On Parabolic and Hyperbolic Automorphisms of the Unit Disc

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Abstract

In this paper, we will determine whether an automorphism of the unit disc without fixed point is parabolic or hyperbolic in terms of the Poincaré metric.

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1 Introduction

The uniformization theorem for Riemann surfaces due to H. Poincaré [8] and P. Koebe [6] implies that a compact Riemann surface of genus greater than 1 should be a quotient space of the unit disc $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. This means that such Riemann surface is biholomorphically equivalent to $\Delta / \Gamma$ for some discrete subgroup $\Gamma$ of the automorphism group $\text{Aut}(\Delta)$. Here the automorphism group $\text{Aut}(X)$ of a complex manifold $X$ is the topological group of holomorphic automorphisms (self-biholomorphisms) of $X$ with the law of the composition and the compact-open topology. If a complex manifold is hyperbolic in the sense of S. Kobayashi [4], its automorphism group is a real Lie group. Automorphism groups of biholomorphically equivalent complex manifolds are isomorphic to each others under the conjugation by biholomorphisms.
From the uniformization theorem it is important to study the Lie group structure of $\text{Aut}(\Delta)$ and its action on $\Delta$. The Lie group structure of $\text{PSL}(2, \mathbb{R})$ which is isomorphic to $\text{Aut}(\Delta)$ has been studied in wide mathematical point of view (for instance, see [7]). In such theory, it is known that there are three kinds of elements in $\text{PSL}(2, \mathbb{R})$. An element $\mathcal{H}$ of $\text{PSL}(2, \mathbb{R})$ is elliptic, parabolic or hyperbolic if $|\text{Tr } \mathcal{H}| < 2$, $= 2$ or $> 2$, with respectively. Then for an automorphism $f$ of the unit disc or its biholomorphically same Riemann surface, $f$ is called elliptic, parabolic or hyperbolic if its representation $\mathcal{H}_f$ in $\text{PSL}(2, \mathbb{R})$ is so. This is independent of a choice of a representation. This trichotomy is also characterized by the action of a non-identity automorphism $f$ of $\Delta$ (for instance, see [3]):

1. $f$ is elliptic $\iff$ it has a fixed point in $\Delta$,
2. $f$ is parabolic $\iff$ it has only one fixed point in $\partial \Delta$,
3. $f$ is hyperbolic $\iff$ it has only two fixed points in $\partial \Delta$.

Every automorphism of $\Delta$ extends to a diffeomorphism of $\overline{\Delta}$; thus we can consider the fixed point of $f \in \text{Aut}(\Delta)$ on $\overline{\Delta}$.

The aim of this paper is to characterize this classification in an intrinsic way. Let us consider $\Delta$ as a manifold without boundary. Suppose that there is $f \in \text{Aut}(\Delta)$ without fixed point in $\Delta$ so it should be parabolic or hyperbolic. One may ask:

*How can we determine the type of $f$ not using the boundary extension of $f$?*

In order to achieve, it is natural to consider the Poincaré metric of the unit disc which is a hermitian metric of a negative constant curvature, so which is a complete metric. The unit disc has many its own geometric and analytic aspects in the geometry of the Poincaré metric (see [4, 2]). We will study the parabolic and hyperbolic automorphisms in terms of the Poincaré metric.

For a hermitian metric $h$ of a Riemann surface $S$, the metric distance function of $h$ will be denoted by $d_h(\cdot, \cdot)$. For an automorphism $f$ of $S$, we consider the two-sided iterations $\{f^{(n)} : n \in \mathbb{Z}\}$ where $f^{(1)} = f$, $f^{(0)}$ is the identity mapping of $S$ and

$$f^{(n)} = f \circ \cdots \circ f_n, \quad f^{(-n)} = f^{-1} \circ \cdots \circ f^{-1}_n$$

for $n > 0$.

