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Arbitrary Contract Supersubdivision of Paths and Cycles are Graceful

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Abstract

We investigate some new results for graceful labeling of graph. In this paper we define an arbitrary contract supersubdivision of disjoint union of paths and cycles and furthermore we prove that an arbitrary contract supersubdivision of disjoint union of paths and cycles are graceful.

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Keywords: Graceful labeling, Parallel Cycles, Arbitrary Supersubdivision, Contract Supersubdivision

1 Introduction

We begin with simple, finite, undirected and connected graph $G = (p, q)$. A graceful labeling of G is an injection from the set of its p vertices to the set

$\{0, 1, 2, \ldots, q\}$ such that the values of the edges are all integers from 1 to q, the value of an edge being the absolute value of the difference between the integers attributed to its end vertices.

Sethuraman and Selvaraju [7] have introduced a new method of construction called supersubdivision of a graph and they proved that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They also raised a question that whether some supersubdivision is valid for disconnected graphs [4]. After that Sekar and Ramachandran [6] proved that arbitrary supersubdivision of disconnected graph is graceful. Ambili and Singh [2] defined an arbitrary strong supersubdivision of a graph and they proved that arbitrary strong supersubdivision of paths, cycles and stars are graceful.

Contracting a pair of vertices v_i and v_j replaces them by one vertex v such that v is adjacent to the union of the nodes to which v_i and v_j were originally adjacent. In vertex contraction, it doesn't matter if v_i and v_j are connected by an edge; if they are, then the edge disappear when v_i and v_j are contracted. Pemmaraju and Skiena [5] used the concept of contracting the vertices and they are limited to studying time complexity of the isomorphism problem. Ivanco and Semanicova [3] proved that every 3-regular triangle-free supermagic graph has an edge such that the graph obtained by contracting that edge is also supermagic and the graph obtained by contracting one of the edges joining the two n-cycles of $C_n \times K_2$ $(n \geq 3)$ is supermagic. In [1] Al-Addasi, AbuGhneim, and Al-Ezeh proved that the contraction of a divisor graph along a bridge is a divisor graph; if e is an edge of a divisor graph that lies on an induced even cycle of length at least 6, then the contraction along e is not a divisor graph; and they introduced a special type of vertex splitting that yields a divisor graph when applied to a cut vertex of a given divisor graph. For detail survey on graph labeling in the field of arbitrary supersubdivision one can refer to Gallian [4].

In this paper we define an arbitrary contract supersubdivision of disjoint union of paths and cycles and furthermore we prove that an arbitrary contract supersubdivision of disjoint union of paths and cycles are graceful.

Definition 1.1. *Let G be a graph with q edges. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily).*

Definition 1.2. *The contraction of a pair of vertices v_i and v_j of a graph produces a graph in which the two nodes v_i and v_j are replaced with a single* *node v such that v is adjacent to the union of the nodes to which v_i and v_j were originally adjacent. In vertex contraction, it doesn't matter if v_i and v_j are connected by an edge; if they are, then the edge disappear when v_i and v_j are contracted. Also v_i and v_j are from two components of a disconnected graph.*

Definition 1.3. *Let G be the disjoint union of k copies of paths $P_n^j = v_1^j, v_2^j, \ldots, v_n^j$, $1 \leq j \leq k$. A graph H is called a contract supersubdivision of G if H is obtained from G by replacing every edge $e_i^j = v_i^j v_{i+1}^j$, $1 \leq i \leq n-1$, $1 \leq j \leq k$ of G by a complete bipartite graph K_{2,m_i^j} for some m_i^j, $1 \leq i \leq n-1$ ($m_i^j \geq 2$ if $j = 1, k$ and $m_i^j \geq 3$ if $2 \leq j \leq k-1$) in such a way that the end vertices of each e_i^j are merged with the two vertices of 2-vertices part of K_{2,m_i^j} after removing the edge e_i^j from graph G, then labeling the m_i^j vertices part of K_{2,m_i^j}, $1 \leq i \leq n$, $1 \leq j \leq k$ by $N_{i,1}^j, N_{i,2}^j, N_{i,3}^j, \ldots, N_{i,m_i^j}^j$ and contracting the vertices $N_{i,m_i^j}^j$ and $N_{i,1}^{j+1}$ (say W_i^j), $1 \leq i \leq n-1$, $1 \leq j \leq k-1$, so that H is a connected graph. [For example refer Figure 1]*

