* \times X$; N are natural numbers; Z are integers; C is a complex plane; $Z_+ = N \bigcup \{0\}$; $\Rightarrow$ means follows; $\Leftrightarrow$ $-$equivalent; $cardM-$number of elements of M; δ_{ik} is the Kronecker symbol.

Definition 2.2. *A system $\{x_n\}_{n\in N} \subset X$ is called ω-linearly independent in B-space X, if $\sum_{n=1}^\infty a_n x_n = 0 \Rightarrow a_n = 0$, $\forall n \in N$.*

It is easy to establish validity of the following statement.

Proposition 2.3. *Let X be a B-space with a basis $\{x_n\}_{n\in N} \subset X$ and $F: X \to X$ be a Fredholm operator (i.e., index of the operator F is equal to zero). Then the following properties of $\{y_n\}_{n\in N} : y_n = Fx_n$, $\forall n \in N$, in X are equivalent:*

1. *$\{y_n\}_{n\in N}$ is complete;*
2. *$\{y_n\}_{n\in N}$ is minimal;*
3. *$\{y_n\}_{n\in N}$ is ω-linearly independent;*
4. *$\{y_n\}_{n\in N}$ is the basis isomorphic to $\{x_n\}_{n\in N}$.*

There holds the following

Lemma 2.4. *Let X be a B-space with a basis $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}\subset X$: card $\{n: x_n\neq y_n\}<+\infty$. Then the linear operator $F: X\to X$, defined as*

$$F(\cdot)=\sum_{n=1}^{\infty}x_n^*(\cdot)\,y_n,$$

is Fredholm, where the system $\{x_n^*\}_{n\in N}\subset X^*$ is conjugate to $\{x_n\}_{n\in N}$ system.

Indeed, we have

$$Fx=\sum_{n=1}^{\infty}x_n^*(x)\,x_n+\sum_{n=1}^{\infty}x_n^*(x)\,(y_n-x_n)=(I+T)(x),$$

where $I: X\to X$ is an identity operator and $Tx=\sum_{n=1}^{\infty}x_n^*(x)\,(y_n-x_n)$. It is clear that the operator T is finite dimensional and as a result, $F=I+T$ is Fredholm.

Using this lemma and Proposition 2.3 we have following

Corollary 2.5. *Let X be B-space with a basis $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}\subset$ $\subset X$: card $\{n: x_n\neq y_n\}<+\infty$. Then the properties 1)-4) of $\{y_n\}_{n\in N}$ in Proposition 2.1 are equivalent.*

Using the results in [14, 27] it is easy to prove the following

Theorem 2.6. *Let the system $\{\cos\lambda_n x\}_{n\in N}$ ($\{\sin\lambda_n x\}_{n\in N}$) is complete in $L_{p(\cdot)}$, $1\leq p^-\leq p^+<+\infty$. Then the system $\{\cos\mu_n x\}_{n\in N}$ ($\{\sin\mu_n x\}_{n\in N}$) is also complete in $L_{p(\cdot)}$, if card $\{n:\lambda_n\neq\mu_n\}<+\infty$ and $\mu_i\neq\pm\mu_j$, for $i\neq j$.*

We will also use the concept of the space of coefficients. We define it as follows. Let $\vec{x}\equiv\{x_n\}_{n\in\mathbb{N}}\subset X$ be a non-degenerate system in a B-space X, i.e. $x_n\neq 0$, $\forall n\in\mathbb{N}$. Define

$$\mathscr{K}_{\vec{x}}\equiv\left\{\{\lambda_n\}_{n\in\mathbb{N}}: \textit{ the series } \sum_{n=1}^{\infty}\lambda_n x_n \textit{ is convergent in } X\right\}.$$

Introduce the norm in $\mathscr{K}_{\vec{x}}$:

$$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}}=\sup_m\left\|\sum_{n=1}^{m}\lambda_n x_n\right\|, \text{ where } \vec{\lambda}=\{\lambda_n\}_{n\in\mathbb{N}}.$$

With respect to the usual operations of addition and multiplication by a complex number, $\mathscr{K}_{\vec{x}}$ is a Banach space. Take $\forall\vec{\lambda}\in\mathscr{K}_{\vec{x}}$ and consider the operator $T:\mathscr{K}_{\vec{x}}\to X$:

$$T\vec{\lambda}=\sum_{n=1}^{\infty}\lambda_n x_n,\ \ \vec{\lambda}=\{\lambda_n\}_{n\in\mathbb{N}}.$$

Denote by $\{e_n\}_{n\in\mathbb{N}}\subset\mathscr{K}_{\vec{x}}$ a canonical system in $\mathscr{K}_{\vec{x}}$, where $e_n=\{\delta_{nk}\}_{k\in\mathbb{N}}$. It is absolutely clear that $Te_n=x_n$, $\forall n\in\mathbb{N}$. The following statement is true.