The main result is the following.
Theorem 1.1. Let $S$ be a simply connected Riemann surface admitting a hermitian metric $h$ of constant curvature $-4$. For a point $p \in S$, let $\varphi : S \to \mathbb{R}$ be a negative function defined by

$$\varphi(q) = \tanh^2(d_h(p,q)) - 1$$

for $q \in S$. Then for any $f \in \text{Aut}(S)$ without fixed point, the two-sided sequence $\varphi \circ f^{(n)}/(\varphi \circ f^{(n)})(p)$ converges to a positive function $\hat{\varphi}$ such that

$$\hat{\varphi} \circ f \equiv c$$

for some positive $c$. Moreover, the automorphism $f$ is parabolic if and only if $c = 1$.

The Riemann surface in the theorem is biholomorphic to the unit disc and the function $\varphi$ is a purely intrinsic quantity. Thus this gives an answer of the question. Note that a convergence of two-sided sequence $a_n (n \in \mathbb{Z})$ to $b$ means that $a_n \to b$ as $n \to +\infty$ or $-\infty$.

We will mainly devote to confirm the main theorem for $S = \Delta$ and $p = 0$ in Section 2. Then a short proof for the theorem will be placed in Section 3.

2 Automorphisms of the unit disc

In this section, we will study non-elliptic automorphisms of the unit disc $\Delta$ and give a simple version of Theorem 1.1.

2.1 Parabolic and hyperbolic action

Let $f$ be an automorphism of $\Delta$ leaving the boundary point $1 \in \partial \Delta$ fixed. When we write $f$ by

$$f(z) = e^{i\theta} \frac{z + \alpha}{1 + \bar{\alpha}z}$$

for some $\alpha \in \Delta$ and $\theta \in \mathbb{R}$, the rotational factor $e^{i\theta}$ is uniquely determined by $\alpha$ in the sense of $e^{i\theta} = (1 + \bar{\alpha})/(1 + \alpha)$. Thus $f$ will be denoted by $f_\alpha$ where

$$f_\alpha(z) = \frac{1 + \bar{\alpha}}{1 + \alpha} \frac{z + \alpha}{1 + \bar{\alpha}z} .$$

Let us consider another fixed point of $f_\alpha$. Rewriting the equation $f_\alpha(z) = z$, we have

$$0 = \bar{\alpha}(1 + \alpha)z^2 + (\alpha - \bar{\alpha})z - \alpha(1 + \bar{\alpha}) = (z - 1)(\bar{\alpha}(1 + \alpha)z + \alpha(1 + \bar{\alpha})) .$$
Thus another fixed point of $f_\alpha$ is only

$$
\beta_\alpha = \frac{-\bar{\alpha}(1 + \alpha)}{\alpha(1 + \bar{\alpha})},
$$

which also belongs to $\partial \Delta$. Thus $f_\alpha$ is parabolic ($\beta_\alpha = 1$) if and only if $2|\alpha|^2 + \alpha + \bar{\alpha} = 0$ equivalently,

$$
\frac{1 - |\alpha|^2}{|1 + \alpha|^2} = 1.
\tag{2.2}
$$

**Remark 2.1.** Since $|f'_\alpha(z)| = (1 - |\alpha|^2)/|1 + \bar{\alpha}z|^2$, values of $|f'_\alpha|$ at fixed points of $f_\alpha$ are

$$
|f'_\alpha(1)| = \frac{1 - |\alpha|^2}{|1 + \alpha|^2} \quad \text{and} \quad |f'_\alpha(\beta_\alpha)| = \frac{|1 + \alpha|^2}{1 - |\alpha|^2}.
\tag{2.3}
$$

Thus $f_\alpha$ is parabolic if and only if $|f'_\alpha(1)| = 1$.

### 2.2 Testing function

Let us consider the real-valued function $\rho$ on $\Delta$ defined by

$$
\rho(z) = |z|^2 - 1.
\tag{2.4}
$$

This is a negative subharmonic function on $\Delta$ which corresponds to the function $\varphi$ in Theorem 1.1 for $p = 0$ (see Section 3).