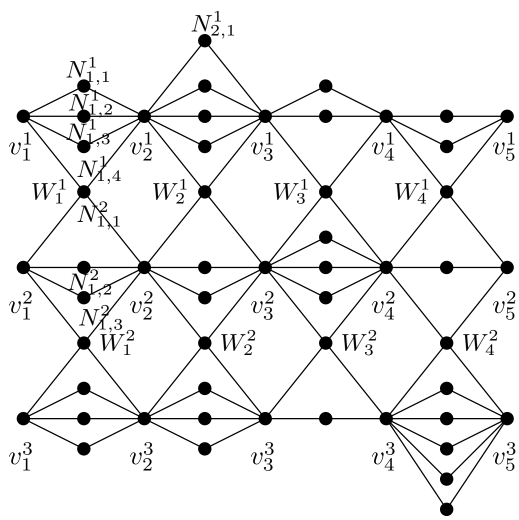

Figure 1

Definition 1.4. *Let G be the disjoint union of k copies of cycles $C_n^j = v_1^j, v_2^j, \ldots, v_n^j, v_1^j$, $1 \leq j \leq k$. A graph H is called a contract supersubdivision of G if H is obtained from G by replacing every edge e_i^j, $1 \leq i \leq n$, $1 \leq j \leq k$ of G by a complete bipartite graph K_{2,m_i^j} for some m_i^j, $1 \leq i \leq n-1$ ($m_i^j \geq 2$ if $j = 1, k$ and $m_i^j \geq 3$ if $2 \leq j \leq k-1$) in such a way that the end vertices of each e_i^j are merged with the two vertices of 2-vertices part of K_{2,m_i^j} after removing the edge e_i^j from graph G, then labeling the m_i^j vertices part of K_{2,m_i^j},*

$1 \leq i \leq n$, $1 \leq j \leq k-1$ by $N^j_{i,1}, N^j_{i,2}, N^j_{i,3}, \ldots, N^j_{i,m^j_i}$ and contracting the vertices N^j_{i,m^j_i} and $N^{j+1}_{i,1}$ (say W^j_i), $1 \leq i \leq n$, $1 \leq j \leq k-1$, so that H is a connected graph. [For example refer Figure 2]

A contract supersubdivision H of G is said to be an arbitrary contract supersubdivision of G if every edge of G is replaced by an arbitrary K_{2,m^j} (m^j may vary for each edge arbitrarily).

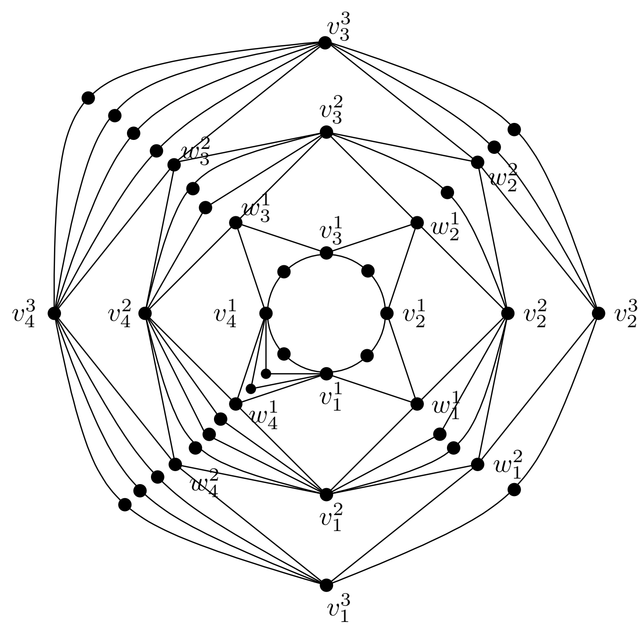

Figure 2

2 Main Results

The graph obtained by arbitrary contract supersubdivision of disjoint union of paths and cycles are graceful.

Theorem 2.1. *The graph obtained by arbitrary contract supersubdivision of paths is graceful.*

Proof. Let G be the disjoint union of k copies of paths $P^j_n = v^j_1, v^j_2, \ldots, v^j_n$, $1 \leq j \leq k$. Let H be an arbitrary contract supersubdivision of G. Then by Definition 1.3, H contains M edges, where M is defined below.
Let

$$M_j = \sum_{i=1}^{n-1} m^j_i,\ 1 \leq j \leq k$$

$E_1 = M = 2\left[\sum_{l=1}^{k} M_l\right]$

$E_j = M - 2\sum_{l=1}^{j-1} M_l + 2(n-1)(j-2),\ 2 \le j \le k$

Define

$v_i^j = (i-1) + 2(j-1)(n-1),\ 1 \le i \le n,\ 1 \le j \le k$

$w_i^j = M - (i-1) - 2\sum_{l=1}^{j} M_l + 2j(n-1),\ 1 \le i \le n-1,\ 1 \le j \le k-1$

$N_i^j = \begin{cases} E_j - 2\sum_{l=1}^{i-1} m_l^j + 3(i-1), & \text{if } j = 1\&k, 1 \le i \le n-1 \\ E_j - 2\sum_{l=1}^{i-1} m_l^j + 5(i-1), & \text{if } 2 \le j \le k-1, 1 \le i \le n-1 \end{cases}$