Corollary 2.7. *Space of coefficients $\mathscr{K}_{\vec{x}}$ is a Banach space with the canonical basis $\{e_n\}_{n\in\mathbb{N}}$. Moreover, the system $\vec{x}$ forms a basis for $X \Leftrightarrow T$ performs an isomorphism between $\mathscr{K}_{\vec{x}}$ and X.*

More information about the above facts can be found in [13, 30, 31, 9].

3 Close bases of cosines and sines in $L_{p(\cdot)}$

We denote the sequence $\{\lambda_n\}_{n\in Z_+} \subset C : \bar{\lambda} \equiv \{\lambda_n\}_{n\in Z_+}$ by $\bar{\lambda}$. Let $c_{\bar{\lambda}} \equiv \{\cos \lambda_n x\}_{n\in Z_+}$, $s_{\bar{\lambda}} \equiv \{\sin \lambda_n x\}_{n\in N}$. Denote by $\mathscr{K}_{p(\cdot)}\left(\{f_n\}_{n\in N}\right)$ the space of coefficients of the basis $\{f_n\}_{n\in N}$ in the space $L_{p(\cdot)}$. If the basis $\{f_n\}_{n\in N}$ and $\{g_n\}_{n\in N}$ are isomorphic in $L_{p(\cdot)}$, we denote this as $\{f_n\}_{n\in N} \overset{p(\cdot)}{\sim} \{g_n\}_{n\in N}$. The following proposition is true.

Proposition 3.1. $\{f_n\}_{n\in N} \overset{p(\cdot)}{\sim} \{g_n\}_{n\in N}$ *if and only if* $\mathscr{K}_{p(\cdot)}\left(\{f_n\}_{n\in N}\right) \equiv$ $\equiv \mathscr{K}_{p(\cdot)}\left(\{g_n\}_{n\in N}\right)$.

Let $\bar{\lambda}; \bar{\mu} \subset C$ are some sequences. Denote by $\tilde{p} : \tilde{p} = \min\{p^-; 2\}$. There holds the following

Theorem 3.2. *Let $c_{\bar{\lambda}}$ forms an isomorphic to $\{\cos nx\}_{n\in Z_+}$ basis in $L_{p(\cdot)}$. Assume that $p \in WL_\pi, p^- > 1$, and $\sum_n |\lambda_n - \mu_n|^{\tilde{p}} < +\infty$. If $\mu_i \neq \pm\mu_j$, for $i \neq j$, then $c_{\bar{\mu}}$ is also a basis in $L_{p(\cdot)}$ isomorphic to the basis $\{\cos nx\}_{n\in Z_+}$.*

Proof. Everywhere we will denote constants without an index by c, generally speaking, different in different inequalities. There holds

$$|\cos \lambda_n x - \cos \mu_n x| \leq c\,|\lambda_n - \mu_n|\,.$$

Then according to Property A we have

$$\sum_n \|\cos \lambda_n x - \cos \mu_n x\|_{p(\cdot)}^{\tilde{p}} \leq c \sum_n |\lambda_n - \mu_n|^{\tilde{p}} < +\infty. \tag{1}$$

Denote by P the automorphism in $L_{p(\cdot)}$, which transforms the basis $c_{\bar{\lambda}}$ to $\{\cos nx\}_{n\in Z_+}$. We take $f \in L_{p(\cdot)}$ and put $g = Pf$. Let $\{f_n\}_{n\in Z_+}$ is a system of biorthogonal coefficients of function f by the system $c_{\bar{\lambda}}$. Therefore $\{f_n\}_{n\in Z_+}$ is a system of biorthogonal coefficients of function g by the system $\{\cos nx\}_{n\in Z_+}$. Consider the expression

$$\sum_n \left(\cos \lambda_n t - \cos \mu_n t\right)\left(Pf; \cos nx\right), \tag{2}$$