Let $\alpha \in \Delta$. For the automorphism $f_\alpha$ of $\Delta$ as in (2.1), we will consider the sequence $(\rho_n)$ of positive functions defined by

$$
\rho_n = \frac{\rho \circ f_\alpha^{(n)}}{(\rho \circ f_\alpha^{(n)})(0)}.
$$

Since each iteration $f_\alpha^{(n)}$ of $f_\alpha$ for $n \in \mathbb{Z}$ is also an automorphism of $\Delta$ leaving 1 fixed, there is $\alpha_n \in \Delta$ such that

$$
f_\alpha^{(n)} = f_{\alpha_n}.
$$

Note that $\alpha_0 = 0$ and $\alpha_{-1} = -\bar{\alpha}(1 + \alpha)/\alpha(1 + \bar{\alpha})$. Then we have

$$
(\rho \circ f_\alpha^{(n)})(z) = \left|\frac{1 + \bar{\alpha}_n z + \alpha_n}{1 + \alpha_n 1 + \bar{\alpha}_n z}\right|^2 - 1 = \frac{(|z|^2 - 1)(1 - |\alpha_n|^2)}{|1 + \bar{\alpha}_n z|^2},
$$

thus

$$
\rho_n(z) = \frac{1 - |z|^2}{|1 + \bar{\alpha}_n z|^2}.
$$
Proposition 2.2. If the two-sided sequence \((\alpha_n)\) converges to \(\beta\), then \(\rho_n\) converges to \(\hat{\rho}\) defined by
\[
\hat{\rho}(z) = \frac{1 - |z|^2}{|1 + \beta z|^2}.
\]

In order to study the limit function \(\hat{\rho}\), we will describe the limit of \(\alpha_n\) for a parabolic or hyperbolic \(f_\alpha\).

2.2.1 The parabolic case
The following implies that a parabolic automorphism \(f_\alpha\) is of the form:
\[
f_\alpha(z) = \frac{(2 + is)z - is}{isz + (2 - is)}
\]
for some \(s \in \mathbb{R}\).

Proposition 2.3. For each \(\alpha \in \Delta\) with (2.2), there is a unique \(s \in \mathbb{R}\) such that
\[
\alpha = \frac{-is}{2 + is}.
\]
Moreover the point \(\alpha_n\) for \(f^{(n)}_\alpha = f_{\alpha_n}\) can be written by
\[
\alpha_n = \frac{-ins}{2 + ins}
\]
for \(n \in \mathbb{Z}\).

Proof. Let us consider the left-half plane \(H = \{z \in \mathbb{C} : \Re z < 0\}\) which is biholomorphic to the unit disc \(\Delta\) by the Cayley transform \(F : H \to \Delta\) defined by
\[
F(z) = \frac{1 + z}{1 - z}.
\]
Note that its inverse is \(F^{-1}(z) = (z - 1)/(z + 1)\). Since
\[
\frac{1 - |F(z)|^2}{|1 + F(z)|^2} = \frac{-2(z + \bar{z})|1 - z|^2}{|1 - z|^2} \frac{4}{4} = -\frac{1}{2}(z + \bar{z}) = -\Re z,
\]
the complex number \(\alpha = F(-1 - si) = -si/(2 + si)\) satisfies (2.2) for each \(s \in \mathbb{R}\). From
\[
\Re F^{-1}(z) = \frac{1}{2}(F^{-1}(z) + \overline{F^{-1}(z)}) = -\frac{1 - |z|^2}{|1 + z|^2},
\]
we can conclude that for \(\alpha \in \Delta\) with (2.2), the image \(F^{-1}(\alpha)\) has \(-1\) as its real part. This complete the unique existence of the proposition.
Now we consider another Cayley transform $G : H \to \Delta$ defined by
\[ F(z) = \frac{z + 1}{z - 1}. \]