Note that the two vertices of the 2-vertices part of K_{2,m_i^j}, $1 \le i \le n-1$ get the labels v_i^j and v_{i+1}^j. Now we label the non-contracting $m_i^j - 1$ vertices of K_{2,m_i^j} for $j = 1\&k$ by $N_i^j, N_i^j - 2, N_i^j - 4, \ldots, N_i^j - 2(m_i^j - 2)$ and the non-contracting $m_i^j - 2$ vertices of K_{2,m_i^j} for $2 \le j \le k-1$ by $N_i^j, N_i^j - 2, N_i^j - 4, \ldots, N_i^j - 2(m_i^j - 3)$ as shown Figure 3. It is clear from the above labeling that all the vertices of H have distinct labels and as well as all the edges of H have distinct labels as $M, M-1, \ldots, 3, 2, 1$, so H is graceful.

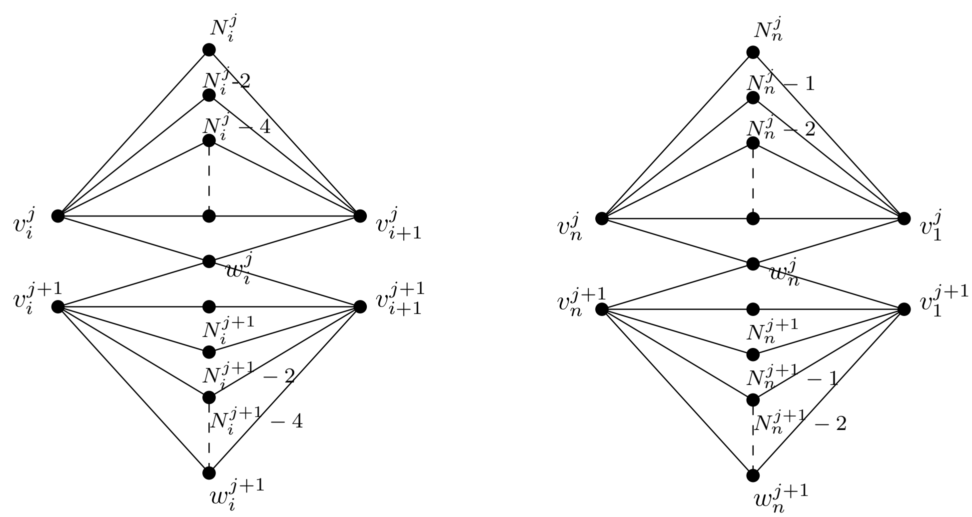

Figure 3

□

Example 2.2. *We can give graceful labeling for some arbitrary contract supersubdivision of the graph which is in Figure 1 as Figure 4.*

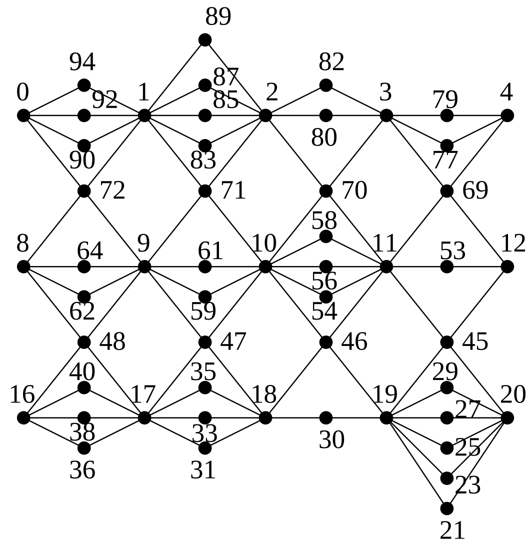

Figure 4

Theorem 2.3. *For $n = 2l$ $(l \geq 2)$, there exist a graph obtained by arbitrary contract supersubdivision of cycles C_n that is graceful.*

Proof. Let G be the disjoint union of k copies of cycles $C_n^j = v_1^j, v_2^j, \ldots, v_n^j, v_1^j$, $1 \leq j \leq k$. Let H be an arbitrary contract supersubdivision of G. Then by Definition 1.4, H contains M edges, where M is defined below. Let H be obtained from G by replacing every edge e_i^j, $1 \leq i \leq n-1$, $1 \leq j \leq k$ of G by a complete bipartite graph K_{2,m_i^j} for some m_i^j, $1 \leq i \leq n-1$ ($m_i^j \geq 2$ if $j = 1, k$ and $m_i^j \geq 3$ if $2 \leq j \leq k-1$) and also replacing every edge $e_n^j = v_n^j v_1^j$, $1 \leq j \leq k$ by a complete bipartite graph K_{2,m_n^j} for some m_n^j ($m_n^j = n$ if $j = 1, k$ and $m_n^j = n+1$ if $2 \leq j \leq k-1$) in such a way that the end vertices of each e_i^j are merged with the two vertices of 2-vertices part of K_{2,m_i^j} after removing the edge e_i^j from graph G.