where $(f;g)=\int_0^\pi f(t)\,\overline{g(t)}dt$. We have

$$\left\|\sum_n(\cos\lambda_n t-\cos\mu_n t)f_n\right\|_{p(\cdot)}\leq\sum_n\|\cos\lambda_n t-\cos\mu_n t\|_{p(\cdot)}|f_n|\leq$$
$$\leq\left(\sum_n\|\cos\lambda_n t-\cos\mu_n t\|_{p(\cdot)}^{\tilde p}\right)^{\frac{1}{\tilde p}}\left(\sum_n|f_n|^{\tilde q}\right)^{\frac{1}{\tilde q}},$$

where $\frac{1}{\tilde p}+\frac{1}{\tilde q}=1$. It‘s obvious that $1<\tilde p\leq 2$. Then by the Hausdorff-Young theorem [32] we have

$$\left(\sum_n|f_n|^{\tilde q}\right)^{\frac{1}{\tilde q}}\leq c\,\|Pf\|_{\tilde p}\leq c\,\|f\|_{\tilde p}\,.$$

Since , $\tilde p\leq p^-$, then continuous embeddings $L_{p(\cdot)}\subset L_{p^-}\subset L_{\tilde p}$ imply the estimate $\|f\|_{\tilde p}\leq c\,\|f\|_{p(\cdot)}$. We get

$$\left\|\sum_n(\cos\lambda_n t-\cos\mu_n t)f_n\right\|_{p(\cdot)}\leq c\left(\sum_n\|\cos\lambda_n t-\cos\mu_n t\|_{p(\cdot)}^{\tilde p}\right)^{\frac{1}{\tilde p}}\|f\|_{p(\cdot)}\,. \quad (3)$$

From (1), (3) we obtain that the expression (2) is a function from $L_{p(\cdot)}$, denote it by Tf. Define

$$\delta_{n_0}=\left(\sum_{n>n_0}\|\cos\lambda_n t-\cos\mu_n t\|^{\tilde p}\right)^{\frac{1}{\tilde p}}.$$

We put

$$\omega_n=\begin{cases}\lambda_n\,, & n\leq n_0\,,\\ \mu_n\,, & n>n_0\,.\end{cases}$$

Consequently

$$\left\|\sum_n(\cos\lambda_n t-\cos\omega_n t)f_n\right\|_{p(\cdot)}\leq c\delta_{n_0}\|f\|_{p(\cdot)}\,. \quad (4)$$

It‘s clear that there exists $n_0\in N$ such that $c\delta_{n_0}<1$. Let $\tilde T$ be the operator

$$\tilde Tf=\sum_n(\cos\lambda_n t-\cos\omega_n t)(Pf;\cos nx)\,.$$

From (4) we obtain $\left\|\tilde T\right\|<1$. Denote by $\{\vartheta_n\}_{n\in Z_+}\subset L_{q(\cdot)}$ the system biorthogonal to the system $c_{\bar\lambda}$. We have $f_n=(Pf;\cos nx)=(f;\vartheta_n)$,$\forall n\in Z_+$. The operator $\left(I-\tilde T\right)$ is invertible in $L_{p(\cdot)}$ and moreover $\left(I-\tilde T\right)(\cos\lambda_n t)=\cos\omega_n t$, $\forall n\in Z_+$. Thus the system $c_{\bar\omega}$ is a basis in $L_{p(\cdot)}$ isomorphic to $c_{\bar\lambda}$. It‘s clear that the systems $c_{\bar\omega}$ and $c_{\bar\mu}$ differ by finite number of elements. The result follows from Theorem 2.6.

The theorem is proved.

Similar statement holds for the system of sines.

Theorem 3.3. *Let $s_{\bar{\lambda}}$ forms an isomorphic to $\{\sin nx\}_{n\in N}$ basis for $L_{p(\cdot)}$, $p \in WL_\pi$, $p^- > 1$ and $\sum_n |\lambda_n - \mu_n|^{\tilde{p}} < +\infty$. If $\mu_i \neq \pm\mu_j$, for $i \neq j$, then $s_{\bar{\mu}}$ also forms a basis in $L_{p(\cdot)}$ isomorphic to $\{\sin nx\}_{n\in N}$.*

4 Bases in $L_{p(\cdot)}$ consisting of the solutions of the Cauchy problem for the Sturm-Liouville equation

Consider the Cauchy problem

$$L_\lambda u\,(t) = 0\,,\ t \in (0, \pi)\,, \tag{5}$$

$$u\,(0) = 1\,,\ u'\,(0) = \lambda\,, \tag{6}$$

where λ is a constant.