In this case $G^{-1} = G$. For $\alpha = -si/(2 + si)$, we have the explicit formula for $g = G^{-1} \circ f_\alpha \circ G \in \text{Aut}(H)$:
\[ g(z) = (G^{-1} \circ f_\alpha \circ G)(z) = z + is. \]

Thus we have
\[ g^{(n)}(z) = (G^{-1} \circ f_\alpha^{(n)} \circ G)(z) = z + ins \]
for any $n \in \mathbb{Z}$. Taking $f_\alpha^{(n)} = f_\alpha = G \circ g^{(n)} \circ G^{-1}$ completes the proof. \( \square \)

As a conclusion, the sequence $(\alpha_n)$ for a parabolic $f_\alpha$ converges to $-1$ as $n \to \pm \infty$. Proposition 2.2 implies

**Lemma 2.4.** If $f_\alpha$ is a parabolic automorphism, then $\rho_n$ converges to
\[ \hat{\rho}(z) = \frac{1 - |z|^2}{|1 - z|^2} \] (2.5)
as $n$ tends to both of $+\infty$ and $-\infty$.

### 2.2.2 The hyperbolic case

Let $f_\alpha$ be a hyperbolic automorphism of $\Delta$. Since each iteration $f_\alpha^{(n)}$ has the common fixed point 1, we have $|f_\alpha^{(n)}(1)|^n = |f'_\alpha(1)|^n$ for each $n \in \mathbb{Z}$ so that
\[ \frac{1 - |\alpha_n|^2}{|1 + \alpha_n|^2} = \left( \frac{1 - |\alpha|^2}{|1 + \alpha|^2} \right)^n \]
from (2.3). If $(1 - |\alpha|^2)/(|1 + \alpha|^2) > 1$, then the left-handed term above tends to $+\infty$ as $n \to +\infty$. Since $1 - |\alpha_n|^n$ is bounded by 1, we can conclude that $\alpha_n \to -1$ as $n \to +\infty$. In case of $(1 - |\alpha|^2)/(|1 + \alpha|^2) < 1$, taking $n \to -\infty$ we get that $\alpha_n \to -1$ as $n \to -\infty$. Using Proposition 2.2 again, we have

**Lemma 2.5.** If $(1 - |\alpha|^2)/(|1 + \alpha|^2) > 1$ or $< 1$, then $\rho_n$ converges to $\hat{\rho}$ as in (2.5) as $n \to +\infty$ or $n \to -\infty$, with respectively.
2.3 A conclusion

Let $f_\alpha$ be an automorphism of $\Delta$ leaving 1 fixed. By Lemma 2.4 and Lemma 2.5, we have the same function $\hat{\rho}(z) = (1 - |z|^2)/|1 - z|^2$ as the limit of $\rho_n$. By an elementary computation, we have

$$1 - |f_\alpha(z)|^2 = \frac{(1 - |\alpha|^2)(1 - |z|^2)}{|1 + \bar{\alpha}z|^2},$$

$$|1 - f_\alpha(z)|^2 = \frac{(1 - |\alpha|^2)^2 |1 - z|^2}{|1 + \alpha|^2 |1 + \bar{\alpha}z|^2},$$

so

$$(\hat{\rho} \circ f_\alpha)(z) = \frac{1 - |f_\alpha(z)|^2}{|1 - f_\alpha(z)|^2} = \frac{|1 + \alpha|^2 |1 - z|^2}{1 - |\alpha|^2} \hat{\rho}(z).$$

This gives the unit disc version of Theorem 1.1.

**Theorem 2.6.** For any $\alpha \in \Delta$, the two-sided sequence $\rho_n = (\rho \circ f^{(n)}_\alpha)/(\rho \circ f^{(n)}_\alpha)(0))$ converges to $\hat{\rho}$ in (2.5) which satisfies

$$\frac{\hat{\rho}}{\rho \circ f_\alpha} = \frac{1 - |\alpha|^2}{|1 + \alpha|^2}.$$

Therefore $f_\alpha$ is parabolic if and only if $\hat{\rho}/(\rho \circ f_\alpha) = 1$.