Let

$$M_j = \sum_{i=1}^{n} m_i^j,\ 1 \leq j \leq k$$

$$E_1 = M = 2\left[\sum_{l=1}^{k} M_l\right]$$

$$E_j = M - 2\sum_{l=1}^{j-1} M_l + 2n(j-2),\ 2 \leq j \leq k$$

Define

$$v_i^j = (i-1) + 2(j-1)n,\ 1 \leq i \leq n,\ 1 \leq j \leq k$$

$$w_i^j = \begin{cases} M-(i-1)-2\sum_{l=1}^{j} M_l + 2jn, & 1 \le i \le \frac{n}{2}, 1 \le j \le k-1 \\ M-i-2\sum_{l=1}^{j} M_l + 2jn, & \frac{n+2}{2} \le i \le n, 1 \le j \le k-1 \end{cases}$$

$N_1^j = E_j$, $1 \le j \le k$

$$N_i^j = \begin{cases} E_j - 2\sum_{l=1}^{i-1} m_l^j + 3(i-1), & \text{if } j = 1\&k, 2 \le i \le n-1 \\ E_j - 2\sum_{l=1}^{i-1} m_l^j + 5(i-1), & \text{if } 2 \le j \le k-1, 2 \le i \le n-1 \end{cases}$$

$N_n^j = E_{j+1} + 4n - 2$, $1 \le j \le k-1$, $N_n^k = 2(nk-1)$

Note that the two vertices of the 2-vertices part of K_{2,m_i^j}, $1 \le i \le n-1$ get the labels v_i^j & v_{i+1}^j and the two vertices of the 2-vertices part of K_{2,m_n^j} get the labels v_n^j & v_1^j. Now we label the non-contracting $m_i^j - 1$ vertices of K_{2,m_i^j} for $j = 1\&k$ by N_i^j, $N_i^j - 2$, $N_i^j - 4$, $\ldots$, $N_i^j - 2(m_i^j - 2)$, the non-contracting $m_i^j - 2$ vertices of K_{2,m_i^j} for $2 \le j \le k-1$ by N_i^j, $N_i^j - 2$, $N_i^j - 4$, $\ldots$, $N_i^j - 2(m_i^j - 3)$ and the non-contracting $n-1$ vertices of K_{2,m_n^j} for $1 \le j \le k$ by N_n^j, $N_n^j - 1$, $N_n^j - 2$, $\ldots$, $N_n^j - (n-2)$ as shown Figure 3. It is clear from the above labeling that all the vertices of H have distinct labels and as well as all the edges of H have distinct labels as $M, M-1, \ldots, 3, 2, 1$, so H is graceful. □

Example 2.4. *We can give graceful labeling for some arbitrary contract supersubdivision of the 4-block cycles C_4 as Figure 5.*

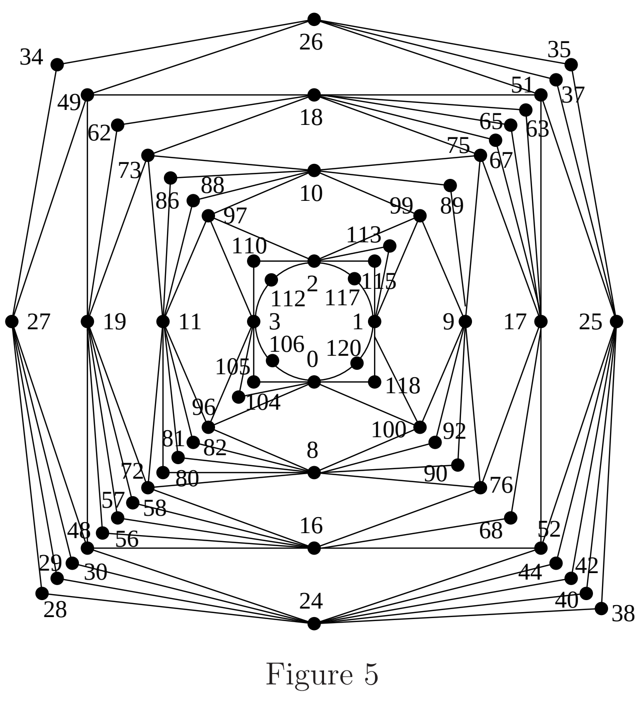

Figure 5

Conclusion

In this paper we defined an arbitrary contract supersubdivision of disjoint union of paths and cycles and furthermore we proved that an arbitrary contract supersubdivision of disjoint union of paths and cycles are graceful.

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