Theorem 4.1. *The system of solutions $\{u\,(x, \lambda_n)\}_{n\in Z_+}$ of the problem (5)-(6) corresponding to the sequence $\{\lambda_n\}_{n\in Z_+}$ forms a basis for $L_{p(\cdot)}$ isomorphic to the system $\{\cos \lambda_n x\}_{n\in Z_+}$ only if the system of cosines $c_{\bar{\lambda}}$ forms a basis for $L_{p(\cdot)}$ isomorphic to $\{\cos \lambda_n x\}_{n\in Z_+}$.*

Indeed, it is well known that the following representation is true

$$u\,(x, \lambda_n) = \cos \lambda_n x + \int_0^x K\,(x, t) \cos \lambda_n t\, dt, \tag{7}$$

where $K(x, t)$ is continuous kernel on $[0, \pi] \times [0, \pi]$. See, for example, the monograph [22]. Then the truth of the theorem follows from (7). In fact, denote by $K : L_{p(\cdot)} \to L_{p(\cdot)}$ the following integral operator

$$(Kf)\,(x) = \int_0^x K\,(x, t)\, f\,(t)\, dt,\ x \in [0, \pi]\,.$$

Let $I : L_{p(\cdot)} \to L_{p(\cdot)}$ be an identity operator. From the continuity of the kernel $K\,(\cdot;\cdot)$, it follows that K is a completely continuous operator with spectrum $\{0\}$. Therefore, it is clear that the operator $(I + K)$ is invertible in $L_{p(\cdot)}$, and furthermore, as follows from (7) we have

$$u\,(x, \lambda_n) = (I + K)\, cos \lambda_n x, \forall n \in Z_+.$$

This immediately implies the assertion.

We have similar statement for the Cauchy problem

$$L_\lambda u\,(t) = 0\,,\ t \in (0, \pi)\,, \tag{8}$$

$$u\,(0) = 0\,,\ u'\,(0) = \lambda. \tag{9}$$

Theorem 4.2. *The system of solutions* $\{u(x,\lambda_n)\}_{n\in N}$ *of the Cauchy problem* (8) − (9) *forms a basis for* $L_{p(\cdot)}$ *isomorphic to* $\{\sin nx\}_{n\in N}$ *only if the system* $s_{\bar{\lambda}}$ *forms a basis for* $L_{p(\cdot)}$ *isomorphic to* $\{\sin nx\}_{n\in N}$.

Validity of this theorem follows from the relation

$$u(x,\lambda_n) = \sin\lambda_n x + \int_0^x R(x,t)\sin\lambda_n t\,dt,$$

where $R(x,t)$ is continuous on $[0,\pi]\times[0,\pi]$ (see. [22]). A further reasoning is given in the previous case.

It should be noted that in the particular case $p(t)\equiv 2$, we obtain the results in [12] and in the case $p(\cdot)\equiv const$ we obtain the results of [15].

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References

[1] B. T. Bilalov, On uniform convergence of series with regard to some system of sines,*Diff. uravneniya*, **24(1)** (1988) 175-177. (in Russian)

[2] B. T. Bilalov, Basicity of some systems of functions, *Diff. Uravneniya*, **25** (1989) 163-164. (in Russian)

[3] B. T. Bilalov, Basicity of some exponents, cosines and sines systems, *Diff. uravneniya*, **26** (1990) 10-16.

[4] B. T. Bilalov, On the Basis Property of Systems of Exponentials, Cosines, and Sines in L_p, *Doklady Mathematics*, **59(2)**(1999).

[5] B. T. Bilalov, On Bases for Some Systems of Exponentials, Cosines, and Sines in L_p, *Doklady Mathematics*, **379(2)** (2001) 7-9.

[6] B. T. Bilalov, The basis properties of some systems of exponential functions, cosines and sines, *Sibirsk. Mat. Zh.* **45(2)** (2004) 214-221. http://dx.doi.org/10.1023/b:simj.0000021278.64966.d7

[7] B. T. Bilalov, Z. G. Guseynov, Basicity of a system of exponents with a piece-wise linear phase in variable spaces, *Mediterr. J. Math.* **9(3)**(2012) 487-498. http://dx.doi.org/10.1007/s00009-011-0135-7

[8] B. T. Bilalov, Z. G. Guseynov, $\mathscr{K}$-Bessel and $\mathscr{K}$-Hilbert systems and $\mathscr{K}$-bases, *Dokl. Math.*, **80(3)** (2009) 826-828. http://dx.doi.org/10.1134/s1064562409060118

[9] B. T. Bilalov, Z. V. Mamedova, On the frame properties of some degenerate trigonometric system, *Dokl. Acad. Nauk*, **LXVIII(5)** (2012), 14-18.