3 Proof of Theorem 1.1

Let $S$ be a simple connected Riemann surface with a hermitian metric $h_S$ of constant curvature $-4$. Since the complex plane $\mathbb{C}$ and the Riemann sphere $\mathbb{P} \mathbb{C}^1$, the other simply connected Riemann surfaces up to the biholomorphic equivalence, can not admit such metric, $S$ is biholomorphic to the unit disc $\Delta$. Moreover the Schwarz lemma for negatively curved Riemann surfaces due to L. V. Ahlfors [1] (see also [2]) implies that a biholomorphism between $\Delta$ and $S$ is an isometry with respect to the Poincaré metric $h_\Delta$ of $\Delta$ and the hermitian metric $h$. The Poincaré metric is defined by

$$ds^2_\Delta = \frac{1}{(1 - |z|^2)^2} |dz|^2$$

and its distance function is given by

$$d_{h_\Delta}(\zeta, \xi) = \tanh^{-1} \left| \frac{\zeta - \xi}{1 - \xi \bar{\zeta}} \right|$$

for $\zeta, \xi \in \Delta$ (see [5, 2]).
Let $f$ be an automorphism of $S$ without fixed point and let $F : \Delta \to S$ be a biholomorphism. Then the automorphism $F^*f = F^{-1} \circ f \circ F$ of $\Delta$ has at least one fixed point at $\partial \Delta$. Since $\Delta$ is homogeneous and rotationally symmetric, we may assume that $F(0) = p$ and the extension of $F^*f$ to $\overline{\Delta}$ leaves $1$ fixed. That means that $F^*f = f_{\alpha}$ for some $\alpha \in \Delta$. Thus it needs to determine whether $(1 - |\alpha|^2)/|1 + \alpha|^2 = 1$ or not.

Since $F : (\Delta, h_\Delta) \to (S, h_S)$ is isometric, we have $F^*d_{h_S} = d_{h_\Delta}$; more precisely
\[ d_{h_S}(F(0), F(z)) = d_{h_\Delta}(0, z) = \tanh^{-1} |z| \]
for any $z \in \Delta$. This implies
\[ F^*\varphi = \rho \]
where $\rho(z) = |z|^2 - 1$ is the test function in (2.4) and $\varphi(q) = \tanh^2(d_h(p, q)) - 1$ as in Theorem 1.1.

Since $F^*f^{(n)} = f_{\alpha}^{(n)}$ and $F^*(\varphi \circ f^{(n)}) = (F^*\varphi) \circ (F^*f^{(n)})$ for each $n \in \mathbb{Z}$, Theorem 2.6 implies that
\[ F^* \left( \frac{\varphi \circ f^{(n)}}{(\varphi \circ f^{(n)})(p)} \right) = \frac{\rho \circ f_{\alpha}^{(n)}}{(\rho \circ f_{\alpha}^{(n)})(0)} \to \hat{\rho} \]
as $n \to +\infty$ or $-\infty$. Here $\hat{\rho}$ as defined in (2.5) satisfies $\hat{\rho}/(\hat{\rho} \circ f_{\alpha}) = (1 - |\alpha|^2)/|1 + \alpha|$. Since $F$ is a biholomorphism, the two-sided sequence $\varphi \circ f^{(n)}/(\varphi \circ f^{(n)})(p)$ converges to $\hat{\phi} = (F^{-1})^* \hat{\rho}$ and
\[ \frac{\hat{\phi}}{\varphi \circ f} = (F^{-1})^* \left( \frac{\hat{\rho}}{\hat{\rho} \circ f_{\alpha}} \right) = \frac{1 - |\alpha|^2}{|1 + \alpha|^2}. \]
Thus $f$ is parabolic if and only if $\hat{\phi}/(\hat{\phi} \circ f) = 1$. This complete the proof. $\square$

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References


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