[10] L. Diening, M. Ruzicka, Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, *J. Reine Angeu. Math.*, **563** (2003) 197-220. http://dx.doi.org/10.1515/crll.2003.081

[11] X. Fan, D. Zhao, On the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, *J. Math. Anal. Appl.*, **263(2)** (2001) 424-446. http://dx.doi.org/10.1006/jmaa.2000.7617

[12] X. He, H. Volkmer, Riesz Bases of Solutions of Sturm-Liouville Equations, *The Journal of Fourier Anal. Appl.*, **7(3)** (2001) 297-307. http://dx.doi.org/10.1007/bf02511815

[13] Ch. Heil, A Basis Theory Primer, Springer, 2011. http://dx.doi.org/10.1007/978-0-8176-4687-5

[14] H. Horvath, Inverse spectral problems and closed exponential systems, *Annals of Mathematics*, **162** (2005) 885-918. http://dx.doi.org/10.4007/annals.2005.162.885

[15] A. A. Huseynli, On the stability of basisness in $L_p\,(1 < p < +\infty)$ of cosines and sines, *Turk. J. Math.*, **35** (2011) 47-54.

[16] N. Yu. Kapustin, Oscillation properties of solutions of a non-selfadjoint spectral problem with spectral parameter in boundary condition, *Diff. uravneniya*, **35(8)** (1999) 1024-1027. (Russian)

[17] N. Yu. Kapustin, E. I. Moiseev, On spectral problems with spectral parameter in boundary condition, *Diff. uravneniya,* **33(1)** (1997), 115-119. (in Russian)

[18] N. Yu. Kapustin, E. I. Moiseev, On basicity in space L_p of eigenfunctions for the two problems with spectral parameter in boundary condition, *Diff. Uravneniya*, **36(10)** (2000) 1357-1360. (in Russian)

[19] V. Kokilashvili, V. Paatashvili, and S. Samko, Boundary Value Problems for Analytic Functions in the class of Cauchy Type Integrals With Density in $L^{p(\cdot)}(\Gamma)$, *Bound. Value Probl.* **1-2** (2005) 43-71. http://dx.doi.org/10.1155/bvp.2005.43

[20] V. Kokilashvili, S. Samko, Singular Integrals in Weighted Lebesgue Spaces with Variable Exponent, *Georgian Math.J.*, **10(1)** (2003) 145-156.

[21] O. Kovacik, J. Rakosnik, On spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Czechoslovak Math. J.*, **41(116)** (1991) 592-618.

[22] B. M. Levitan, Generalized shift operators theory, M., Nauka, 1972, 312 p. (in Russian)

[23] E. I. Moiseev, On basicity of systems of sines and cosines, *DAN SSSR* **275(4)** (1984) 794-798.

[24] E. I. Moiseev, On basicity of systems of sines, *Diff. Uravnenia,* **23(1)** (1987) 177-179.

[25] M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory, *Springer, Lecture Notes in Math.*, **1748** (2000) 176 p. http://dx.doi.org/10.1007/bfb0104029

[26] A. M. Sedletskii, On convergence of non-harmonic Fourier series by the systems of exponents, cosines and sines, *DAN SSSR*, **301(5)** (1988) 501-504.

[27] A. M. Sedletskii, Classes of analytic Fourier transforms and exponential approximations, *Moscow, Physmatlit*, 2005. (in Russian)

[28] I. I. Sharapudinov, Uniform boundedness in $L^p\,(p = p(x))$ of some families of convolution operators, *Mat. Zametki*, **59(2)** (1996) 291-302. (in Russian). http://dx.doi.org/10.1007/bf02310962

[29] I. I. Sharapudinov, The topology of the space $L^{p(\cdot)}\,(0,1)$, *Math. Zametki*, **26(4)**(1979) 613-632. (in Russian). http://dx.doi.org/10.1007/bf01159546

[30] I. Singer, Bases in Banach spaces, v.1, Springer, 1970. http://dx.doi.org/10.1007/978-3-642-51633-7

[31] I. Singer, Bases in Banach spaces, v.2, Springer, 1981.

[32] A. Zygmund, Trigonometric series, **1**, M., Mir, 1965, 616 p. (in Russian)

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